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Research Article

Lifts of L-valued powerset mapping systems

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Abstract

Motivated by Zadeh's extension of mappings, this paper introduces a concept of lifts of powerset mapping systems. In this framework, L-subsets are lifts of ordinary subsets, L-Zadeh functions are lifts of ordinary mappings, L-fuzzy rough set operators are lifts of classical rough set operators, L-interior/closure operators of L-topology are lifts of interior/closure operators of level topologies. Consequently, lifts of powerset mapping systems can be considered as a useful fuzzy structures for future study.

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1. Introduction and preliminaries

The decomposition theorems of fuzzy sets are the most basic tools in fuzzy set theory [4]. They strongly connect levelwise approaches to fuzzy mathematical structures, which is used by a lot of researches in the past 40 years (there are too many related literature, we omitted here). In the viewpoint of fuzzy set researchers, everything is mapping, for example, fuzzy sets are mappings from the background set to the valued lattice [8]; fuzzy relations are mappings from Cartesian product of the background set to the valued lattice [2], fuzzifying structures (eg. topologies, matroids, etc) are mappings from the powerset of the background set to the valued lattice [3, 5]; fuzzy groups are mappings from a group to the valued lattices [1].

This short paper aims to provide a common framework of fuzzy subsets, Zadeh functions of ordinary mappings, fuzzy rough operators, fuzzy interior operators of fuzzy meet structures, fuzzy closure operators of fuzzy join structures. For details, we will introduce a concept of lattice-valued powerset mapping systems and then propose the concept of lifts, so that we can represent all the structures mentioned above to lifts of some kinds of powerset mapping systems.

Let L be a complete lattice and $a, b \in L$. The element a is called wedge-blow b, in symbols $a \lhd b$, if for all $S \subseteq L$, $b \leq \bigvee S$ always implies $a \leq s$ for some $s \in S$; and a is called co-wedge-blow b, in symbols $a \prec b$, if for all $S \subseteq L$, $\bigwedge S \leq a$ always implies $s \leq b$ for some $s \in S$. Both $a \lhd b$ and $a \prec b$ imply $a \leq b$ for all $a, b \in L$.

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A complete lattice L is called completely distributive if $a = \bigvee\{b \in L \mid b \lhd a\}$ for all $a \in L$, or equivalently, $a = \bigwedge\{c \in L \mid a \prec c\}$ for all $a \in L$. It is well-known that a complete lattice L is completely distributive iff its opposite poset L^{op} is completely distributive. In this paper, L always denotes a completely distributive lattice unless otherwise stated.

Let X be a nonempty set and $a \in L$, $A \in L^X$. Denote the following four cut sets [4] by

$$\begin{array}{rcl} A_{[a]} & = & \{x \in X \mid A(x) \geq a\}; \\ A_{(a)} & = & \{x \in X \mid a \lhd A(x)\}, \\ A^{[a]} & = & \{x \in X \mid A(x) \nleq a\}, \\ A^{(a)} & = & \{x \in X \mid A(x) \nprec a\}. \end{array}$$

It is routine to show that $A(\lambda) \subseteq A_{[\lambda]}$ and $A^{(\lambda)} \subseteq A^{[\lambda]}$ hold for all $\lambda \in L$. For a completely distributive lattice L, if $a \lhd b$, then $A_{[b]} \subseteq A_{(a)}$, and if $b \prec a$, then $A^{[a]} \subseteq A^{(b)}$. For $L = [0,1]$, we have $A_{(\lambda)} = A^{(\lambda)}$ and $A_{[\lambda]} = A^{[\lambda]}$ $(\forall \lambda \in [0,1])$.

Proposition 1.1. Let $\{A_i \mid i \in I\} \subseteq L^X$ and $\lambda \in L$. Then
(1) $(\bigvee_i A_i)_{(\lambda)} = \bigcup_i (A_i)_{(\lambda)}$;
(2) $(\bigvee_i A_i)^{(\lambda)} = \bigcup_i (A_i)^{(\lambda)}$;
(3) $(\bigwedge_i A_i)_{[\lambda]} = \bigcap_i (A_i)_{[\lambda]}$;
(4) $(\bigwedge_i A_i)^{[\lambda]} = \bigcap_i (A_i)_{[\lambda]}$.

