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Research Article

New developments in fractional Hermite-Hadamard type inequalities through (α, m)-convex functions

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Abstract

This work aims to establish new fractional integral inequalities of the Hermite-Hadamard type. These results are explored by combining the Riemann-Liouville fractional integrals in a new identity and then applying extended class of convex functions. The limiting cases of the novel results are presented in terms of remarks that connect our findings to the body of existing literature. Furthermore, we present inequalities in the form of some special means as applications of the main results.

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1. Introduction

The establishment of mathematical inequalities involving fractional integral operators, such as the Riemann-Liouville integrals, is a significant area of research in fractional calculus [17]. These inequalities are essential for understanding the behavior of fractional integrals and their applications in various fields, including physics, engineering, and mathematical analysis. The Riemann-Liouville fractional integral operator [14], which extends the classical concept of integration to fractional orders, plays a central role in defining these inequalities. By developing inequalities for Riemann-Liouville (R-L) fractional integrals, researchers can provide rigorous bounds and constraints that are crucial for solving fractional differential equations, ensuring the stability and robustness of solutions in systems where traditional calculus falls short.

The integration of generalized convexity concepts into the study of fractional integral operators further enhances the establishment of these mathematical inequalities. Generalized convexity, which includes concepts such as convex functions and their generalizations, provides powerful tools for deriving sharper and more comprehensive inequalities for fractional integrals. By combining the fractional operators like the Riemann-Liouville and

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generalized convexity allows for the development of more refined inequalities that can better describe the complex behaviors of systems governed by fractional dynamics. Fractional integral inequalities have been established by a huge number of researchers for more details one can see [5,10,12,19,22]. These convexity-based inequalities help in characterizing the relationships between fractional integrals and other mathematical entities, leading to deeper insights into the solutions of fractional differential equations.

By combining the principles of fractional calculus with generalized convexity, researchers can establish a broader class of mathematical inequalities. These inequalities not only contribute to the theoretical understanding of fractional integrals but also have practical implications for the analysis and optimization of complex systems in science and engineering. The interplay between fractional integral operators and generalized convexity opens new avenues for research, enabling the development of more robust mathematical frameworks for modeling, analyzing, and solving problems involving fractional dynamics.

Within this framework, fractional Hermite-Hadamard inequalities emerge as a significant extension of classical inequalities to the fractional domain. The classical Hermite-Hadamard inequality provides bounds on the integral of a convex function, and its fractional counterpart extends this concept to fractional integrals, particularly those defined by operators like the Riemann-Liouville integral. Fractional Hermite-Hadamard inequalities leverage the principles of convexity and fractional calculus to establish bounds on the fractional integrals of convex functions. These inequalities are very important for theoretical investigations and have practical applications in various fields, such as optimization, control theory, and mathematical physics.

Let $\zeta : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $\xi, \nu \in I$ with $\xi < \nu$, then the inequality

$$\zeta\left(\frac{\xi+\nu}{2}\right) \leq \frac{1}{\nu-\xi}\int_{\xi}^{\nu}\zeta(r)dr \leq \frac{\zeta(\xi)+\zeta(\nu)}{2},$$

referred to as Hermite-Hadamard inequality [8], is one of the most famous results for convex mappings.

Let $\zeta : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a positive function and $\xi, \nu \in I$ with $0 \leq \xi < \nu$ such that $\zeta \in L[\xi, \nu]$. If ζ is convex on $[\xi, \nu]$, then the inequality

$$\zeta\left(\frac{\xi+\nu}{2}\right) \leq \frac{\Gamma(\aleph+1)}{2(\nu-\xi)^{\aleph}}\left[I^{\aleph}_{\xi+}\zeta(\nu) + I^{\aleph}_{\nu-}\zeta(\xi)\right] \leq \frac{\zeta(\xi)+\zeta(\nu)}{2},$$

with $\aleph > 0$ referred to as fractional Hermite-Hadamard inequality [29], where $I^{\aleph}_{\xi+}$ and $I^{\aleph}_{\nu-}$ stand for the right-sided and the left-sided Riemann-Liouville fractional integrals of the order $\aleph$. Observe that, the fractional Hermite-Hadamard inequality reduces to the original Hermite-Hadamard inequality for $\aleph = 1$.

By connecting fractional Hermite-Hadamard inequalities with the broader framework of fractional integral operators and generalized convexity, researchers can develop more powerful tools for analyzing and optimizing systems that exhibit fractional dynamics. The interplay between these concepts allows for the formulation of new mathematical inequalities that are more precise and applicable to a wide range of complex systems. This interconnected approach opens new research directions, offering deeper insights into the mathematical structures governing fractional calculus and its applications in science and engineering. One can refer to some recent findings related to the fractional integral inequalities of Hermite-Hadamard type by visiting [1,3,4,6,9,11,15,20,21,27,31–33].

In a recent paper [13], the authors established fractional Hermite-Hadamard integral inequalities for functions whose derivatives' absolute values are s-convex. Similarly, in another work [2], Hermite-Hadamard type inequalities were derived using R-L fractional integrals and extended convexity.

The theory of fractional calculus has significant applications in mathematical physics and engineering. Moreover, the Hermite-Hadamard inequality a corner stone in convex analysis motivates to explore more deeper connections between fractional calculus and convexity theory. In main results section, we establish the novel developments of Hermite-Hadamard type fractional inequalities by exploring an identity and then utilizing fractional integrals through the use of (α, m)-convexity.

2. Preliminaries

Definition 2.1 ([23]). A function $\zeta : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$\zeta(\varpi\xi + (1-\varpi)\nu) \leq \varpi\zeta(\xi) + (1-\varpi)\zeta(\nu),$$

is valid for all $\xi, \nu \in I$ and $\varpi \in [0,1]$.

Definition 2.2 ([30]). A function $\zeta : [0,d] \to \mathbb{R}$ is called m-convex $(0 \leq m \leq 1)$ if the inequality

$$\zeta(\varpi\xi + m(1-\varpi)\nu) \leq \varpi\zeta(\xi) + m(1-\varpi)\zeta(\nu),$$

is satisfied for all $\xi, \nu \in [0,d]$ and $\varpi \in [0,1]$.

Clearly, m-convexity is exactly the classical convexity at $m = 1$.

Definition 2.3 ([16]). Assume $0 \leq \alpha, m \leq 1$. A function $\zeta : [0,d] \to \mathbb{R}$ is called (α, m)-convex if the inequality

$$\zeta(\varpi\xi + m(1-\varpi)\nu) \leq \varpi^{\alpha}\zeta(\xi) + m(1-\varpi^{\alpha})\zeta(\nu),$$

is valid for all $\xi, \nu \in [0,d]$ for $d > 0$ and $\varpi \in [0,1]$.

Observe that (α, m)-convexity is exactly the m-convexity at $\alpha = 1$ and classical convexity for $\alpha = m = 1$.

Definition 2.4 ([26]). The right-sided and left-sided R-L fractional integrals having order $\aleph \in [0, \infty)$ are given as follows

$$I^{\aleph}_{\xi+}\zeta(r) = \frac{1}{\Gamma(\aleph)}\int_{\xi}^{r}(r-\varpi)^{\aleph-1}\zeta(\varpi)d\varpi; \qquad 0 \leq \xi < r \leq \nu,$$
$$I^{\aleph}_{\nu-}\zeta(r) = \frac{1}{\Gamma(\aleph)}\int_{r}^{\nu}(\varpi-r)^{\aleph-1}\zeta(\varpi)d\varpi; \qquad 0 \leq \xi < r \leq \nu,$$

where $\Gamma(\cdot)$ is the classical Euler gamma function defined in [28] as

$$\Gamma(x) = \int_{0}^{\infty}\varpi^{x-1}e^{-\varpi}d\varpi, \qquad x > 0.$$

Definition 2.5 ([28]). , The beta function is defined by

$$\mathbb{B}(x,y) = \int_{0}^{1}\varpi^{x-1}(1-\varpi)^{y-1}d\varpi, \qquad x, y > 0.$$

In closing of this section, we review well-known inequalities that are needed at various points throughout our research.

Theorem 2.6 ([18]). *(Hölder's integral inequality) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. We assume that the real functions ζ and ξ defined on the interval $[\xi, \nu]$ in such a way that $|\zeta|^q, |\xi|^q$ $(q \geq 1)$ are functions integrable on the interval $[\xi, \nu]$. Then the inequality*

$$\int_{\xi}^{\nu}|\zeta(r)\xi(r)|dr \leq \left(\int_{\xi}^{\nu}|\zeta(r)|^{p}dr\right)^{\frac{1}{p}}\left(\int_{\xi}^{\nu}|\xi(r)|^{q}dr\right)^{\frac{1}{q}},$$

is true.

