International Electronic Journal of Algebra
Volume 37 (2025) 273-296
DOI: 10.24330/ieja.1554197

DECOMPOSITIONS OF MULTIPLICATIVE SEMIGROUPS OF M-DOMAIN RINGS AND REDUCED RICKART RINGS

Insa Cremer

Received: 12 November 2023; Revised: 14 July 2024; Accepted 22 July 2024
Communicated by Meltem Altun Özarslan

Dedicated to the memory of Professor Syed M. Tariq Rizvi

Abstract. The main result of this article is that the multiplicative semigroup of an m-domain ring is a strong semilattice of certain subsemigroups, each of which turns out to be a right-cancellative monoid, and that this presentation of the semigroup as a strong semilattice of right-cancellative semigroups is essentially unique. As a consequence, it is shown that, given an m-domain ring $\langle R, +, \cdot \rangle$ with the unary operation $^\circ$ mapping every element to its minimal idempotent duplicator (in the sense of N.V. Subrahmanyam), the algebra $\langle R, \cdot, ^\circ \rangle$ is a strong semilattice of right-cancellative D-semigroups (in the sense of T. Stokes), also essentially unique. Implications for reduced Rickart rings, which can be seen as a subclass of m-domain rings, are also described.

Mathematics Subject Classification (2020): 16W99, 20M10, 20M25, 20M99, 08A62, 08A05, 08A99
Keywords: Reduced Rickart ring, m-domain ring, strong semilattice

1. Introduction

In 1960, N. V. Subrahmanyam introduced a class of rings with a special unary operation $^\circ$ which he called m-domain rings in [16]. They were a generalization of associate rings as defined by I. Sussman in [17].

Rickart rings have been investigated since the middle of the 20th century. They are also called PP-rings. In the early 70s, the class of commutative Rickart rings (a subclass of reduced Rickart rings) was studied independently by W. Cornish in [5] and by T.P. Speed in a series of papers (see, for example, [14]). Janowitz proved in 1976 in [10] that a reduced Rickart ring with the so-called Abian order is a semi-Boolean algebra (i.e., a meet semilattice in which every principal ideal is a Boolean algebra). Some necessary and sufficient conditions for a Rickart ring to be reduced were given later in [8] by J. A. Fraser and W. K. Nicholson.

In [3], it was proved that a ring is a reduced Rickart ring if and only if it is isomorphic to an associate ring in the sense of Sussman. We will see that the class of reduced Rickart rings is essentially the subclass of m-domain rings consisting of those rings which have the multiplicative identity.

In 1975, Penning [11] introduced minimal duplicator rings as a generalization of, among others, m-domain rings. Later they were called C-rings by Cornish (see, for example, [6]). Of course, every reduced Rickart ring is a C-ring, because every m-domain ring is a C-ring.

In [15], a semigroup equipped with a unary operation $^\circ$ satisfying three particular identities is called a D-semigroup, and a ring whose multiplicative semigroup is a D-semigroup is called D-ring. C-rings (and therefore also m-domain rings and reduced Rickart rings) are a subclass of D-rings.

The multiplicative semigroup of any m-domain ring can be decomposed into mutually disjoint semigroups which are called m-domains in [16] (following [17]). Every idempotent of an m-domain ring R is the identity of some m-domain. In particular, the multiplicative semigroup of a reduced Rickart ring can be decomposed in this way.

An important subclass of D-semigroups are D-abundant D-semigroups. It was also shown in [15] that a D-ring with unity is D-abundant if and only if it is a left Rickart ring. We will see that also the operation $^\circ$ on an m-domain ring satisfies the necessary conditions to make $\langle R, \cdot, ^\circ \rangle$ a D-abundant D-semigroup as defined in [15].

A reduced Rickart ring admits a particular unary operation $'$ which we call a focal operation following [2] and [3]. Also C-rings, m-domain rings and D-rings are equipped with some special unary operations. These unary operations correspond to the "double" focal operation $^\circ$ defined by $a^\circ = (a')'$ in the reduced Rickart ring.

Since Rickart rings are a far more common research field than m-domain rings, the focus of this article is on reduced Rickart rings. However, the main results of this paper (Theorems 4.1 and 8.1) hold even for the slightly more general m-domain rings. The proofs are very similar and not longer, but since there are some differences in the details, it seemed that it would not be enough to prove everything for reduced Rickart rings and then claim without proof that similar results hold for m-domain rings. Therefore, we prove all the results for m-domain rings and then derive their reduced Rickart ring versions as corollaries.