2. L-valued powerset mapping systems

In this section, we will introduce a concept of lattice-valued powerset mapping systems.

Definition 2.1. Let X, Y be two nonempty sets. A family $F = \{F_\lambda \mid \lambda \in L\}$ is called an L-valued *powerset mapping system* from X to Y if every F_λ is a mapping from 2^X to 2^Y, where F_λ is called the λ-component of F.

An L-valued powerset mapping system F from X to Y is called:
(1) *constant* if every λ-component is a fixed mapping $h : 2^X \longrightarrow 2^Y$, denoted by $\overline{h}$.
(2) *natural* if every F_λ is monotone w.r.t the set-inclusion order for every $\lambda \in L$ and $b \leq a$ implies $F_a \leq F_b$ for all $a, b \in L$.

Definition 2.2. Let F be an L-valued powerset mapping system from X to Y. Define $F^+, F^- : L^X \longrightarrow L^Y$ by

$$F^+(A) = \bigvee_{\lambda \in L} \lambda \wedge F_\lambda(A_{[\lambda]}), \qquad F^-(A) = \bigwedge_{\lambda \in L} \lambda \vee F_\lambda(A^{(\lambda)}).$$

For a complete lattice L, the decomposition theorems of L-subsets say that every L-subset A of a set X can be represented as the forms of $\bigvee_{\lambda \in L} \lambda \wedge A_{[\lambda]}$ and $\bigwedge_{\lambda \in L} \lambda \vee A^{(\lambda)}$. We can formulate them as lifts of L-valued powerset mapping systems.

Example 2.3 (Representation of L-subsets)**.** Let X be a set and let $A \in L^X$. For every $\lambda \in L$, define $[A]_\lambda,\ (A)_\lambda : 2^\varnothing \longrightarrow 2^X$ by

$$[A]_\lambda(\varnothing) = A_{[\lambda]}, \quad (A)_\lambda(\varnothing) = A^{(\lambda)} \ (\forall \lambda \in L).$$

Then $[A]^+ = A = (A)^-$.

Proof. Since $|L^\varnothing| = 1$, we denote $L^\varnothing = \{*\}$. Then

$$[A]^+(*) = \bigvee_{\lambda \in L} \lambda \wedge [A]_\lambda(*_{[\lambda]}) = \bigvee_{\lambda \in L} \lambda \wedge [A]_\lambda(\varnothing) = \bigvee_{\lambda \in L} \lambda \wedge A_{[\lambda]} = A,$$

$$(A)^-(*) = \bigwedge_{\lambda \in L} \lambda \vee (A)_\lambda(*^{(\lambda)}) = \bigwedge_{\lambda \in L} \lambda \vee (A)_\lambda(\varnothing) = \bigwedge_{\lambda \in L} \lambda \wedge A^{(\lambda)} = A.$$

Hence, $[A]^+ = A = (A)^-$. □

Let $f : X \longrightarrow Y$ be a mapping. The L-Zadeh functions $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ are defined by $\forall A \in L^X$, $\forall B \in L^Y$,

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) \mid f(x) = y\} \ (\forall y \in Y);$$
$$f_L^{\leftarrow}(B) = B \circ f.$$

They are exactly a kind of lifts of the ordinary mapping f. While we also can formulate them as lifts of L-valued powerset mapping systems.