Theorem 2.7 ([18])**.** (*Power-Mean inequality for integrals*) *Let $q \geq 1$. We assume that the real functions ζ and ξ defined on the interval $[\xi, \nu]$ in such a way that $|\zeta|, |\zeta||\xi|^q$ are integrable functions on $[\xi, \nu]$. Then we have the inequality*

$$\int_{\xi}^{\nu} |\zeta(r)\xi(r)|dr \leq \left(\int_{\xi}^{\nu} |\zeta(r)|dr\right)^{1-\frac{1}{q}} \left(\int_{\xi}^{\nu} |\zeta(r)||\xi(r)|^q dr\right)^{\frac{1}{q}}.$$

3. Fractional integral Hermite-Hadamard inequalities for differentiable functions

In this section, we first establish an identity involving Riemann-Lioville fractional integral. The we establish an integral inequality involving beta function. We begin with the following lemma which is used to explore integral inequality.

Lemma 3.1. *Let ζ be a differentiable mapping on $int(I)$ and $\xi, \nu \in int(I)$ with $\xi < \nu$. If $\zeta' \in L[\xi, \nu]$, then the equality given below involving fractional integral with $\aleph > 0$ holds:*

$$\begin{aligned} &\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I_{r^-}^{\aleph}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I_{r^+}^{\aleph}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right] \\ &= \left(\frac{r-m\xi}{2}\right)\int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\xi)d\varpi \\ &+ \left(\frac{r-m\nu}{2}\right)\int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\nu)d\varpi, \end{aligned}$$

for all $r \in (\xi, \nu)$.

Proof. Consider

$$\begin{aligned} I_1 &= \int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\xi)d\varpi \\ &= \frac{\varpi^{\aleph}\zeta(\varpi r + m(1-\varpi)\xi)}{(r-m\xi)}\Big|_0^1 - \int_0^1 \frac{\aleph\varpi^{\aleph-1}\zeta(\varpi r + m(1-\varpi)\xi)}{(r-m\xi)}d\varpi \\ &= \frac{\zeta(r)}{(r-m\xi)} - \frac{\aleph}{(r-m\xi)}\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r + m(1-\varpi)\xi)d\varpi \\ &= \frac{1}{(r-m\xi)}\left[\zeta(r) - \aleph\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r + m(1-\varpi)\xi)d\varpi\right]. \end{aligned}$$

Assume $\ell = \varpi r + m(1-\varpi)\xi$. Then $\varpi = \frac{\ell - m\xi}{r-m\xi}$ and $d\varpi = \frac{d\ell}{r-m\xi}$. By substituting these values in above equation, we get

$$\begin{aligned} I_1 &= \frac{1}{(r-m\xi)}\left[\zeta(r) - \frac{\Gamma(\aleph+1)}{\Gamma(\aleph)} - \int_{m\xi}^{r}\left(\frac{\ell-m\xi}{r-m\xi}\right)^{\aleph-1}\zeta(\ell)\frac{d\ell}{(r-m\xi)}\right] \\ &= \frac{1}{(r-m\xi)}\left[\zeta(r) - \frac{\Gamma(\aleph+1)}{(r-m\xi)^{\aleph}}\left(\frac{1}{\Gamma(\aleph)}\int_{m\xi}^{r}(\ell-m\xi)^{\aleph-1}\zeta(\ell)d\ell\right)\right] \\ &= \frac{1}{(r-m\xi)}\left[\zeta(r) - \frac{\Gamma(\aleph+1)}{(r-m\xi)^{\aleph}}I_{r^-}^{\aleph}\zeta(m\xi)\right]. \end{aligned}$$

Thus, we have

$$(r-m\xi)I_1 = \zeta(r) - \frac{\Gamma(\aleph+1)}{(r-m\xi)^{\aleph}}I_{r^-}^{\aleph}\zeta(m\xi). \tag{3.1}$$

Now, consider

$$\begin{aligned}
I_2 &= \int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\nu)d\varpi \\
&= \frac{\varpi^{\aleph}\zeta(\varpi r + m(1-\varpi)\nu)}{(r-m\nu)}\Big|_0^1 - \int_0^1 \frac{\aleph\varpi^{\aleph-1}\zeta(\varpi r + m(1-\varpi)\nu)}{(r-m\nu)}d\varpi \\
&= \frac{\zeta(r)}{(r-m\nu)} - \frac{\aleph}{(r-m\nu)}\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r + m(1-\varpi)\nu)d\varpi \\
&= \frac{1}{(r-m\nu)}\Big[\zeta(r) - \aleph\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r + m(1-\varpi)\nu)d\varpi\Big].
\end{aligned}$$

Assume $\hbar = \varpi r + m(1-\varpi)\nu$. Then $\varpi = \frac{m\nu-\hbar}{m\nu-r}$ and $d\varpi = \frac{d\hbar}{r-m\nu}$. By plugging these values into the above equation, we obtain

$$\begin{aligned}
I_2 &= \frac{1}{(r-m\nu)}\Big[\zeta(r) - \frac{\Gamma(\aleph+1)}{\Gamma(\aleph)} - \int_{m\nu}^{r}\Big(\frac{m\nu-\hbar}{m\nu-r}\Big)^{\aleph-1}\zeta(\hbar)\frac{d\hbar}{(r-m\nu)}\Big] \\
&= \frac{1}{(r-m\nu)}\Big[\zeta(r) - \frac{\Gamma(\aleph+1)}{(m\nu-r)^{\aleph}}\Big(\frac{1}{\Gamma(\aleph)}\int_r^{m\nu}(m\nu-\hbar)^{\aleph-1}\zeta(\hbar)d\hbar\Big)\Big] \\
&= \frac{1}{(r-m\nu)}\Big[\zeta(r) - \frac{\Gamma(\aleph+1)}{(m\nu-r)^{\aleph}}I_{r+}^{\aleph}\zeta(m\nu)\Big].
\end{aligned}$$

Thus, we have

$$(r-m\nu)I_2 = \zeta(r) - \frac{\Gamma(\aleph+1)}{(m\nu-r)^{\aleph}}I_{r+}^{\aleph}\zeta(m\nu). \tag{3.2}$$

Adding equations (3.1) and (3.2), we have

$$\begin{aligned}
&(r-m\xi)\int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\xi)d\varpi + (r-m\nu)\int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\nu)d\varpi \\
&= 2\zeta(r) - \Gamma(\aleph+1)\Big[\frac{I_{r-}^{\aleph}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I_{r+}^{\aleph}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\Big].
\end{aligned}$$

By simplification, we obtain our desired result. □

Remark 3.2. If we set $m = 1$, then we get the result [13, Lemma 2] .

Our first main result is given as follows.

Theorem 3.3. *Let $\aleph \geq 1$ and ζ be a positive real function on $int(I)$ and $\xi, \nu \in int(I)$ with $0 \leq \xi < r < \nu$ and $\zeta \in L[\xi, \nu]$. If ζ is (α, m)-convex on $[\xi, \nu]$, then the inequality*

$$\Gamma(\aleph+1)\Big[\frac{I_{r-}^{\aleph}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I_{r+}^{\aleph}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\Big] \leq \aleph[2\zeta(r)\mathbb{B}(\aleph+\alpha, 1) + m\mathbb{B}(\aleph, \alpha+1)(\zeta(\xi)+\zeta(\nu))],$$

involving fractional integral holds.

