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On the Frobenius Norm of Commutator of Cauchy-Hankel Matrix and Exchange Matrix

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Abstract. This paper gives upper and lower bounds for the Frobenius norm of the commutator of the exchange matrix and the Cauchy-Hankel matrix of the form $H_n = \left(\frac{2}{1+2(i+j)}\right)_{i,j=1}^{n}$.

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1. Introduction

In recent years, several studies have aimed to establish an upper bound for the Frobenius norm of matrix commutators. Motivated by this, this paper aims to give an upper and lower bound for the Frobenius norm of the commutator of the Cauchy-Hankel matrix and the exchange matrix.

Matrices of order n in the following forms are called Cauchy and Hankel matrices,

$$C_n = \left(\frac{1}{x_i - y_j}\right)_{i,j=1}^{n},$$

where $x_i \neq y_j$ for all i, j and

$$H_n = \left(h_{i+j}\right)_{i,j=1}^{n},$$

respectively. In general, the Cauchy-Hankel matrix is defined as follows:

$$H_n = \left(h_{ij}\right)_{i,j=1}^{n} = \left(\frac{1}{g+(i+j)h}\right)_{i,j=1}^{n}, \tag{1.1}$$

where $h \neq 0$, g and h are some numbers and g/h is not an integer. Recently, several studies have also been conducted on the norms of the Cauchy-Hankel matrix and the Cauchy-Toeplitz matrix [2, 13, 15]. Solak and Bozkurt have found bounds for the spectral norm and Euclidean norm of the Cauchy-Hankel matrix [12]. Turkmen and Bozkurt have obtained an upper bound for the spectral norm of Cauchy-Hankel matrices [14]. In addition, Solak and Bahşi have

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obtained upper and lower bounds for the Frobenius norm of the commutator of the Cauchy-Toeplitz matrix and the exchange matrix [11].

The commutator of any two complex matrices $A, B \in M_n(\mathbb{C})$ is defined as $[A, B] = AB - BA$, and the anticommutator is defined as $(A, B) = AB + BA$. Commutators play an important role in many branches of mathematics, physics, as well as quantum physics and quantum chemistry [1,5]. In recent studies, some bounds have been obtained for the norms of matrix commutators. Böttcher and Wenzel [4] showed that

$$\|[A, B]\|_F \leq \sqrt{2}\,\|A\|_F\,\|B\|_F \tag{1.2}$$

for the Frobenius norm of the commutator of two nxn complex matrices A and B. Böttcher and Wenzel first conjectured this inequality for real 2x2 matrices [3]. Later, László proved this inequality for 3x3 real matrices [8], and Lu [9] and, independently, Vong and Jin [16] proved it for nxn real matrices. The proof of the inequality for nxn complex matrices was made by Böttcher and Wenzel. Additionally, a different proof of the inequality was given by Audenaert [1]. Wu and Liu [17] improved the Böttcher and Wenzel inequality for real nxn matrices and proposed the upper bound

$$\|[X, Y]\|_F^2 \leq 2\,\|X\|_F^2\,\|Y\|_F^2 - 2\left[tr\left(X^T Y\right)\right]^2. \tag{1.3}$$

Gil' [6] derived a sharp bound for the Frobenius norm of the commutator $[A, A^*]$ (self-commutator) for an arbitrary nxn matrix A, where A^* is the conjugate transpose of the matrix A.

Now, we give some preliminaries related to our study. The Frobenius norm of a mxn matrix A is

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)^{1/2}. \tag{1.4}$$

The nxn exchange matrix is the permutation matrix

$$K_n = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} = (k_{ij}) \in M_n$$

in which $k_{i,n-i+1} = 1$ for $i = 1, \ldots, n$ and all other entries are zero [7]. A function ψ is called as psi (or digamma) function if

$$\psi(x) = \frac{d}{dx}\left(\ln \Gamma(x)\right),$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

It is called as polygama function the nth derivative of the psi function [10], i.e.,

$$\psi(n, x) = \frac{d}{dx^n}\psi(x) = \frac{d}{dx^n}\left(\frac{d}{dx}\ln \Gamma(x)\right).$$

In addition, if $a > 0$ and positive integer b, then [14]

$$\lim_{n\to\infty} \psi(a, n + b) = 0.$$

The paper is organized as follows. In Section 2, we give upper and lower bounds for the Frobenius norm of the commutator of the exchange matrix and the Cauchy-Hankel matrix of the form $H_n = \left(\frac{2}{1 + 2(i + j)}\right)_{i,j=1}^{n}$. Then, we confirm these bounds with tables and graphs.