Example 2.4 (Representation of L-Zadeh functions)**.** Every mapping $f : X \longrightarrow Y$ can induce two constant L-valued powerset mapping system: the first one is $\overline{f^{\rightarrow}}$ from X to Y, and the second one is $\overline{f^{\leftarrow}}$ from Y to X. Then

$$\overline{f^{\rightarrow}}^{+} = f_L^{\rightarrow} \text{ and } \overline{f^{\leftarrow}}^{-} = f_L^{\leftarrow}.$$

Proof. This result has been proved by Theorem 5.1(1,2) in [4] for L being a completely distributive lattice. In this paper, we give a proof for L just being a complete lattice. For $f^{\rightarrow}$, let $A \in L^X$ and $y \in Y$, we have

$$\overline{f^{\rightarrow}}^{+}(A)(y) = \bigvee_{\lambda\in L} \lambda \wedge f^{\rightarrow}(A_{[\lambda]})(y) = \bigvee_{y\in f^{\rightarrow}(A_{[\lambda]})} \lambda.$$

If $y \in f^{\rightarrow}(A_{[\lambda]})$, then there exists $x \in X$ such that $f(x) = y$ and $A(x) \geq \lambda$. Therefore,

$$\bigvee_{y\in f^{\rightarrow}(A_{[\lambda]})} \lambda \leq \bigvee_{f(x)=y} A(x) = f_L^{\rightarrow}(A)(y).$$

Conversely, for all $x \in X$ with $f(x) = y$, put $\lambda = A(x)$, then we have $y \in f^{\rightarrow}([A]_{\lambda})$. Therefore,

$$f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x) \leq \bigvee_{y\in f^{\rightarrow}(A_{[\lambda]})} \lambda.$$

Hence, $\overline{f^{\rightarrow}}^{+}(A)(y) = f_L^{\rightarrow}(A)(y)$ and the arbitrariness of y and A, $\overline{f^{\rightarrow}}^{+} = f_L^{\rightarrow}$.

For $f^{\leftarrow}$, let $B \in L^Y$ and $x \in X$, we have

$$\begin{aligned}\overline{f^{\leftarrow}}^{-}(B)(x) &= \bigwedge_{\lambda\in L} \lambda \vee f^{\leftarrow}(B^{(\lambda)})(x)\\ &= \bigwedge_{x\notin f^{\leftarrow}(B^{(\lambda)})} \lambda = \bigwedge_{f(x)\notin B^{(\lambda)}} \lambda\\ &= \bigwedge_{B(f(x))\leq\lambda} \lambda = B(f(x)) = f_L^{\leftarrow}(B)(x).\end{aligned}$$

Hence, $\overline{f^{\leftarrow}}^{-} = f_L^{\leftarrow}$. □

In [7], by the unit interval as the valued lattice, Wu used the decomposition form to introduce a pair of L-fuzzy rough operators. In [6], more types of decomposition forms of L-fuzzy rough operators have been constructed and studied. In fact, those definitions can be formulated as lifts of L-valued powerset mapping systems.

Let $(L, ')$ be a De Morgan algebra (that is, $' : L \longrightarrow L$ is an order-reversing involution) and let $R : X \times Y \longrightarrow L$ be an L-relation from X to Y. A kind of L-fuzzy rough operators $\overline{\mathrm{Apr}}_R$, $\underline{\mathrm{Apr}}_R : L^Y \longrightarrow L^X$ by $\forall A \in L^Y$, $\forall x \in X$,

$$\overline{\mathrm{Apr}}_R(A)(x) = \bigvee_{y\in X} A(y) \wedge R(x, y),$$

$$\underline{\mathrm{Apr}}_R(A)(x) = \bigwedge_{y\in X} R'(x, y) \vee A(y).$$

If R is a classical binary relation from X to Y, i.e., $R \subseteq X \times Y$, then

$$\overline{\mathrm{Apr}}_R(A) = \{x \in X \mid \exists y \in A \text{ s.t. } (x, y) \in R\},$$
$$\underline{\mathrm{Apr}}_R(A) = \{x \in X \mid \forall y \in Y,\ (x, y) \in R \Rightarrow y \in A\}.$$

Example 2.5. (Representation of L-fuzzy rough operators) Let $(L,')$ be a De Morgan algebra and let $R : X \times Y \longrightarrow L$ be an L-relation from X to Y. Define $[R] = \{\overline{\mathrm{Apr}}_{R_{[\lambda]}} \mid \lambda \in L\}$ and $(R) = \{\underline{\mathrm{Apr}}_{R_{[\lambda']}}\}$. Then

$$[R]^+ = \overline{\mathrm{Apr}}_R, \quad (R)^- = \underline{\mathrm{Apr}}_R(A).$$