Strong semilattices of semigroups are a well-known structure in semigroup theory (see, for example, [9]). In this article, we will equip the family of the m-domains $\mathcal{M}$ of an m-domain ring R with a set of semigroup homomorphisms Φ such that

$\langle \mathcal{M}, \Phi \rangle$ is an inverse system over the semilattice of idempotents of the ring. The strong semilattice of m-domains which we obtain from this inverse system turns out to be the multiplicative semigroup of the ring R.

Unlike in the case of multiplication, the m-domains are not closed under sums. They are, however, closed under the operation $^\circ$. Therefore, our focus will be on strong semilattice constructions involving the multiplication and the operation $^\circ$ (which, in the special case of a reduced Rickart ring, can be derived from the focal operation (see [3])).

In Section 2, we first define reduced Rickart rings and m-domain rings and state some preliminary results about them. Then we prove that a ring is a reduced Rickart ring if and only if it is an m-domain ring with multiplicative identity.

In Section 3, we deal with inverse systems and strong semilattices of semigroups, and we show as an example that the m-domains of an m-domain ring (with suitably chosen homomorphisms) form an inverse system of semigroups.

In Section 4, we prove that the multiplicative semigroup of an m-domain ring is the strong semilattice induced by this inverse system of m-domains. In the opposite direction, we show that, if the strong semilattice induced by a particular inverse system $\langle \mathcal{A}, \mathcal{H} \rangle$ of right-cancellative semigroups happens to be the multiplicative semigroup of some m-domain ring, then the inverse system $\langle \mathcal{A}, \mathcal{H} \rangle$ essentially equals the inverse system of m-domains of the ring.

The results of Section 4 give rise to the question of whether it is possible to obtain similar results which are not only about the ring multiplication, but also include the unary operation $^\circ$. To answer this question, instead of dealing with inverse systems of semigroups, we need to deal with inverse systems of algebras having two operations (multiplication and $^\circ$). Therefore, in Section 5, we recall the definition and basic properties of D-semigroups, which are suitable for this purpose. We also recall D-rings and prove that every m-domain ring is a D-abundant D-ring.

In Section 6, we equip the m-domains with an additional unary operation which turns them into D-semigroups (or D-monoids, if we include also the multiplicative identities into the signatures) in order to obtain an inverse system $\langle \mathcal{M}, \Phi \rangle$ of D-semigroups (or of D-monoids) in an m-domain ring.

In Section 7, we define in a standard way strong semilattices of D-semigroups and D-monoids analogously to the definition of strong semilattices of semigroups.

Finally, in Section 8, we obtain D-semigroup analogues of the results of Section 4. That is, we show that the strong semilattice of D-semigroups induced by the inverse system of D-semigroups of an m-domain ring equals the D-semigroup reduct of the ring. Moreover, if an inverse system of right-cancellative D-semigroups with identities $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$ over a lower semilattice induces a strong semilattice of D-semigroups that happens to be the D-semigroup reduct of some m-domain ring, then the $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$ is the inverse system of D-semigroups of the ring. In the case of a reduced Rickart ring, the multiplicative identity is also included into the constructions, i.e., the D-monoid reduct of a reduced Rickart ring is a strong semilattice induced by its inverse system of D-monoids.

2. Reduced Rickart rings and m-domain rings

In this section, we first define reduced Rickart rings and state some of their basic properties (Section 2.1), then we define m-domain rings and also state some of their basic properties in Section 2.2. In Section 2.3, we prove that a ring is a reduced Rickart ring if and only if it is an m-domain ring with multiplicative identity.

2.1. Reduced Rickart rings. This subsection is a short summary of the most important preliminaries from [3]. For more details on Rickart rings and reduced Rickart rings in particular, see Sections 2 and 3 of that article.

A ring is called *reduced* if it has no nonzero nilpotent elements, i.e., $x^n = 0$ implies $x = 0$. It is known that a ring R is reduced if and only if, for all $x \in R$, $x^2 = 0$ implies $x = 0$. Moreover, every reduced ring is commutative at zero, i.e., for all elements x, y of a reduced ring R, $xy = 0$ if and only if $yx = 0$ (because from $xy = 0$ follows $(yx)^2 = y(xy)x = 0$, whence $yx = 0$).

We will use the following fact which is well-known and easy to prove.