Proof. By exploiting (α, m)-convexity of ζ, we have

$$\begin{aligned}
&\zeta(\varpi r + m(1-\varpi)\xi) + \zeta(\varpi r + m(1-\varpi)\nu) \\
&\leq \varpi^{\alpha}\zeta(r) + m(1-\varpi^{\alpha})\zeta(\xi) + \varpi^{\alpha}\zeta(r) + m(1-\varpi^{\alpha})\zeta(\nu).
\end{aligned}$$

Multiplying by $\varpi^{\aleph-1}$ and integrating w.r.t. ϖ over [0,1] yields

$$\begin{aligned}
&\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\xi)d\varpi+\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\nu)d\varpi\\
&\leq \zeta(r)\int_0^1 \varpi^{\aleph+\alpha-1}d\varpi+m\zeta(\xi)\int_0^1 \varpi^{\aleph-1}(1-\varpi^{\alpha})d\varpi\\
&+\zeta(r)\int_0^1 \varpi^{\aleph+\alpha-1}d\varpi+m\zeta(\nu)\int_0^1 \varpi^{\aleph-1}(1-\varpi^{\alpha})d\varpi.
\end{aligned}$$

Since $\varpi^{\alpha}\geq\varpi$ for $\alpha\in(0,1]$ and $\varpi\in[0,1]$, therefore we have $1-\varpi^{\alpha}\leq 1-\varpi\leq(1-\varpi)^{\alpha}$. By utilizing this fact, the above inequality takes the form

$$\begin{aligned}
&\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\xi)d\varpi+\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\nu)d\varpi\\
&\leq \zeta(r)\int_0^1 \varpi^{\aleph+\alpha-1}d\varpi+m\zeta(\xi)\int_0^1 \varpi^{\aleph-1}(1-\varpi)^{\alpha}d\varpi\\
&+\zeta(r)\int_0^1 \varpi^{\aleph+\alpha-1}d\varpi+m\zeta(\nu)\int_0^1 \varpi^{\aleph-1}(1-\varpi)^{\alpha}d\varpi,\\
&\leq 2\zeta(r)\mathbb{B}(\aleph+\alpha,1)+m\mathbb{B}(\aleph,\alpha+1)\big[\zeta(\xi)+\zeta(\nu)\big].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\xi)d\varpi+\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\nu)d\varpi\\
&\leq 2\zeta(r)\mathbb{B}(\aleph+\alpha,1)+m\mathbb{B}(\aleph,\alpha+1)\big[\zeta(\xi)+\zeta(\nu)\big].
\end{aligned}$$

Next, we extract the left side of above inequality and substitute $\ell=\varpi r+m(1-\varpi)\xi$ and $\hbar=\varpi r+m(1-\varpi)\nu$, we can write

$$\begin{aligned}
&\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\xi)d\varpi+\int_0^1 \varpi^{\aleph-1}\zeta(\varpi r+m(1-\varpi)\nu)d\varpi\\
&=\int_{m\xi}^{r}\left(\frac{\ell-m\xi}{r-m\xi}\right)^{\aleph-1}\zeta(\ell)\frac{d\ell}{r-m\xi}+\int_{m\nu}^{r}\left(\frac{m\nu-\hbar}{m\nu-r}\right)^{\aleph-1}\zeta(\hbar)\frac{d\hbar}{r-m\nu}\\
&=\frac{1}{(r-m\xi)^{\aleph}}\int_{m\xi}^{r}(\ell-m\xi)^{\aleph-1}\zeta(\ell)d\ell+\frac{1}{(m\nu-r)^{\aleph}}\int_{r}^{m\nu}(m\nu-\hbar)^{\aleph-1}\zeta(\hbar)d\hbar\\
&=\frac{\aleph}{\aleph(r-m\xi)^{\aleph}}\int_{m\xi}^{r}(\ell-m\xi)^{\aleph-1}\zeta(\ell)d\ell+\frac{\aleph}{\aleph(m\nu-r)^{\aleph}}\int_{r}^{m\nu}(m\nu-\hbar)^{\aleph-1}\zeta(\hbar)d\hbar\\
&=\frac{\Gamma(\aleph+1)}{\aleph}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}}+\frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right].
\end{aligned}$$

As a result, we get

$$\Gamma(\aleph+1)\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}}+\frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right]\leq\aleph\big[2\zeta(r)\mathbb{B}(\aleph+\alpha,1)+m\mathbb{B}(\aleph,\alpha+1)\big(\zeta(\xi)+\zeta(\nu)\big)\big].$$

Hence, the theorem is fully proven. □

Remark 3.4. If we set $m=1$ and $\alpha=s$, we get the result [13, Theorem 4].

Next, we give another main result.

Theorem 3.5. *Let ζ be a differentiable mapping on $int(I)$ and $\xi, \nu \in int(I)$ with $\xi < r < \nu$ s.t. $\zeta' \in L[\xi, \nu]$. If $|\zeta'|$ is (α, m)-convex on $[\xi, \nu]$, then the inequality*

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu - r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{r-m\xi}{2}\right)\Big[|\zeta'(r)|\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\xi)|\mathbb{B}(\aleph+1,\alpha+1)\Big] \\
&+ \left(\frac{r-m\nu}{2}\right)\Big[|\zeta'(r)|\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\nu)|)\mathbb{B}(\aleph+1,\alpha+1)\Big],
\end{aligned} \tag{3.3}$$

involving fractional integral holds.

Proof. By using Lemma 3.1, we have

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu - r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{r-m\xi}{2}\right)\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\xi)|d\varpi \\
&+ \left(\frac{r-m\nu}{2}\right)\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\nu)|d\varpi.
\end{aligned}$$

By using (α, m)-convexity of $|\zeta'|$, we obtain

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu - r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{r-m\xi}{2}\right)\left[\int_0^1 \Big(\varpi^{\aleph+\alpha}|\zeta'(r)| + m\varpi^{\aleph}(1-\varpi^{\alpha})|\zeta'(\xi)|\Big)d\varpi\right] \\
&+ \left(\frac{r-m\nu}{2}\right)\left[\int_0^1 \Big(\varpi^{\aleph+\alpha}|\zeta'(r)| + m\varpi^{\aleph}(1-\varpi^{\alpha})|\zeta'(\nu)|\Big)d\varpi\right], \\
&\leq \left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|\int_0^1 \varpi^{\aleph+\alpha}d\varpi + m|\zeta'(\xi)|\int_0^1 \varpi^{\aleph}(1-\varpi^{\alpha})d\varpi\right] \\
&+ \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|\int_0^1 \varpi^{\aleph+\alpha}d\varpi + m|\zeta'(\nu)|\int_0^1 \varpi^{\aleph}(1-\varpi^{\alpha})d\varpi\right].
\end{aligned}$$

Since $\varpi^{\alpha} \geq \varpi$ for $\alpha \in (0,1]$ and $\varpi \in [0,1]$, therefore we have $1-\varpi^{\alpha} \leq 1-\varpi \leq (1-\varpi)^{\alpha}$. By utilizing this fact, the above inequality takes the form

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu - r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\xi)|\int_0^1 \varpi^{\aleph}(1-\varpi)^{\alpha}d\varpi\right] \\
&+ \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\nu)|\int_0^1 \varpi^{\aleph}(1-\varpi)^{\alpha}d\varpi\right] \\
&\leq \left(\frac{r-m\xi}{2}\right)\Big[|\zeta'(r)|\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\xi)|\mathbb{B}(\aleph+1,\alpha+1)\Big] \\
&+ \left(\frac{r-m\nu}{2}\right)\Big[|\zeta'(r)|\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\nu)|\mathbb{B}(\aleph+1,\alpha+1)\Big].
\end{aligned}$$

Hence, the theorem is complete. □

Remark 3.6. If we set $m = 1$ and $\alpha = s$, we get the result [13, Theorem 5].

Our next result is given as follows.

Theorem 3.7. *Let ζ be a real differentiable mapping on $int(I)$ and $\xi, \nu \in int(I)$ with $\xi < r < \nu$ s.t. $\zeta' \in L[\xi, \nu]$. If $|\zeta'|^q$ $(q > 1)$ is (α, m)-convex on $[\xi, \nu]$, then the inequality given below involving fractional integral holds.*

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r^-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r^+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{1}{\aleph+1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\xi)|^q\mathbb{B}(\aleph+1,\alpha+1)\right]^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\nu)|^q\mathbb{B}(\aleph+1,\alpha+1)\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.4}$$

Proof. By utilizing Lemma 3.1, Power-Mean inequality and (α, m)-convexity of $|\zeta'|^q$, we obtain

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r^-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r^+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right]\right| \\
&= \left|\left(\frac{r-m\xi}{2}\right)\int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\xi)d\varpi + \left(\frac{r-m\nu}{2}\right)\int_0^1 \varpi^{\aleph}\zeta'(\varpi r + m(1-\varpi)\nu)d\varpi\right| \\
&\leq \left(\frac{r-m\xi}{2}\right)\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\xi)|d\varpi + \left(\frac{r-m\nu}{2}\right)\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\nu)|d\varpi \\
&\leq \left(\frac{r-m\xi}{2}\right)\left(\int_0^1 \varpi^{\aleph}d\varpi\right)^{1-\frac{1}{q}}\left[\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\xi)|^q d\varpi\right]^{\frac{1}{q}} \\
&+ \left(\frac{r-m\nu}{2}\right)\left(\int_0^1 \varpi^{\aleph}d\varpi\right)^{1-\frac{1}{q}}\left[\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\nu)|^q d\varpi\right]^{\frac{1}{q}} \\
&\leq \left(\frac{1}{\aleph+1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[\int_0^1 \left(\varpi^{\aleph+\alpha}|\zeta'(r)|^q + m\varpi^{\aleph}(1-\varpi^{\alpha})|\zeta'(\xi)|^q\right)d\varpi\right]^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{r-m\nu}{2}\right)\left[\int_0^1 \left(\varpi^{\aleph+\alpha}|\zeta'(r)|^q + m\varpi^{\aleph}(1-\varpi^{\alpha})|\zeta'(\nu)|^q\right)d\varpi\right]^{\frac{1}{q}}\right\} \\
&\leq \left(\frac{1}{\aleph+1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q\int_0^1 \varpi^{\aleph+\alpha}d\varpi + m|\zeta'(\xi)|^q\int_0^1 \varpi^{\aleph}(1-\varpi^{\alpha})d\varpi\right]^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q\int_0^1 \varpi^{\aleph+\alpha}d\varpi + m|\zeta'(\nu)|^q\int_0^1 \varpi^{\aleph}(1-\varpi^{\alpha})d\varpi\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\varpi^\alpha \geq \varpi$ for $\alpha \in (0,1]$ and $\varpi \in [0,1]$, therefore we have $1-\varpi^\alpha \leq 1-\varpi \leq (1-\varpi)^\alpha$. By utilizing this fact, the above inequality takes the form