Proof. This result has been proved by Theorems 4.13 and 4.19 in [6] for L being a completely distributive De Morgan algebra. While they hold for a usual De Morgan algebra. For every $x \in X$,

$$\begin{aligned}
[R]^+(A)(x) &= [\bigvee_{\lambda\in L} \lambda \wedge \overline{\mathrm{Apr}}_{R_{[\lambda]}}(A_{[\lambda]})](x) \\
&= \bigvee\{\lambda \in L \mid x \in \overline{\mathrm{Apr}}_{R_{[\lambda]}}(A_{[\lambda]})\} \\
&= \bigvee\{\lambda \in L \mid \exists y \in A_{[\lambda]}\ s.t.\ (x,y) \in R_{[\lambda]}\} \\
&= \bigvee\{\lambda \in L \mid \exists y \in X\ s.t.\ A(y) \geq \lambda,\ R(x,y) \geq \lambda\} \\
&= \bigvee\{\lambda \in L \mid \exists y \in X\ s.t.\ R(x,y) \wedge A(y) \geq \lambda\} \\
&= \bigvee_{y\in X} A(y) \wedge R(x,y) \\
&= \overline{\mathrm{Apr}}_R(A)(x).
\end{aligned}$$

$$\begin{aligned}
(R)^-(A)(x) &= [\bigwedge_{\lambda\in L} \lambda \vee \underline{\mathrm{Apr}}_{R_{[\lambda']}}(A^{(\lambda)})](x) \\
&= \bigwedge\{\lambda \in L \mid x \notin \underline{\mathrm{Apr}}_{R_{[\lambda']}}(A^{(\lambda)})\} \\
&= \bigwedge\{\lambda \in L \mid \exists y \in R_{[\lambda']} - A^{(\lambda)}\} \\
&= \bigwedge\{\lambda \in L \mid \exists y \in X\ s.t.\ R(x,y) \geq \lambda',\ A(y) \leq \lambda\} \\
&= \bigwedge\{\lambda \in L \mid \exists y \in X\ s.t.\ R'(x,y) \leq \lambda,\ A(y) \leq \lambda\} \\
&= \bigwedge\{\lambda \in L \mid \exists y \in X\ s.t.\ R'(x,y) \vee A(y) \leq \lambda\} \\
&= \bigwedge_{y\in X} R'(x,y) \vee A(y) \\
&= \underline{\mathrm{Apr}}_R(A)(x).
\end{aligned}$$

Hence, $[R]^+ = \overline{\mathrm{Apr}}_R$, $(R)^- = \underline{\mathrm{Apr}}_R(A)$. □

The L-fuzzy rough operators are in fact the interior and closure operators of an L-topology on X induced by R. In a broad sense, we can formulate the interior and closure operators of ordinary L-topology as lifts of L-valued mapping systems.

Let X be a set. A family $\mathcal{U} \subseteq L^X$ (resp., $\mathcal{N} \subseteq L^X$) is called an *$L$-join structure* (resp., *L-meet structure*) on X if $\bigvee_{i\in I} A_i \in \mathcal{U}$ (resp., $\bigwedge_{i\in I} A_i \in \mathcal{N}$) for all $\{A_i \mid i \in I\} \subseteq \mathcal{U}$ (resp., $\{A_i \mid i \in I\} \subseteq \mathcal{N}$). It is routine to show that if L is a completely distributive lattice, then for every *L-join structure* $\mathcal{U}$ (resp., L-meet structure $\mathcal{N}$) on X and for every $\lambda \in L$, the family $\mathcal{U}_{(\lambda)} = \{A_{(\lambda)} \mid A \in \mathcal{N}\}$ is an ordinary join-structure (resp., $\mathcal{N}^{[\lambda]} = \{A^{[\lambda]} \mid A \in \mathcal{N}\}$ is an ordinary meet-structure) on X.