Proposition 2.1. *All idempotents of a reduced ring are central.*

It is well-known that the set of idempotents E of a semigroup A is ordered by

$$e \leq f \text{ iff } ef = e = fe. \tag{1}$$

In the set of central idempotents, this reduces to $e \leq f$ iff $ef = e$. This order turns the set of central idempotents of a semigroup into a lower semilattice with

$$e \wedge f = ef. \tag{2}$$

In a unital reduced ring, the set of idempotents (all of which are central, because of reducedness) E with the natural order of idempotents given by Equation (1) is not only a lower semilattice but even a Boolean algebra with the meet as in Equation (2), the join $e \vee f = e + f - ef$ and the complementation $e^\perp = 1 - e$.

Following [3], we define a Rickart ring in the following way.

Definition 2.2. A ring R is called a *Rickart ring* if it admits unary operations $'$ and $^{\backprime}$ such that for every $a \in R$, the elements a' and $a^{\backprime}$ are idempotents such that, for all $x \in R$,

$$ax = 0 \quad \text{iff} \quad a'x = x, \tag{3}$$

$$xa = 0 \quad \text{iff} \quad xa^{\backprime} = x. \tag{4}$$

The operations $'$ and $^{\backprime}$ are called *focal operations.*

See also [2] for the focal operations and [1, page 65] for a more standard definition of Rickart ring (which differs from the preceding one only by technical details). It can be easily verified that any Rickart ring is unital.

If a Rickart ring R is reduced and a, a', $a^{\backprime}$ are as in Definition 2.2, then the operations $'$ and $^{\backprime}$ are uniquely determined and $a' = a^{\backprime}$ for all $a \in R$ (see, for example, [3]). Hence, a reduced Rickart ring has only one focal operation, which we will denote by $'$.

The most simple example of a reduced Rickart ring is an arbitrary Boolean ring R with the focal operation defined as $a' = 1 - a$ for every element $a \in R$. More general examples include certain subdirect products of (possibly non-commutative) domains (see [3] for the details).

In this paper we will use some properties of the focal operation, which we state in the following proposition.

Proposition 2.3. *Let R be a reduced Rickart ring and $'$ its focal operation. For all $a, b \in R$*

(a) *$aa'' = a$,*

(b) *if $e \in R$ is idempotent, then $e' = 1 - e$*

(c) *a'' is an idempotent such that $ax = 0 \Leftrightarrow a''x = 0$ for all $x \in R$,*

(d) *$(ab)'' = a''b''$,*

(e) *$a''' = a'$.*

Proof. See [3] (Proposition 2.4 (h) for (a), p. 381 for (b), Equation (2.2) for (c), Proposition 3.8 (g) for (d) and Proposition 2.4 (e) for (e)). $\square$

In [3], Theorem 6.5, reduced Rickart rings are characterised using another unary operation which is closely related to the focal operation. Unfortunately, there is a mistake in that theorem. The correct version of it (see [4]) is as follows:

Lemma 2.4. *A ring with unity is a reduced Rickart ring if and only if it admits a unary operation $^\circ$ such that*

(a) *$xx^\circ = x = x^\circ x$,*

(b) $(xy)^\circ = x^\circ y^\circ$,

(c) $0^\circ = 0$.

In this case, the focal operation of the ring is given by $x' := 1 - x^\circ$.

Note that x° is always an idempotent, because $x^\circ = 1 - x' = x''$ by Proposition 2.3(b).

In the sequel, for a reduced Rickart ring $\langle R, +, \cdot, 1\rangle$ with the operation $^\circ$ from Lemma 2.4, we will call the algebra $\langle R, +, \cdot, ^\circ, 1\rangle$ *enriched reduced Rickart ring.*

2.2. M-domain rings. Cancellative (from both sides) semigroups are sometimes called *multiplicative domains.* In [17], associate rings in the sense of Sussman were decomposed into such semigroups, which were called *m-domains* in that paper. This was generalized for a class of rings which was therefore named *m-domain rings* in [16].

In Section 2.3 we will see that every reduced Rickart ring is a unital m-domain ring and vice-versa, which will enable us to apply the properties mentioned in this subsection to reduced Rickart rings.

Definition 2.5. [16] A ring R is called a *multiplicative domain ring*, or shorter, an *m-domain ring* if, for every $a \in R$, there exists a central idempotent a° such that, for all $a, b \in R$,

(a) $aa^\circ = a$,

(b) if $e \in R$ is idempotent and $ea = ae = a$, then $a^\circ e = a^\circ$,

(c) $(ab)^\circ = a^\circ b^\circ$.