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{1}{\aleph+1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\xi)|^q\int_0^1 \varpi^{\aleph}(1-\varpi)^\alpha d\varpi\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\nu)|^q\int_0^1 \varpi^{\aleph}(1-\varpi)^\alpha)d\varpi\right]^{\frac{1}{q}}\right\} \\
&\leq \left(\frac{1}{\aleph+1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\xi)|^q\mathbb{B}(\aleph+1,\alpha+1)\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q\mathbb{B}(\aleph+\alpha+1,1) + m|\zeta'(\nu)|^q\mathbb{B}(\aleph+1,\alpha+1)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

The proof is complete. □

Remark 3.8.

(1) If we set $m=1$ and $\alpha=s$, we get the result [13, Theorem 6].
(2) By setting $q=1$, the inequality (3.4) reduces to the inequality (3.3).

We wrap this section by providing the following theorem.

Theorem 3.9. *Let ζ be a real differentiable mapping on $int(I)$ and $\xi,\nu \in int(I)$ with $\xi < r < \nu$ s.t. $\zeta' \in L[\xi,\nu]$. If $|\zeta'|^q$ $(q>1)$ is (α,m)-convex on $[\xi,\nu]$, then the inequality given below involving fractional integral holds.*

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{1}{p\aleph+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q + m\alpha|\zeta'(\xi)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q + m\alpha|\zeta'(\nu)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 3.1, Hölder's inequality and (α, m)-convexity of $|\zeta'|^q$, we get

$$\begin{aligned}
&\left|\zeta(r) - \frac{\Gamma(\aleph+1)}{2}\left[\frac{I^{\aleph}_{r-}\zeta(m\xi)}{(r-m\xi)^{\aleph}} + \frac{I^{\aleph}_{r+}\zeta(m\nu)}{(m\nu-r)^{\aleph}}\right]\right| \\
&\leq \left(\frac{r-m\xi}{2}\right)\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\xi)|d\varpi \\
&+ \left(\frac{r-m\nu}{2}\right)\int_0^1 \varpi^{\aleph}|\zeta'(\varpi r + m(1-\varpi)\nu)|d\varpi \\
&\leq \left(\frac{r-m\xi}{2}\right)\left(\int_0^1 \varpi^{p\aleph}d\varpi\right)^{\frac{1}{p}}\left[\int_0^1 |\zeta'(\varpi r + m(1-\varpi)\xi)|^q d\varpi\right]^{\frac{1}{q}} \\
&+ \left(\frac{r-m\nu}{2}\right)\left(\int_0^1 \varpi^{p\aleph}d\varpi\right)^{\frac{1}{p}}\left[\int_0^1 |\zeta'(\varpi r + m(1-\varpi)\nu)|^q d\varpi\right]^{\frac{1}{q}} \\
&\leq \left(\frac{1}{p\aleph+1}\right)^{\frac{1}{p}}\left\{\left(\frac{r-m\xi}{2}\right)\left[\int_0^1 \left(\varpi^{\alpha}|\zeta'(r)|^q + m(1-\varpi^{\alpha})|\zeta'(\xi)|^q\right)d\varpi\right]^{\frac{1}{q}}\right. \\
&\left.+ \left(\frac{r-m\nu}{2}\right)\left[\int_0^1 \left(\varpi^{\alpha}|\zeta'(r)|^q + m(1-\varpi^{\alpha})|\zeta'(\nu)|^q\right)d\varpi\right]^{\frac{1}{q}}\right\} \\
&\leq \left(\frac{1}{p\aleph+1}\right)^{\frac{1}{p}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q\int_0^1 \varpi^{\alpha}d\varpi + m|\zeta'(\xi)|^q\int_0^1(1-\varpi^{\alpha})d\varpi\right]^{\frac{1}{q}}\right. \\
&\left.+ \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q\int_0^1 \varpi^{\alpha}d\varpi + m|\zeta'(\nu)|^q\int_0^1(1-\varpi^{\alpha})d\varpi\right]^{\frac{1}{q}}\right\} \\
&\leq \left(\frac{1}{p\aleph+1}\right)^{\frac{1}{p}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q\left(\frac{1}{\alpha+1}\right) + m|\zeta'(\xi)|^q\left(\frac{\alpha}{\alpha+1}\right)\right]^{\frac{1}{q}}\right. \\
&\left.+ \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q\left(\frac{1}{\alpha+1}\right) + m|\zeta'(\nu)|^q\left(\frac{\alpha}{\alpha+1}\right)\right]^{\frac{1}{q}}\right\} \\
&\leq \left(\frac{1}{p\aleph+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\left\{\left(\frac{r-m\xi}{2}\right)\left[|\zeta'(r)|^q + m\alpha|\zeta'(\xi)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+ \left(\frac{r-m\nu}{2}\right)\left[|\zeta'(r)|^q + m\alpha|\zeta'(\nu)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Hence, the proof is complete. □

Remark 3.10. If we set $m = 1$ and $\alpha = s$, we get the result [13, Theorem 8].

4. Fractional Integral Hermite-Hadamard Inequalities for Twice Differentiable Functions

In order to obtain our main results of this section, we first establish the following lemma.

Lemma 4.1. *Let ζ be a twice differentiable mapping on $int(I)$. Assume that $\xi, \nu \in int(I)$ with $\xi < \nu$. If $\zeta'' \in L[\xi, \nu]$, then the equality given below involving fractional integral with $\aleph > 0$ holds.*

$$\begin{aligned}
&\int_0^1 \varpi(1-\varpi^{\aleph})[\zeta''(\varpi\xi + m(1-\varpi)\nu)d\varpi + \zeta''(m(1-\varpi)\xi + \varpi\nu)d\varpi] \\
&= \frac{\aleph\zeta(\xi) + \zeta(m\nu)}{(m\nu-\xi)^2} + \frac{\aleph\zeta(\nu) + \zeta(m\xi)}{(\nu-m\xi)^2} - (\aleph+1)\Gamma(\aleph+1)\left[\frac{I^{\aleph}_{\xi+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}} + \frac{I^{\aleph}_{\nu-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\right].
\end{aligned}$$

Proof. Consider

$$\begin{aligned}
I_1 &= \int_0^1 \varpi(1-\varpi^{\aleph})\big[\zeta''(\varpi\xi + m(1-\varpi)\nu)d\varpi\big] \\
&= \frac{\varpi(1-\varpi^{\aleph})\zeta'(\varpi\xi + m(1-\varpi)\nu)}{(\xi - m\nu)}\bigg|_0^1 - \int_0^1 \frac{(1-(\aleph+1))\varpi^{\aleph}\zeta'(\varpi\xi + m(1-\varpi)\nu)}{(\xi - m\nu)}d\varpi \\
&= -\frac{1}{(m\nu - \xi)}\left[\frac{(1-(\aleph+1)\varpi^{\aleph})\zeta(\varpi\xi + m(1-\varpi)\nu)}{(\xi - m\nu)}\bigg|_0^1\right. \\
&\left. - \int_0^1 \frac{(-\aleph(\aleph+1))\varpi^{\aleph-1}\zeta(\varpi\xi + m(1-\varpi)\nu)}{(\xi - m\nu)}d\varpi\right] \\
&= \frac{\aleph\zeta(\xi) + \zeta(m\nu)}{(m\nu - \xi)^2} - \frac{\aleph(\aleph+1)}{(m\nu-\xi)^2}\int_0^1 \zeta(\varpi\xi + m(1-\varpi)\nu)\varpi^{\aleph-1}d\varpi \\
&= \frac{\aleph\zeta(\xi) + \zeta(m\nu)}{(m\nu - \xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(m\nu-\xi)^2}\left[\frac{1}{\Gamma(\aleph)}\int_0^1 \zeta(\varpi\xi + m(1-\varpi)\nu)\varpi^{\aleph-1}d\varpi\right].
\end{aligned}$$