Example 2.6. (Representation of interior operators of L-join structures) Let $\mathcal{U}$ be an L-join-structure on X with the property of $a \wedge B_{(\lambda)} \in \mathcal{U}$ $(\forall \lambda \in L, \forall B \in \mathcal{U})$. For all $\lambda \in L$, denote int_λ as its interior operator. Then $\mathrm{int}^+ = \mathrm{int}_{\mathcal{U}}$, that is,

$$\mathrm{int}_{\mathcal{U}}(A) = \bigvee_{\lambda\in L} \lambda \wedge \mathrm{int}_\lambda(A_{[\lambda]})\ (\forall A \in L^X).$$

Proof. Let $T = \mathrm{int}_{\mathcal{U}}(A)$ and $x \in X$. For every $\lambda \in L$, if $x \in \mathrm{int}_\lambda(A_{[\lambda]})$, then there exists $B \in \mathcal{U}$ such that $x \in B_{(\lambda)} \subseteq A_{[\lambda]}$. It is easily seen that $\lambda \wedge B_{(\lambda)} \in \mathcal{U}$ and $\lambda \wedge B_{(\lambda)} \leq A$. Then $\lambda \wedge B_{(\lambda)} \leq T$ and then $\lambda \leq T(x)$. Then we have, $\mathrm{int}_{\mathcal{U}}(A)(x) \geq [\bigvee_{\lambda\in L} \lambda \wedge \mathrm{int}_\lambda(A_{[\lambda]})](x)$. Conversely, let $b = T(x)$ and arbitrarily choose $\lambda \lhd b$. Then $x \in T_{(\lambda)} \subseteq A_{(\lambda)} \subseteq A_{[b]}$ and then $x \in \mathrm{int}_b(A_{[b]})$ and then $[\bigvee_{\lambda\in L} \lambda \wedge \mathrm{int}_\lambda(A_{[\lambda]})](x) \geq b = \mathrm{int}_{\mathcal{U}}(A)(x)$. Hence, $\mathrm{int}_{\mathcal{U}}(A) = \bigvee_{\lambda\in L} \lambda \wedge \mathrm{int}_\lambda(A_{[\lambda]})$. □

Example 2.7. (Representation of interior operators of L-meet structures) Let $\mathcal{N}$ be an L-meet-structure on X with the property of $a \vee B^{[\lambda]} \in \mathcal{N}$ $(\forall \lambda \in L, \forall B \in \mathcal{N})$. For all $\lambda \in L$, denote cl_λ as its closure operator. Then $\mathrm{cl}^+ = \mathrm{cl}_{\mathcal{N}}$, that is,

$$\mathrm{cl}_{\mathcal{N}}(A) = \bigwedge_{\lambda \in L} \lambda \vee \mathrm{cl}_\lambda(A^{(\lambda)}).$$

Proof. Let $T = \mathrm{cl}_{\mathcal{N}}(A)$ and $x \in X$. For every $\lambda \in L$, if $x \notin \mathrm{cl}_\lambda(A^{(\lambda)})$, then there exists $B \in \mathcal{N}$ such that $x \notin B^{[\lambda]}$ and $A^{(\lambda)} \subseteq B^{[\lambda]}$. It is easy to check that $A \leq \lambda \vee B^{[\lambda]} \in \mathcal{N}$. Then $T \leq \lambda \vee B^{[\lambda]}$ and then $T(x) \leq \lambda$. Then we have, $\mathrm{cl}_{\mathcal{N}}(A)(x) \leq [\bigwedge_{\lambda \in L} \lambda \vee \mathrm{cl}_\lambda(A^{(\lambda)})](x)$. Conversely, let $b = T(x)$ and arbitrarily choose $b \prec a$. Clearly, $x \notin T^{[a]} \supseteq T^{(a)} \supseteq A^{(a)}$. Then $x \notin \mathrm{cl}_a(A^{(a)})$ and $\bigwedge_{\lambda \in L} \lambda \vee \mathrm{cl}_\lambda(A^{(\lambda)}) \leq a$. By the arbitrariness of a, we have $\bigwedge_{\lambda \in L} \lambda \vee \mathrm{cl}_\lambda(A^{(\lambda)}) \leq b = \mathrm{cl}_{\mathcal{N}}(A)(x)$. Hence, $\mathrm{cl}_{\mathcal{N}}(A) = \bigwedge_{\lambda \in L} \lambda \vee \mathrm{cl}_\lambda(A^{(\lambda)})$. □

3. Properties of L-valued powerset mapping systems

The lifts F^+, F^- have another description for L being a completely distributive lattice.