2. Commutator of the Matrices H_n and P

If we substitute $g = 1/2$ and $h = 1$ into (1.1), then we have

$$H_n = \left(h_{ij}\right)_{i,j=1}^{n} = \left(\frac{2}{1+2(i+j)}\right)_{i,j=1}^{n}.$$

The commutator and anticommutator of the matrices H_n and P are

$$[H_n, P] = H_nP - PH_n = (a_{ij})_{i,j=1}^{n}$$

and

$$(H_n, P) = H_nP + PH_n = (b_{ij})_{i,j=1}^{n},$$

respectively. Since left (or right) multiplication by the exchange matrix P against the matrix H_n reverses the row (or column) order of H_n, $H_nP = (h_{i,n+1-j})$ and $PH_n = (h_{n+1-i,j})$. Thus,

$$a_{ij} = h_{i,n+1-j} - h_{n+1-i,j} = 2\frac{(i-j)}{(i-j)^2 - \left(\frac{3}{2}+n\right)^2}$$

and

$$b_{ij} = h_{i,n+1-j} + h_{n+1-i,j} = \frac{(3+2n)}{\left(\frac{3}{2}+n\right)^2 - (i-j)^2}.$$

Here are the main results we obtained.

Theorem 2.1. *The Frobenius norm of the commutator matrix* $[H_n, P]$ *holds*

$$\|[H_n, P]\|_F^2 = 2\,\|H\|_F^2 - \frac{2}{(3+2n)}x_n,$$

where x_n *is the sum of the elements of the anticommutator matrix* (H_n, P)*, that is*

$$x_n = \sum_{i,j=1}^{n} b_{ij} = 10 - 3\gamma - 6\ln 2 + (4n+3)\psi\left(2n+\frac{3}{2}\right) - (4n+6)\psi\left(\frac{5}{2}+n\right).$$

Proof. By considering the definition of the Frobenius norm, we have

$$\begin{aligned}
\|[H_n, P]\|_F^2 &= \sum_{i,j=1}^{n} a_{ij}^2 = \sum_{i,j=1}^{n}(h_{i,n+1-j} - h_{n+1-i,j})^2 = \sum_{i,j=1}^{n}\left(\frac{1}{\frac{3}{2}+n+i-j} - \frac{1}{\frac{3}{2}+n-i+j}\right)^2 \\
&= \sum_{i,j=1}^{n}\left(\frac{1}{\frac{3}{2}+n+i-j}\right)^2 + \sum_{i,j=1}^{n}\left(\frac{1}{\frac{3}{2}+n-i+j}\right)^2 - 2\sum_{i,j=1}^{n}\frac{1}{\left(\frac{3}{2}+n\right)^2 - (i-j)^2} \\
&= \|H_n\|_F^2 + \|H_n\|_F^2 - 2\sum_{i,j=1}^{n}\frac{b_{ij}}{3+2n} \\
&= 2\,\|H_n\|_F^2 - \frac{2}{3+2n}x_n,
\end{aligned}$$

where

$$\begin{aligned}
x_n &= \sum_{i,j=1}^{n} b_{ij} = \sum_{i,j=1}^{n} 2\frac{\left(\frac{3}{2}+n\right)}{\left(\frac{3}{2}+n\right)^2 - (i-j)^2} \\
&= 10 - 3\gamma - 6\ln 2 + (4n+3)\psi\left(2n+\frac{3}{2}\right) - (4n+6)\psi\left(\frac{5}{2}+n\right).
\end{aligned}$$

□

From the definition of Forbenius norm in (1.4), we write

$$\|H_n\|_F^2 = \left\{\sum_{k=1}^{n} \frac{k}{(g+(k+1)\,h)^2} + \sum_{k=1}^{n} \frac{n-k}{(g+(n+k+1)\,h)^2}\right\}$$

$$\|H_n\|_F^2 = 4 - \frac{3\pi^2}{4} + \gamma + 2\ln 2 + 2\psi\left(\frac{5}{2}+n\right) - \left(2n+\frac{3}{2}\right)\psi\left(1, 2n+\frac{5}{2}\right) - \psi\left(2n+\frac{5}{2}\right) + (2n+3)\,\psi\left(1, \frac{5}{2}+n\right). \tag{2.1}$$

Thus, the Frobenius norm of the commutator matrix $[H_n, P]$ can also be expressed in terms of psi and polygamma functions as follows:

Corollary 2.2. *Theorem 2.1 and Equation (2.1) yield*

$$\begin{aligned}\|[H_n,P]\|_F^2 =& 8 + 2\gamma + 4\ln 2 + 8\psi\left(\frac{5}{2}+n\right) - \frac{3\pi^2}{2} - (4n+3)\,\psi\left(1, \frac{5}{2}+2n\right) \\ & - 2\psi\left(\frac{5}{2}+2n\right) + (4n+6)\,\psi\left(1, \frac{5}{2}+n\right) + \frac{-20+6\gamma+12\ln 2}{2n+3} - (8n+6)\,\frac{\psi\left(2n+\frac{3}{2}\right)}{2n+3}.\end{aligned}$$

Before moving on to our work's main theorem, we give some bounds for the Frobenius norm of the commutator matrix $[H_n, P]$. According to upper bound of Böttcher and Wenzel in (1.2), we have

$$\|[H_n,P]\|_F \leq \sqrt{2}\,\|H_n\|_F\,\|P\|_F = \sqrt{2n}\,\|H_n\|_F\,, \tag{2.2}$$

where $\|P\|_F = \sqrt{n}$. Additionally, according to the upper bound proposed by Wu and Liu in (1.3), we get

$$\|[H_n,P]\|_F^2 \leq 2n\,\|H_n\|_F^2 - 2\,[tr\,(H_nP)]^2\,, \tag{2.3}$$

where $H_n^T = H_n$ and $\|P\|_F = \sqrt{n}$.
Let's now present the fundamental theorem that is essential for our work.

Theorem 2.3. *For the Frobenius norm of the commutator matrix $[H_n, P]$, we have*

$$\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n\ln 2}{3+2n}} \leq \|[H_n,P]\|_F \leq \sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n}{15+10n}}, \tag{2.4}$$

where $n \geq 8$.

Proof. Consider the sequence $\{x_n\}_{n\geq 1}$, where

$$x_n = 10 - 3\gamma - 6\ln 2 + (4n+3)\psi\left(2n+\frac{3}{2}\right) - (4n+6)\psi\left(\frac{5}{2}+n\right).$$

If we divide both sides by n of this equality and take the limit as $n \to \infty$, then we have

$$\lim_{n\to\infty} \frac{1}{n}x_n = \lim_{n\to\infty} \frac{1}{n}\left\{10 - 3\gamma - 6\ln 2 + (4n+3)\psi\left(2n+\frac{3}{2}\right) - (4n+6)\psi\left(\frac{5}{2}+n\right)\right\} = 4\ln 2.$$

Also, the sequence $\left\{\frac{1}{n}x_n\right\}_{n\geq 1}$ strictly increasing and $\frac{1}{n}x_n = \frac{4}{5}$ for $n = 1$. Thus,

$$\frac{4}{5} \leq \frac{1}{n}x_n \leq 4\ln 2$$

and

$$\frac{4n}{5} \leq x_n \leq 4n\ln 2\,.$$

Therefore, from Theorem 2.1 we get

$$\|[H_n,P]\|_F^2 = 2\,\|H_n\|_F^2 - \frac{2}{3+2n}x_n$$

$$\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n\ln 2}{3+2n}} \leq \|[H_n,P]\|_F \leq \sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n}{15+10n}}\,.$$

□

Table 1 compares our obtained bounds for the Frobenius norm of the commutator matrix $[H_n, P]$ with the other bounds. The values in the second column of the table denote our lower bounds given in Theorem 2.3, the values in the third column denote the values of $\|[H_n, P]\|_F$, the values in the fourth column denote our upper bounds given in Theorem 2.3, the values in the fifth column denote the upper bound proposed by Wu and Liu in (2.3), and the values in the sixth column denote the upper bound of Böttcher and Wenzel in (2.2).

Table 1. Bounds for Frobenius norm of $[H_n, P]$ for $g = 1/2$ and $h = 1$.

n	$\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n\ln 2}{3+2n}}$	$\|[H_n, P]\|_F$	$\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n}{15+10n}}$	$\sqrt{2n\,\|H_n\|_F^2 - 2\,[tr\,(H_nP)]^2}$	$\sqrt{2n}\,\|H_n\|_F$
8	0.2674	0.8644	1.3163	4.2230	4.3876
9	0.4616	0.9244	1.3798	4.6732	4.8279
10	0.5891	0.9784	1.4361	5.1057	5.2517
11	0.6879	1.0274	1.4863	5.5225	5.6609
12	0.7699	1.0722	1.5318	5.9254	6.0573
20	1.1683	1.3346	1.7888	8.7835	8.8815
30	1.4298	1.5379	1.9807	11.7783	11.8551
40	1.5974	1.6781	2.1107	14.3915	14.4559
50	1.7196	1.784	2.2034	16.7499	16.8061
100	2.0639	2.0962	2.4873	26.4043	26.4411
200	2.3671	2.3835	2.7611	40.8546	40.8788