Assume $\hbar = \varpi\xi + m\nu(1-\varpi)$. Then $\varpi = \frac{m\nu - \hbar}{m\nu - \xi}$ and $d\varpi = \frac{d\hbar}{\xi - m\nu}$. We substitute these values in above equation and we get

$$\begin{aligned}
I_1 &= \frac{\aleph\zeta(\xi) + \zeta(m\nu)}{(m\nu - \xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(m\nu-\xi)^2}\left[\frac{1}{\Gamma(\aleph)}\int_{m\nu}^{\xi}\left(\frac{m\nu-\hbar}{m\nu-\xi}\right)^{\aleph-1}\zeta(\hbar)\frac{d\hbar}{\xi - m\nu}\right] \\
&= \frac{\aleph\zeta(\xi) + \zeta(m\nu)}{(m\nu - \xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(m\nu-\xi)^{\aleph+2}}\left[\frac{1}{\Gamma(\aleph)}\int_{\xi}^{m\nu}(m\nu-\hbar)^{\aleph-1}\zeta(\hbar)d\hbar\right].
\end{aligned}$$

Thus, we obtain

$$I_1 = \frac{\aleph\zeta(\xi) + \zeta(m\nu)}{(m\nu - \xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(m\nu-\xi)^{\aleph+2}} I_{\xi^+}^{\aleph}\zeta(m\nu). \tag{4.1}$$

Now, consider

$$\begin{aligned}
I_2 &= \int_0^1 \varpi(1-\varpi^{\aleph})\big[\zeta''(m(1-\varpi)\xi + \varpi\nu)d\varpi\big] \\
&= \frac{\varpi(1-\varpi^{\aleph})\zeta'(m(1-\varpi)\xi + \varpi\nu)}{(\nu - m\xi)}\bigg|_0^1 - \int_0^1 \frac{(1-(\aleph+1))\varpi^{\aleph}\zeta'(m(1-\varpi)\xi + \varpi\nu)}{(\nu - m\xi)}d\varpi \\
&= -\frac{1}{(\nu - m\xi)}\left[\frac{(1-(\aleph+1)\varpi^{\aleph})\zeta(m(1-\varpi)\xi + \varpi\nu)}{(\nu - m\xi)}\bigg|_0^1\right. \\
&\left. - \int_0^1 \frac{(-\aleph(\aleph+1))\varpi^{\aleph-1}\zeta(m(1-\varpi)\xi + \varpi\nu)}{(\nu - m\xi)}d\varpi\right] \\
&= \frac{\aleph\zeta(\nu) + \zeta(m\xi)}{(\nu - m\xi)^2} - \frac{\aleph(\aleph+1)}{(\nu - m\xi)^2}\int_0^1 \zeta(m(1-\varpi)\xi + \varpi\nu)\varpi^{\aleph-1}d\varpi \\
&= \frac{\aleph\zeta(\nu) + \zeta(m\xi)}{(\nu - m\xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(\nu - m\xi)^2}\left[\frac{1}{\Gamma(\aleph)}\int_0^1 \zeta(m(1-\varpi)\xi + \varpi\nu)\varpi^{\aleph-1}d\varpi\right].
\end{aligned}$$

Assume $\ell = m(1-\varpi)\xi + \varpi\nu$. Then $\varpi = \frac{\ell - m\xi}{\nu - m\xi}$ and $d\varpi = \frac{d\ell}{\nu - m\xi}$. By substituting these values in above equation, we get

$$\begin{aligned}
I_2 &= \frac{\aleph\zeta(\nu) + \zeta(m\xi)}{(\nu - m\xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(\nu - m\xi)^2}\left[\frac{1}{\Gamma(\aleph)}\int_{m\xi}^{\nu}\left(\frac{\ell - m\xi}{\nu - m\xi}\right)^{\aleph-1}\zeta(\ell)\frac{d\ell}{\nu - m\xi}\right] \\
&= \frac{\aleph\zeta(\nu) + \zeta(m\xi)}{(\nu - m\xi)^2} - \frac{(\aleph+1)\Gamma(\aleph+1)}{(\nu - m\xi)^{\aleph+2}}\left[\frac{1}{\Gamma(\aleph)}\int_{m\xi}^{\nu}(\ell - m\xi)^{\aleph-1}\zeta(\ell)d\ell\right].
\end{aligned}$$

Thus, we have

$$I_2=\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-\frac{(\aleph+1)\Gamma(\aleph+1)}{(\nu-m\xi)^{\aleph+2}}I^{\aleph}_{\nu^-}\zeta(m\xi). \tag{4.2}$$

By adding (4.1) and (4.2), we obtain the result

$$\begin{aligned}&\int_0^1\varpi(1-\varpi^{\aleph})\left[\zeta''(\varpi\xi+m(1-\varpi)\nu)d\varpi+\zeta''(m(1-\varpi)\xi+\varpi\nu)d\varpi\right]\\&=\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\left[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\right].\end{aligned}$$

Hence, the proof is complete. □

The first theorem of this section is given as follows.

Theorem 4.2. *Let ζ be a real valued twice differentiable mapping on $int(I)$ and $\xi,\nu\in int(I)$ with $\xi<\nu$ such that $\zeta''\in L[\xi,\nu]$. If $|\zeta''|$ is (α,m)-convex on $[\xi,\nu]$, then the inequality given below involving fractional integral with $\aleph\in(0,1]$*

$$\begin{aligned}&\left|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\left[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\right]\right|\\&\leq\left(\mathbb{B}(\alpha+2,\aleph+1)+m\mathbb{B}(2,\aleph+\alpha+1)\right)\left[|\zeta''(\xi)|+|\zeta''(\nu)|\right]\end{aligned}$$

holds true.

Proof. By utilizing Lemma 4.1 and (α,m)-convexity of $|\zeta''|$, we get

$$\begin{aligned}&\left|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\left[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\right]\right|\\&\leq\int_0^1\varpi(1-\varpi^{\aleph})\Big[|\zeta''(\varpi\xi+m(1-\varpi)\nu)|+|\zeta''(m(1-\varpi)\xi+\varpi\nu)|\Big]d\varpi\\&\leq\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu)|d\varpi+\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu)|d\varpi\\&\leq\int_0^1\varpi(1-\varpi^{\aleph})\Big(\varpi^{\alpha}|\zeta''(\xi)|+m(1-\varpi^{\alpha})|\zeta''(\nu)|\Big)d\varpi\\&+\int_0^1\varpi(1-\varpi^{\aleph})\Big(m(1-\varpi^{\alpha})|\zeta''(\xi)|+\varpi^{\alpha}|\zeta''(\nu)|\Big)d\varpi\\&\leq\int_0^1\varpi^{\alpha+1}(1-\varpi^{\aleph})|\zeta''(\xi)|d\varpi+m\int_0^1\varpi(1-\varpi^{\aleph})(1-\varpi^{\alpha})|\zeta''(\nu)|d\varpi\\&+m\int_0^1\varpi(1-\varpi^{\aleph})(1-\varpi^{\alpha})|\zeta''(\xi)|d\varpi+\int_0^1\varpi^{\alpha+1}(1-\varpi^{\aleph})|\zeta''(\nu)|d\varpi\\&\leq\left(\int_0^1\varpi^{\alpha+1}(1-\varpi^{\aleph})d\varpi\right)\Big[|\zeta''(\xi)|+|\zeta''(\nu)|\Big]\\&+m\left(\int_0^1\varpi(1-\varpi^{\aleph})(1-\varpi^{\alpha})d\varpi\right)\Big[|\zeta''(\xi)|+|\zeta''(\nu)|\Big].\end{aligned}$$

By using the fact $1-\varpi^t \leq 1-\varpi \leq (1-\varpi)^t$ for $\varpi^t \geq \varpi$ with $t \in (0,1]$ and $\varpi \in [0,1]$, the above inequality takes the following form

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|\\
&\leq \Big(\int_0^1 \varpi^{\alpha+1}(1-\varpi)^{\aleph})d\varpi\Big)\Big[|\zeta''(\xi)|+|\zeta''(\nu)|\Big]\\
&+m\Big(\int_0^1 \varpi(1-\varpi)^{\aleph+\alpha}d\varpi\Big)\Big[|\zeta''(\xi)|+|\zeta''(\nu)|\Big]\\
&\leq (\mathbb{B}(\alpha+2,\aleph+1))\,[|\zeta''(\xi)|+|\zeta''(\nu)|]+m(\mathbb{B}(2,\aleph+\alpha+1))\,[|\zeta''(\xi)|+|\zeta''(\nu)|]\\
&\leq (\mathbb{B}(\alpha+2,\aleph+1)+m\mathbb{B}(2,\aleph+\alpha+1))\,[|\zeta''(\xi)|+|\zeta''(\nu)|].
\end{aligned}$$

Hence, the proof is complete. □

Remark 4.3.