Theorem 3.1. For every natural L-valued powerset mapping system F, it holds that,

(1) $F^+(A) = \bigvee_{\lambda \in L} \lambda \wedge F_\lambda(A_{(\lambda)})$;

(2) $F^-(A) = \bigwedge_{\lambda \in L} \lambda \vee F_\lambda(A^{[\lambda]})$.

Proof. (1) Since F is natural and $A_{[\lambda]} \supseteq A_{(\lambda)}$ for all $\lambda \in L$, we only need to show that for all $x \in X$, it holds that

$$\bigvee\{\lambda \in L \mid x \in F_\lambda(A_{[\lambda]})\} \leq \bigvee\{a \in L \mid x \in F_a(A_{(a)})\}.$$

In fact, let $a \lhd \bigvee\{\lambda \in L \mid x \in F_\lambda(A_{[\lambda]})\}$. Then there exists $\lambda \in L$ such that $a \lhd \lambda$ and $x \in F_\lambda(A_{[\lambda]}) \subseteq F_\lambda(A_{(a)}) \subseteq F_a(A_{(a)})$. This completes the proof.

(2) Since F is natural and $A^{(\lambda)} \subseteq A^{[\lambda]}$ for all $\lambda \in L$, we only need to show that for all $x \in X$, it holds that

$$\bigwedge\{\lambda \in L \mid x \notin F_\lambda(A^{(\lambda)})\} \geq \bigwedge\{a \in L \mid x \notin F_a(A^{[a]})\}.$$

In fact, let $\bigwedge\{\lambda \in L \mid x \notin F_\lambda(A^{(\lambda)})\} \prec a$. Then there exists $\lambda \in L$ such that $\lambda \prec a$ and $x \notin F_\lambda(A^{(\lambda)}) \supseteq F_\lambda(A^{[a]}) \supseteq F_a(A^{[a]})$. This completes the proof. □

Theorem 3.2. Let F be an L-valued powerset mapping system from X to Y. If every component F_λ is join-preserving (resp., meet-preserving), then so is F^+ (resp., F^-).

Proof. (1)

$$\begin{aligned}
F^+(\textstyle\bigvee_i A_i) &= \bigvee_{\lambda \in L} \lambda \wedge F_\lambda((\textstyle\bigvee_i A_i)_{(\lambda)}) \\
&= \bigvee_{\lambda \in L} \lambda \wedge F_\lambda(\textstyle\bigcup_i (A_i)_{(\lambda)}) \\
&= \bigvee_{\lambda \in L} \lambda \wedge [\textstyle\bigcup_i F_\lambda((A_i)_{(\lambda)})] \\
&= \textstyle\bigvee_i \displaystyle\bigvee_{\lambda \in L} \lambda \wedge F_\lambda((A_i)_{(\lambda)}) \\
&= \textstyle\bigvee_i F^+(A_i).
\end{aligned}$$

(2)

$$\begin{aligned}
F^+(\textstyle\bigwedge_i A_i) &= \bigwedge_{\lambda \in L} \lambda \vee F_\lambda((\textstyle\bigwedge_i A_i)^{[\lambda]}) \\
&= \bigwedge_{\lambda \in L} \lambda \vee F_\lambda(\textstyle\bigcap_i (A_i)^{[\lambda]}) \\
&= \bigwedge_{\lambda \in L} \lambda \vee [\textstyle\bigcap_i F_\lambda((A_i)^{[\lambda]})] \\
&= \textstyle\bigwedge_i \displaystyle\bigwedge_{\lambda \in L} \lambda \vee F_\lambda((A_i)^{[\lambda]}) \\
&= \textstyle\bigwedge_i F^-(A_i).
\end{aligned}$$