It is easy to check that for an element a of an m-domain ring, the central idempotent a° is unique. Therefore we treat $^\circ$ as an operation on the m-domain ring. We use the same symbol $^\circ$ as for the operation in Lemma 2.4, because, as we will see in the next subsection, the operation in Lemma 2.4 is a special case of the operation $^\circ$ on an m-domain ring. As for reduced Rickart rings, the algebra $\langle R, +, \cdot, ^\circ\rangle$ will be called an *enriched m-domain ring* if $\langle R, +, \cdot\rangle$ is an m-domain ring and $^\circ$ is the operation from Definition 2.5.

Remark 2.6. Let e be an idempotent in an m-domain ring. From Definition 2.5(b) it follows that $e^\circ e = e^\circ$ (taking $a = e$). Since e° is central, this yields $ee^\circ = e^\circ$. But then, from Definition 2.5(a), we obtain $e^\circ = e$. So the operation $^\circ$ maps every idempotent on itself. In particular, every idempotent in an m-domain ring is central.

Observe that

$$a^\circ = 0 \text{ iff } a = 0 \tag{5}$$

holds in every m-domain ring (one direction follows immediately from Definition 2.5(a), the other one is obtained by chosing $a = e = 0$ in Definition 2.5(b)).

Lemma 2.7. [16, Theorem XIV] *Let R be an m-domain ring and for an idempotent e, let M_e denote the set $\{x \in R \mid x^\circ = e\}$. Then the following statements hold.*

(a) *For every idempotent $e \in R$, M_e is a cancellative subsemigroup of the multiplicative semigroup of the ring, and e is the identity of this subsemigroup.*
(b) *The sets M_e are distinct and form a partition of R.*

Observe that the partition mentioned in Lemma 2.7(b) corresponds to the kernel equivalence of the operation $^\circ$. It is obvious from Equation (5) that $M_0 = \{0\}$.

Since according to Lemma 2.7(a), the semigroups M_e are multiplicative domains (recall that this is just another word for *cancellative semigroups*), we follow the terminology of [17] by using an abbreviated version of this term for the sets M_e.

Definition 2.8. For an idempotent e in an m-domain ring R, the set M_e is called an *m-domain*.

We will use the term not only for the set M_e, but also for the semigroup $\langle M_e, \cdot\rangle$, the monoid $\langle M_e, \cdot, e\rangle$, etc.

Remark 2.9. Obviously, an m-domain M_e is always closed under the operation $^\circ$ from Definition 2.5, since $e \in M_e$ for every idempotent e and $x^\circ = e$ for every $x \in M_e$ by Lemma 2.7.

2.3. Relations between reduced Rickart rings and m-domain rings. After introducing m-domain rings and reduced Rickart rings, we are now able to connect them by the following result.

Theorem 2.10. *A ring is a reduced Rickart ring if and only if it is a unital m-domain ring. The unary operation from Definition 2.5 coincides with the operation from Lemma 2.4 (both are denoted $^\circ$), and they are connected to the focal operation $'$ by $x^\circ = x''$.*

Proof. Let R be a unital m-domain ring and let $^\circ$ be the unary operation from Definition 2.5. By Lemma 2.4 and Definition 2.5, the ring R is a reduced Rickart ring, because the operation $^\circ$ satisfies the identities of Lemma 2.4 (the first condition of Lemma 2.4 follows from Remark 2.6; the third condition follows from Equation (5)). Lemma 2.4 also yields the identity $x^\circ = 1 - x'$. By Proposition 2.3(b), this implies $x^\circ = x''$, since x' is idempotent.

Now let R be a reduced Rickart ring and let $'$ be its focal operation. Recall that every Rickart ring is unital. We consider the operation $^\circ$ given in Lemma 2.4.

As $x^\circ = 1 - x' = x''$ by Lemma 2.4 and Proposition 2.3(b), the element x° is idempotent for all $x \in R$ (and hence also central by Proposition 2.1). Let e be an idempotent such that $ea = ae = a$. Proposition 2.3(b) yields $e'' = e$. Now Proposition 2.3(d) yields $a'' = (ae)'' = a''e'' = a''e$, i.e., $a^\circ = a^\circ e$. Hence, R is an m-domain ring. □

Remark 2.11. It is obvious from Remark 2.9 and Theorem 2.10 that the m-domains M_e of a reduced Rickart ring are closed under the operation $^\circ$ from Lemma 2.4.