(1) By setting $\alpha=1$ and $m=1$, we obtain [7, Theorem 2].
(2) By setting $\alpha=m=\aleph=1$, we obtain [24, Theorem 2].

Theorem 4.4. *Let ζ be a real map twice differentiable on int(I) and $\xi,\nu \in$ int(I) with $\xi<\nu$ s.t. $\zeta'' \in L[\xi,\nu]$. If $|\zeta''|^q$ $(q>1)$ is (α,m)-convex on $[\xi,\nu]$, then the inequality given below involving fractional integral with $\aleph \in (0,1]$ holds.*

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|\\
&\leq \mathbb{B}^{\frac{1}{p}}(p+1,p\aleph+1)\Big\{\Big[\frac{|\zeta''(\xi)|^q+m|\zeta''(\nu)|^q}{\alpha+1}\Big]^{\frac{1}{q}}+\Big[\frac{m|\zeta''(\xi)|^q+|\zeta''(\nu)|^q}{\alpha+1}\Big]^{\frac{1}{q}}\Big\},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By utilizing Lemma 4.1, Hölder's inequality and (α,m)-convexity of $|\zeta''|^q$, we get

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|\\
&\leq \int_0^1 \varpi(1-\varpi^{\aleph})\Big[|\zeta''(\varpi\xi+m(1-\varpi)\nu)|+|\zeta''(m(1-\varpi)\xi+\varpi\nu)|\Big]d\varpi\\
&\leq \int_0^1 \varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu)|d\varpi+\int_0^1 \varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu)|d\varpi\\
&\leq \Big(\int_0^1 \varpi^p(1-\varpi^{\aleph})^p d\varpi\Big)^{\frac{1}{p}}\Big(\int_0^1 |\zeta''(\varpi\xi+m(1-\varpi)\nu)|^q d\varpi\Big)^{\frac{1}{q}}\\
&+\Big(\int_0^1 \varpi^p(1-\varpi^{\aleph})^p d\varpi\Big)^{\frac{1}{p}}\Big(\int_0^1 |\zeta''(m(1-\varpi)\xi+\varpi\nu)|^q d\varpi\Big)^{\frac{1}{q}}\\
&\leq \Big(\int_0^1 \varpi^p(1-\varpi^{\aleph})^p d\varpi\Big)^{\frac{1}{p}}\Big\{\Big[\int_0^1 \Big(\varpi^{\alpha}|\zeta''(\xi)|^q+m(1-\varpi^{\alpha})|\zeta''(\nu)|^q\Big)d\varpi\Big]^{\frac{1}{q}}\\
&+\Big[\int_0^1 \Big(m(1-\varpi^{\alpha})|\zeta''(\xi)|^q+\varpi^{\alpha}|\zeta''(\nu)|^q\Big)^{\frac{1}{q}}d\varpi\Big]^{\frac{1}{q}}\Big\}.
\end{aligned}$$

By using the fact $1-\varpi^t \le 1-\varpi \le (1-\varpi)^t$ for $\varpi^t \ge \varpi$ with $t \in (0,1]$ and $\varpi \in [0,1]$, the above inequality takes the following form

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|\\
&\le\Big(\int_0^1 \varpi^p(1-\varpi)^{p\aleph}d\varpi\Big)^{\frac{1}{p}}\Big\{\Big[\int_0^1\Big(\varpi^\alpha|\zeta''(\xi)|^q+m(1-\varpi)^\alpha|\zeta''(\nu)|^q\Big)d\varpi\Big]^{\frac{1}{q}}\\
&+\Big[\int_0^1\Big(m(1-\varpi)^\alpha|\zeta''(\xi)|^q+\varpi^\alpha|\zeta''(\nu)|^q\Big)^{\frac{1}{q}}d\varpi\Big]^{\frac{1}{q}}\Big\}\\
&\le\Big(\mathbb{B}(p+1,p\aleph+1)\Big)^{\frac{1}{p}}\Big\{\Big[|\zeta''(\xi)|^q\int_0^1\varpi^\alpha d\varpi+m|\zeta''(\nu)|^q\int_0^1(1-\varpi)^\alpha d\varpi\Big]^{\frac{1}{q}}\\
&+\Big[m|\zeta''(\xi)|^q\int_0^1(1-\varpi)^\alpha d\varpi+|\zeta''(\nu)|^q\int_0^1\varpi^\alpha d\varpi\Big]^{\frac{1}{q}}\Big\}\\
&\le\Big(\mathbb{B}(p+1,p\aleph+1)\Big)^{\frac{1}{p}}\Big\{\Big[|\zeta''(\xi)|^q\Big(\frac{1}{\alpha+1}\Big)+m|\zeta''(\nu)|^q\Big(\frac{1}{\alpha+1}\Big)\Big]^{\frac{1}{q}}\\
&+\Big[m|\zeta''(\xi)|^q\Big(\frac{1}{\alpha+1}\Big)+|\zeta''(\nu)|^q\Big(\frac{1}{\alpha+1}\Big)\Big]^{\frac{1}{q}}\Big\}\\
&\le\mathbb{B}^{\frac{1}{p}}(p+1,p\aleph+1)\Big\{\Big[\frac{|\zeta''(\xi)|^q+m|\zeta''(\nu)|^q}{\alpha+1}\Big]^{\frac{1}{q}}+\Big[\frac{m|\zeta''(\xi)|^q+|\zeta''(\nu)|^q}{\alpha+1}\Big]^{\frac{1}{q}}\Big\}.
\end{aligned}$$

Hence, the theorem is complete. □

Remark 4.5.

(1) By setting $\alpha=1$ and $m=1$, we obtain [7, Theorem 3].
(2) For $\aleph=m=1$ and $\alpha=s$, we yield precisely the same result as [25, Corollary 3.6] for $\alpha=1$.
(3) For $\aleph=m=1$ and $\alpha=s$, we yield precisely the same result as [25, Corollary 3.5] for $\alpha=m=1$.

Theorem 4.6. *Let ζ be a real map twice differentiable on int(I) and $\xi,\nu\in$ int(I) with $\xi<\nu$ s.t. $\zeta''\in L[\xi,\nu]$. If $|\zeta''|^q$ $(q>1)$ is (α,m)-convex on $[\xi,\nu]$, then the inequality given below involving fractional integral with $\aleph\in(0,1]$ holds.*

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|\\
&\le\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big\{\Big[|\zeta''(\xi)|^q\mathbb{B}(\alpha+1,q\aleph+1)+m|\zeta''(\nu)|^q\mathbb{B}(1,q\aleph+\alpha+1)\Big]^{\frac{1}{q}}\\
&+\Big[m|\zeta''(\xi)|^q\mathbb{B}(1,q\aleph+\alpha+1)+|\zeta''(\nu)|^q\mathbb{B}(\alpha+1,q\aleph+1)\Big]^{\frac{1}{q}}\Big\},
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. According to Lemma 4.1, Hölder's inequality and (α, m)-convexity of $|\zeta''|^q$, we obtain

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu|^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big|\\
&\le \int_0^1 \varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu|d\varpi+\int_0^1 \varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu)|d\varpi\\
&\le \Big(\int_0^1 \varpi^p d\varpi\Big)^{\frac{1}{p}}\Big[\int_0^1 (1-\varpi^{\aleph})^q|\zeta''(\varpi\xi+m(1-\varpi)\nu)|^q d\varpi\Big]^{\frac{1}{q}}\\
&+\Big(\int_0^1 \varpi^p d\varpi\Big)^{\frac{1}{p}}\Big[\int_0^1 (1-\varpi^{\aleph})^q|\zeta''(m(1-\varpi)\xi+\varpi\nu)|^q d\varpi\Big]^{\frac{1}{q}}\\
&\le \Big(\int_0^1 \varpi^p d\varpi\Big)^{\frac{1}{p}}\Big\{\Big[\int_0^1 \varpi^{\alpha}(1-\varpi^{\aleph})^q|\zeta''(\xi)|^q d\varpi+\int_0^1 (1-\varpi^{\aleph})^q(1-\varpi^{\alpha})m|\zeta''(\nu)|^q d\varpi\Big]^{\frac{1}{q}}\\
&+\Big[\int_0^1 (1-\varpi^{\aleph})^q(1-\varpi^{\alpha})m|\zeta''(\xi)|^q d\varpi+\int_0^1 \varpi^{\alpha}(1-\varpi^{\aleph})^q|\zeta''(\nu)|^q d\varpi\Big]^{\frac{1}{q}}\Big\}\\
&\le \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big\{\Big[|\zeta''(\xi)|^q\int_0^1 \varpi^{\alpha}(1-\varpi^{\aleph})^q d\varpi+m|\zeta''(\nu)|^q\int_0^1 (1-\varpi^{\aleph})^q(1-\varpi^{\alpha})d\varpi\Big]^{\frac{1}{q}}\\
&+\Big[m|\zeta''(\xi)|^q\int_0^1 (1-\varpi^{\aleph})^q(1-\varpi^{\alpha})d\varpi+|\zeta''(\nu)|^q\int_0^1 (1-\varpi^{\aleph})^q\varpi^{\alpha}d\varpi\Big]^{\frac{1}{q}}\Big\}.
\end{aligned}$$