□

Theorem 3.3. Let F be a natural L-valued powerset mapping system from X to Y. Then for all $\in L^X$ and all $\lambda \in L$,

(1) $(F^+(A))_{(\lambda)} \subseteq F_\lambda(A_{(\lambda)}) \subseteq (F^+(A))_{[\lambda]}$;

(2) $(F^-(A))^{(\lambda)} \subseteq F_\lambda(A^{[\lambda]}) \subseteq (F^-(A))^{[\lambda]}$.

Proof. (1) The inclusion $F_\lambda(A_{(\lambda)}) \subseteq (F^+(A))_{[\lambda]}$ is obvious. Let $x \in (F^+(A))_{(\lambda)}$. Then $\lambda \lhd \bigvee\{a \in L \mid x \in F_a(A_{(a)})\}$ and there exists $a \in L$ such that $\lambda \lhd a$ and $x \in F_a(A_{(a)})$. Since F is natural, we have $x \in F_\lambda(A_{(a)}) \subseteq F_\lambda(A_{(\lambda)})$. Hence, $(F^+(A))_{(\lambda)} \subseteq F_\lambda(A_{(\lambda)})$.

(2) The inclusion $(F^-(A))^{(\lambda)} \subseteq F_\lambda(A^{[\lambda]})$ is obvious. Let $x \notin (F^-(A))^{[\lambda]}$, then $\bigwedge\limits_{a\in L} a \vee F_a(A^{[a]})(x) < \lambda$ and then there exists $a < \lambda$ such that $x \notin F_a(A^{[a]}) \supseteq F_\lambda(A^{[\lambda]}) \supseteq F_\lambda(A^{[\lambda]})$. Hence, $F_\lambda(A^{[\lambda]}) \subseteq (F^-(A))^{[\lambda]}$.

□

An L-topological space (X, δ) is called *induced* there is a crisp topology $\mathscr{T}$ on X such that δ is generated by the subbase $\{A \in L^X \mid \forall a \in L,\ A^{(a)} \in \mathscr{T}\}$ [3]. The topology $\mathscr{T}$ is called the base topology of δ. It is well-known that an L-topological space (X, δ) is induced if and only if $A \in \delta$ always implies $A^{(a)} \in \delta$ (considered as the characteristic function) and $a_X \in \delta$ (the constant L-subset) for all $a \in L$.

Example 3.4. Let L be a completely distributive De Morgan algebra and let (X, δ) be an induced L-topological space with $\mathscr{T}$ as the base topology. Then

(1) $\mathrm{cl}_\delta(A) = \bigvee\limits_{\lambda\in L} \lambda \wedge \mathrm{cl}_{\mathscr{T}}(A_{[\lambda]}) = \bigvee\limits_{\lambda\in L} \lambda \wedge \mathrm{cl}_{\mathscr{T}}(A_{(\lambda)})$;

(2) $\mathrm{cl}_\delta(A) = \bigwedge\limits_{\lambda\in L} \lambda \vee \mathrm{cl}_{\mathscr{T}}(A^{[\lambda]}) = \bigwedge\limits_{\lambda\in L} \lambda \vee \mathrm{cl}_{\mathscr{T}}(A^{(\lambda)})$;

(3) $\mathrm{int}_\delta(A) = \bigvee\limits_{\lambda\in L} \lambda \wedge \mathrm{int}_{\mathscr{T}}(A_{[\lambda]}) = \bigvee\limits_{\lambda\in L} \lambda \wedge \mathrm{int}_{\mathscr{T}}(A_{(\lambda)})$;

(4) $\mathrm{int}_\delta(A) = \bigwedge\limits_{\lambda\in L} \lambda \vee \mathrm{int}_{\mathscr{T}}(A^{[\lambda]}) = \bigwedge\limits_{\lambda\in L} \lambda \vee \mathrm{int}_{\mathscr{T}}(A^{(\lambda)})$.