Table 1 indicates that our upper bound (2.4) is more useful than the upper bound of Böttcher-Wenzel and Wu-Liu, for $g = 1/2$, $h = 1$ and $n \geq 8$, and that our upper and lower bounds are quite sharp.

3. Numerical Results

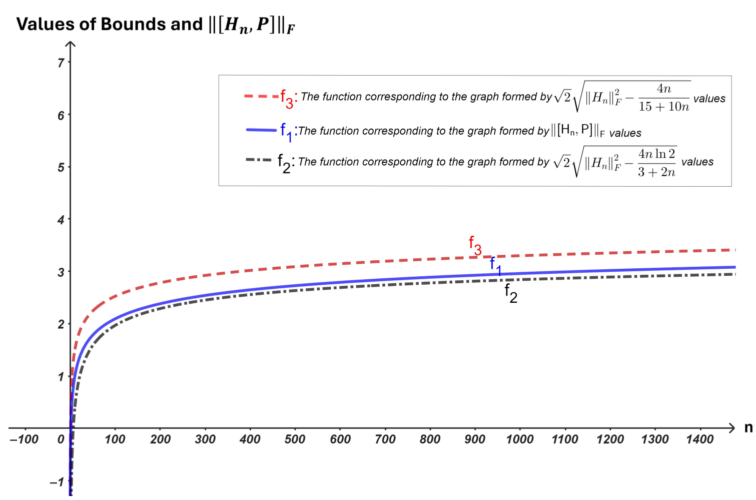

Figure 1. The values of $\|[H_n, P]\|_F$ and the values of its lower and upper bounds

The graphs corresponding to the $\|[H_n, P]\|_F$, $\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n\ln 2}{3+2n}}$ and $\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n}{15+10n}}$ values in Table 1 are in Figure 1. In Figure 1, the horizontal axis represents the order of the matrices, while the vertical axis represents the corresponding values of $\|[H_n, P]\|_F$ and bounds. The general form of the functions corresponding to the graphs above is

$$f(n) = \frac{ab + cn^d}{b + n^d}.$$

The function corresponding to the graph formed by $\|[H_n, P]\|_F$ values is

$$f_1(n) = \frac{-5.1436419468 + 4.5546737739\, n^{0.2412436924}}{1.1107828697 + n^{0.2412436924}},$$

where a=-4.6306457248, b=1.1107828697, c=4.5546737739, d=0.2412436924. The function corresponding to the graph formed by $\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n\ln 2}{3+2n}}$ values is

$$f_2(n) = \frac{-6.1017453675 + 3.8823553162\, n^{0.2650115291}}{0.0000564209 + n^{0.2650115291}},$$

where a=-108146.8988888769, b=0.0000564209, c=3.8823553162, d=0.2650115291. The function corresponding to the graph formed by $\sqrt{2}\sqrt{\|H_n\|_F^2 - \frac{4n}{15+10n}}$ values is

$$f_3(n) = \frac{-5.5216639593 + 5.539526468\, n^{0.1289962945}}{0.0002426437 + n^{0.1289962945}},$$

where a=-22756.2634400381, b=0.0002426437, c=5.5395264680, d=0.1289962945. If we take the limits for the functions f_1, f_2 and f_3 as $n \to \infty$, we get

$$\lim_{n\to\infty} f_1(n) = 4.5546737739,$$

$$\lim_{n\to\infty} f_2(n) = 3.8823553162,$$

$$\lim_{n\to\infty} f_3(n) = 5.5395264680.$$

Hence, we can write

$$\lim_{n\to\infty} f_2(n) \le \lim_{n\to\infty} f_1(n) \le \lim_{n\to\infty} f_3(n).$$

Consequently, the graphs confirm our bounds.

4. Conclusion

In this study, we determined the upper and lower bounds for the Frobenius norm of the Cauchy-Hankel matrix and exchange matrix. Moreover, our numerical examples in Table 1 showed that our bounds are sharp, and the graphs in Figure 1 confirm our bounds.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Both authors have created and written the article.

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