By using the fact $1-\varpi^t \le 1-\varpi \le (1-\varpi)^t$ for $\varpi^t \ge \varpi$ with $t \in (0,1]$ and $\varpi \in [0,1]$, the above inequality takes the following form

$$\begin{aligned}
&\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu|^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph}+2}\Big|\\
&\le \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big\{\Big[|\zeta''(\xi)|^q\int_0^1 \varpi^{\alpha}(1-\varpi)^{q\aleph}d\varpi+m|\zeta''(\nu)|^q\int_0^1 (1-\varpi)^{q\aleph+\alpha}d\varpi\Big]^{\frac{1}{q}}\\
&+\Big[m|\zeta''(\xi)|^q\int_0^1 (1-\varpi)^{q\aleph+\alpha}d\varpi+|\zeta''(\nu)|^q\int_0^1 \varpi^{\alpha}(1-\varpi)^{q\aleph}d\varpi\Big]^{\frac{1}{q}}\Big\}\\
&\le \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big\{\Big[|\zeta''(\xi)|^q\mathbb{B}(\alpha+1,q\aleph+1)+m|\zeta''(\nu)|^q\mathbb{B}(1,q\aleph+\alpha+1)\Big]^{\frac{1}{q}}\\
&+\Big[m|\zeta''(\xi)|^q\mathbb{B}(1,q\aleph+\alpha+1)+|\zeta''(\nu)|^q\mathbb{B}(\alpha+1,q\aleph+1)\Big]^{\frac{1}{q}}\Big\}.
\end{aligned}$$

Hence, the theorem is complete. □

Theorem 4.7. *Let ζ be a real vlued twice differentiable mapping on int(I) and $\xi, \nu \in$ int(I) with $\xi < \nu$ s.t. $\zeta'' \in L[\xi, \nu]$. If $|\zeta''|^q$ $(q \ge 1)$ is (α, m)-convex on $[\xi, \nu]$, then the* *inequality given below involving fractional integral with* $\aleph \in (0,1]$ *holds.*

$$\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|$$
$$\leq\Big(\frac{\aleph}{2(\aleph+2)}\Big)\Big(\frac{2(\aleph+2)}{\aleph}\Big)^{\frac{1}{q}}\Big\{\Big[|\zeta''(\xi)|^q\mathbb{B}(\alpha+2,\aleph+1)+m|\zeta''(\nu)|^q\mathbb{B}(2,\aleph+\alpha+1)\Big]^{\frac{1}{q}}$$
$$+\Big[m|\zeta''(\xi)|^q\mathbb{B}(2,\aleph+\alpha+1)+|\zeta''(\nu)|^q\mathbb{B}(\alpha+2,\aleph+1)\Big]^{\frac{1}{q}}\Big\}.$$

Proof. By using Lemma 4.1, Power-Mean inequality and (α,m)-convexity of $|\zeta''|^q$, we get

$$\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|$$
$$\leq\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu)|d\varpi+\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu)|d\varpi$$
$$\leq\Big(\int_0^1\varpi(1-\varpi^{\aleph})d\varpi\Big)^{1-\frac{1}{q}}\Big[\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu|^qd\varpi\Big]^{\frac{1}{q}}$$
$$+\Big(\int_0^1\varpi(1-\varpi^{\aleph})d\varpi\Big)^{1-\frac{1}{q}}\Big[\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu|^qd\varpi\Big]^{\frac{1}{q}}$$
$$\leq\Big(\int_0^1\varpi(1-\varpi^{\aleph})d\varpi\Big)^{1-\frac{1}{q}}\Big\{\Big[\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu)|^qd\varpi\Big]^{\frac{1}{q}}$$
$$+\Big[\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu)|^qd\varpi\Big]^{\frac{1}{q}}\Big\}$$
$$\leq\Big(\int_0^1\varpi(1-\varpi^{\aleph})d\varpi\Big)^{1-\frac{1}{q}}\Big\{\Big[\int_0^1\varpi(1-\varpi^{\aleph})\Big(\varpi^{\alpha}|\zeta''(\xi)|^q+(1-\varpi^{\alpha})m|\zeta''(\nu)|^q\Big)d\varpi\Big]^{\frac{1}{q}}$$
$$+\Big[\int_0^1\varpi(1-\varpi^{\aleph})\Big((1-\varpi^{\alpha})m|\zeta''(\xi)|^q+\varpi^{\alpha}|\zeta''(\nu)|^q\Big)d\varpi\Big]^{\frac{1}{q}}\Big\}.$$

By using the fact $1-\varpi^t\leq 1-\varpi\leq(1-\varpi)^t$ for $\varpi^t\geq\varpi$ with $t\in(0,1]$ and $\varpi\in[0,1]$, the above inequality takes the following form

$$\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|$$
$$\leq\Big(\frac{\aleph}{2(\aleph+2)}\Big)^{1-\frac{1}{q}}\Big\{\Big[|\zeta''(\xi)|^q\int_0^1\varpi^{\alpha+1}(1-\varpi)^{\aleph}d\varpi+m|\zeta''(\nu)|^q\int_0^1\varpi(1-\varpi)^{\aleph+\alpha}d\varpi\Big]^{\frac{1}{q}}$$
$$+\Big[m|\zeta''(\xi)|^q\int_0^1\varpi(1-\varpi)^{\aleph+\alpha}d\varpi+|\zeta''(\nu)|^q\int_0^1\varpi^{\alpha+1}(1-\varpi)^{\aleph}d\varpi\Big]^{\frac{1}{q}}\Big\}$$
$$\leq\Big(\frac{\aleph}{2(\aleph+2)}\Big)\Big(\frac{2(\aleph+2)}{\aleph}\Big)^{\frac{1}{q}}\Big\{\Big[|\zeta''(\xi)|^q\mathbb{B}(\alpha+2,\aleph+1)+m|\zeta''(\nu)|^q\mathbb{B}(2,\aleph+\alpha+1)\Big]^{\frac{1}{q}}$$
$$+\Big[m|\zeta''(\xi)|^q\mathbb{B}(2,\aleph+\alpha+1)+|\zeta''(\nu)|^q\mathbb{B}(\alpha+2,\aleph+1)\Big]^{\frac{1}{q}}\Big\}.$$

Hence the theorem is complete. □

Remark 4.8. If we set $m=1$ and $\alpha=s$, we get the result [13, Theorem 13].

Theorem 4.9. *Let ζ be a real valued twice differentiable mapping on $int(I)$ and $\xi,\nu \in int(I)$ with $\xi < \nu$ s.t. $\zeta'' \in L[\xi,\nu]$. If $|\zeta''|^q$ $(q \geq 1)$ is (α,m)-convex on $[\xi,\nu]$, then the inequality given below involving fractional integral with $\aleph \in (0,1]$ holds.*

$$\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|$$
$$\leq\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Bigg\{\Big[|\zeta''(\xi)|^q\mathbb{B}(\alpha+2,q\aleph+1)+m|\zeta''(\nu)|^q\mathbb{B}(2,q\aleph+\alpha+1)\Big]^{\frac{1}{q}}$$
$$+\Big[m|\zeta''(\xi)|^q\mathbb{B}(2,q\aleph+\alpha+1)+|\zeta''(\nu)|^q\mathbb{B}(\alpha+2,q\aleph+1)\Big]^{\frac{1}{q}}\Bigg\}.$$