That is to say, $\mathrm{cl}_\delta = \overline{\mathrm{cl}_{\mathscr{T}}}^{+}$, $\mathrm{int}_\delta = \overline{\mathrm{int}_{\mathscr{T}}}^{-}$.

Proof. These results are exactly those of Theorem 5.3 in [4]. □

4. Conclusions

Motivated by Zadeh's extension of mappings and the contents in [4], this paper introduces a concept of lifts of powerset mapping systems. In this framework, L-subsets are lifts of ordinary subsets, L-Zadeh functions are lifts of ordinary mappings, L-fuzzy rough set operators are lifts of classical rough set operators, L-interior/closure operators of L-topology are lifts of interior/closure operators of level topologies. Following the paper [6], more interesting examples can be supplied. These results generalize some interesting results in [4]. Consequently, lifts of powerset mapping systems can be considered as a useful fuzzy structures for future study.

There may be other kind of lifts by means of different kinds of cut sets. For example, for an L-valued powerset mapping system F from X to Y, we can define two new kinds of lifts F_*, $F^* : L^X \longrightarrow L^Y$ as follows:

$$F_*(A) = \bigvee_{\lambda\in L} \lambda \wedge F_\lambda(A^{(\lambda)}), \qquad F^*(A) = \bigwedge_{\lambda\in L} \lambda \vee F_\lambda(A_{[\lambda]}).$$

Clearly, for $L = [0, 1]$, by Theorem 3.1, it holds that $F_* = F^+$, $F^* = F^-$. We have an interesting result:

Theorem 4.1. Let L be a complete lattice, let F be an L-valued powerset mapping system from X to Y and let G be an L-valued powerset mapping system from Y to X. If for every $\lambda \in L$, (F_λ, G_λ) forms a Galois adjunction from 2^X to 2^Y, then (F_*, G^*) forms a Galois adjunction from L^X to L^Y.

Proof. Let $A \in L^X$, $\forall B \in L^Y$. Then

$$
\begin{aligned}
& F_*(A) \leq B \\
\text{iff } & \forall y \in Y,\ \forall \lambda \in L,\ \lambda \wedge F_\lambda(A^{(\lambda)})(y) \leq B(y) \\
\text{iff } & \forall y \in Y,\ \forall \lambda \in L,\ y \in F_\lambda(A^{(\lambda)}) \Longrightarrow \lambda \leq B(y) \\
\text{iff } & \forall y \in Y,\ \forall \lambda \in L,\ y \in F_\lambda(A^{(\lambda)}) \Longrightarrow y \in B_{[\lambda]} \\
\text{iff } & \forall \lambda \in L,\ F_\lambda(A^{(\lambda)}) \subseteq B_{[\lambda]} \\
\text{iff } & \forall \lambda \in L,\ A^{(\lambda)} \subseteq G_\lambda(B_{[\lambda]}) \\
\text{iff } & \forall \lambda \in L,\ \forall x \in X,\ x \notin G_\lambda(B_{[\lambda]}) \Longrightarrow x \notin A^{(\lambda)} \\
\text{iff } & \forall \lambda \in L,\ \forall x \in X,\ x \notin G_\lambda(B_{[\lambda]}) \Longrightarrow A(x) \leq \lambda \\
\text{iff } & \forall x \in X,\ \forall \lambda \in L,\ A(x) \leq \lambda \vee G_\lambda(B_{[\lambda]})(x) \\
\text{iff } & A \leq G^*(B).
\end{aligned}
$$

Hence, (F_*, G^*) is a Galois adjunction from L^X to L^Y. □

Similar to Theorem 3.1, we can show that

Theorem 4.2. Let L be a completely distributive lattice. For every natural powerset mapping systems F, it holds that

(1) $F_*(A) = \bigvee_{\lambda \in L} \lambda \wedge F_\lambda(A^{[\lambda]})$;

(2) $F^*(A) = \bigwedge_{\lambda \in L} \lambda \vee F_\lambda(A_{(\lambda)})$.

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