Proof. According to Lemma 4.1, Power-Mean inequality and (α,m)-convexity of $|\zeta''|^q$, we obtain

$$\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-(\aleph+1)\Gamma(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|$$
$$\leq\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(\varpi\xi+m(1-\varpi)\nu)|d\varpi+\int_0^1\varpi(1-\varpi^{\aleph})|\zeta''(m(1-\varpi)\xi+\varpi\nu)|d\varpi$$
$$\leq\Big(\int_0^1\varpi d\varpi\Big)^{1-\frac{1}{q}}\Big[\int_0^1\varpi(1-\varpi^{\aleph})^q|\zeta''(\varpi\xi+m(1-\varpi)\nu)|^q d\varpi\Big]^{\frac{1}{q}}$$
$$+\Big(\int_0^1\varpi d\varpi\Big)^{1-\frac{1}{q}}\Big[\int_0^1\varpi(1-\varpi^{\aleph})^q|\zeta''(m(1-\varpi)\xi+\varpi\nu)|^q d\varpi\Big]^{\frac{1}{q}}$$
$$\leq\Big(\int_0^1\varpi d\varpi\Big)^{1-\frac{1}{q}}\Bigg\{\Big[\int_0^1\varpi(1-\varpi^{\aleph})^q\Big(\varpi^{\alpha}|\zeta''(\xi)|^q+m(1-\varpi^{\alpha})|\zeta''(\nu)|^q\Big)d\varpi\Big]^{\frac{1}{q}}$$
$$+\Big[\int_0^1\varpi(1-\varpi^{\aleph})^q\Big(m(1-\varpi^{\alpha})|\zeta''(\xi)|^q+\varpi^{\alpha}|\zeta''(\nu)|^q\Big)d\varpi\Big]^{\frac{1}{q}}\Bigg\}.$$

By using the fact $1-\varpi^t\leq 1-\varpi\leq(1-\varpi)^t$ for $\varpi^t\geq\varpi$ with $t\in(0,1]$ and $\varpi\in[0,1]$, the above inequality takes the following form

$$\Big|\frac{\aleph\zeta(\xi)+\zeta(m\nu)}{(m\nu-\xi)^2}+\frac{\aleph\zeta(\nu)+\zeta(m\xi)}{(\nu-m\xi)^2}-\Gamma(\aleph+1)(\aleph+1)\Big[\frac{I^{\aleph}_{\xi^+}\zeta(m\nu)}{(m\nu-\xi)^{\aleph+2}}+\frac{I^{\aleph}_{\nu^-}\zeta(m\xi)}{(\nu-m\xi)^{\aleph+2}}\Big]\Big|$$
$$\leq\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Bigg\{\Big[|\zeta''(\xi)|^q\int_0^1\varpi^{\alpha+1}(1-\varpi)^{q\aleph}d\varpi+m|\zeta''(\nu)|^q\int_0^1\varpi(1-\varpi)^{q\aleph+\alpha}d\varpi\Big]^{\frac{1}{q}}$$
$$+\Big[m|\zeta''(\xi)|^q\int_0^1\varpi(1-\varpi)^{q\aleph+\alpha}d\varpi+|\zeta''(\nu)|^q\int_0^1\varpi^{\alpha+1}(1-\varpi)^{q\aleph}d\varpi\Big]^{\frac{1}{q}}\Bigg\}$$
$$\leq\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Bigg\{\Big[|\zeta''(\xi)|^q\mathbb{B}(\alpha+2,q\aleph+1)+m|\zeta''(\nu)|^q\mathbb{B}(2,q\aleph+\alpha+1)\Big]^{\frac{1}{q}}$$
$$+\Big[m|\zeta''(\xi)|^q\mathbb{B}(2,q\aleph+\alpha+1)+|\zeta''(\nu)|^q\mathbb{B}(\alpha+2,q\aleph+1)\Big]^{\frac{1}{q}}\Bigg\}.$$

Hence, the theorem is complete. □

Remark 4.10. If we set $m=1$ and $\alpha=s$, we get the result [13, Theorem 15].

5. Means-based applications

The importance of means cannot be denied in several fields of life like engineering, physics, economics and business. The means can present data more shortly and appropriately. This section is dedicated to presenting the main results in terms of generalized means which increase the importance of our explored results. For any arbitrary $\phi, \psi \in \mathbb{R}$ and $\phi \neq \psi$, consider the following means

$$\begin{aligned} A(\phi,\psi) &= \frac{\phi+\psi}{2}, \qquad \phi,\psi \in \mathbb{R}, \\ \mathbb{H}(\phi,\psi) &= \frac{2}{\frac{1}{\phi}+\frac{1}{\psi}}, \qquad \phi,\psi \in \mathbb{R}\setminus\{0\}, \\ L(\phi,\psi) &= \frac{\psi-\phi}{ln|\psi|-ln|\phi|}, \qquad |\phi| \neq |\psi|. \end{aligned}$$

In the form of these means, we present the following propositions as applications of the obtained results.

Proposition 5.1. *Let $\xi, \nu \in \mathbb{R}^+$ with $\xi < \nu$ and $\alpha \in (0,1]$. Then we have*

$$\begin{aligned} &\left|\frac{1}{(m\nu-\xi)^2}\Big\{A(e^\xi, e^{m\nu}) - L(e^\xi, e^{m\nu})\Big\} + \frac{1}{(\nu-m\xi)^2}\Big\{A(e^\nu, e^{m\xi}) - L(e^\nu, e^{m\xi})\Big\}\right| \\ &\leq \frac{1}{2}(\mathbb{B}(\alpha+2,2) + m\mathbb{B}(2,\alpha+2))\left[|e^\xi| + |e^\nu|\right]. \end{aligned}$$

Proof. By utilizing $\zeta(\varpi) = e^\varpi$ and $\aleph = 1$ in Theorem 4.2, we obtain the desired result. □

Proposition 5.2. *Let $\xi, \nu \in \mathbb{R}^+$ with $\xi < \nu$ and $\alpha \in (0,1]$. Then we have*

$$\begin{aligned} &\left|\frac{1}{(m\nu-\xi)^2}\Big\{\mathbb{H}^{-1}(\xi, m\nu) - L^{-1}(\xi, m\nu)\Big\} + \frac{1}{(\nu-m\xi)^2}\Big\{\mathbb{H}^{-1}(\nu, m\xi) - L^{-1}(m\xi, \nu)\Big\}\right| \\ &\leq \frac{1}{2}(\mathbb{B}^{\frac{1}{p}}(p+1,p+1))\left\{\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\left[\left(\left|\frac{1}{\xi^3}\right|^q + m\left|\frac{1}{\nu^3}\right|^q\right)^{\frac{1}{q}} + \left(m\left|\frac{1}{\xi^3}\right|^q + \left|\frac{1}{\nu^3}\right|^q\right)^{\frac{1}{q}}\right]\right\}, \end{aligned}$$

Proof. This statement follows by utilizing $\zeta(\varpi) = \frac{1}{\varpi}$, $\varpi \neq 0$ and $\aleph = 1$ in Theorem 4.4. □

Proposition 5.3. *Let $\xi, \nu \in \mathbb{R}^+$ with $\xi < \nu$ and $\alpha \in (0,1]$. Then we have*

$$\begin{aligned} &\left|\frac{1}{(m\nu-\xi)^2}\Big\{\mathbb{H}^{-1}(\xi, m\nu) - L^{-1}(\xi, m\nu)\Big\} + \frac{1}{(\nu-m\xi)^2}\Big\{\mathbb{H}^{-1}(\nu, m\xi) - L^{-1}(m\xi, \nu)\Big\}\right| \\ &\leq \frac{1}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left\{\left(\left|\frac{1}{\xi^3}\right|^q \mathbb{B}(\alpha+2,2) + m\left|\frac{1}{\nu^3}\right|^q \mathbb{B}(2,\alpha+2)\right)^{\frac{1}{q}}\right. \\ &\left. + \left(m\left|\frac{1}{\xi^3}\right|^q \mathbb{B}(2,\alpha+2) + \left|\frac{1}{\nu^3}\right|^q \mathbb{B}(\alpha+2,2)\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. By using $\zeta(\varpi) = \frac{1}{\varpi}$, $\varpi \neq 0$ and $\aleph = 1$ in Theorem 4.7, we get the statement. □

6. Conclusion

In this paper, we have employed two alternative methods to derive Hermite-Hadamard type fractional inequalities involving R-L fractional integrals through the use of (α, m)-convexity. Initially, we established an identity where a differentiable function is expressed in terms of the R-L fractional integrals. Using this identity, we have derived Hermite-Hadamard type inequalities for functions whose absolute first derivatives are (α, m)-convex functions. Subsequently, we have established another identity where a twice differentiable function is expressed involving the R-L fractional integrals. This identity was utilized to obtain Hermite-Hadamard type inequalities for functions whose absolute second derivatives are (α, m)-convex functions. Lastly, we included several mean based applications of the results obtained.

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