3. Inverse systems and strong semilattices

The aim of this section is to recall strong semilattices of semigroups and to settle the corresponding terminology and notation. To state the definition of a strong semilattice of semigroups, it is useful first to define inverse systems of semigroups. At the end of the section, we show as an example that the multiplicative semigroup of a reduced Rickart ring is a strong semilattice of semigroups, and that these semigroups are its m-domains.

The construction which is dual to the strong semilattice is also known as a *Płonka sum* (see [12]).

Definition 3.1. [13] Let $\langle S, \leq\rangle$ be a poset and let $\mathcal{A} = \{A_s | s \in S\}$ be a family of algebras of the same type. Let $\mathcal{H} = \{h_s^t | s, t \in S \text{ and } s \leq t\}$ be a family of homomorphisms $h_s^t : A_t \to A_s$. Suppose that for all $r, s, t \in S$

(a) the homomorphism h_t^t is the identity map

(b) if $r \leq s \leq t$, then $h_r^s h_s^t = h_r^t$.

Then the pair $\langle \mathcal{A}, \mathcal{H}\rangle$ is called an *inverse system* of the algebras A_s and the homomorphisms h_s^t (over the carrier S).

Some authors require the poset $\langle S, \leq\rangle$ to be (upwards) directed, but for our work this is not necessary. In the sequel, $\langle S, \leq\rangle$ will always be a lower semilattice (thus downwards directed). Later it will have to be a lower semilattice which has a greatest element (and thus it will be directed in both directions).

It is also common to define an inverse system as a triple $\langle S, \mathcal{A}, \mathcal{H}\rangle$. However, we do not include the carrier $\langle S, \leq\rangle$ into the signature, because it is determined by $\langle \mathcal{A}, \mathcal{H}\rangle$ up to order isomorphism.

When dealing with inverse systems of monoids, it is sometimes necessary to clarify whether the identities of the monoids are included into their signatures or not. In the first case, we will speak of *inverse systems of monoids*, while in the latter case, we will say *inverse system of semigroups with identities.* I.e., an inverse system of monoids consists of a family of monoids and a family of monoid homomorphisms, while an inverse system of semigroups with identities consists of a family of semigroups that happen to have identities and a family of semigroup homomorphisms.

Example 3.2. Let R be an m-domain ring and let $\langle E, \leq \rangle$ be its semilattice of idempotents. We are going to demonstrate that the family of m-domains together with a suitably chosen family of maps is an inverse system of semigroups over E.

For every idempotent e, let $\cdot_e$ be the restriction of the ring multiplication to the m-domain M_e (recall that by Lemma 2.7 the m-domains are closed with respect to multiplication). Let $\mathcal{M}$ be the family of all the m-domains $\langle M_e, \cdot_e \rangle$ (by Lemma 2.7(b), the family is disjoint).

For each pair $e, f \in E$ such that $e \leq f$, let ϕ_e^f be the map

$$\begin{array}{rccc} \phi_e^f : & M_f & \to & M_e \\ & x & \mapsto & xe. \end{array} \tag{6}$$

This map is well-defined, because, for $x \in M_f$ and $e \leq f$, we have $(xe)^\circ = x^\circ e^\circ = fe = e$ by Definition 2.5(c), so indeed $xe \in M_e$ by the definition of M_e in Lemma 2.7. Moreover, the maps ϕ_e^f are semigroup homomorphisms (by Equation (6), centrality of idempotents and the definition of $\cdot_e$ and $\cdot_f$ as restrictions of the ring multiplication).

Now let Φ be the set of all the maps ϕ_e^f. To demonstrate that $\langle \mathcal{M}, \Phi \rangle$ is an inverse system of semigroups and semigroup homomorphisms, it remains to show that the two conditions on homomorphisms given in Definition 3.1 are satisfied.

(a) If $x \in M_e$ for some $e \in E$, then $x^\circ = e$ (from Lemma 2.7), and therefore $\phi_e^e(x) = xe = xx^\circ = x$ by Definition 2.5(a). Thus, ϕ_e^e is the identity map.
(b) Suppose $e \leq f \leq g$. The definition of the homomorphisms in (6) and the assumption $e \leq f$ (i.e., $ef = e = fe$) yield $\phi_e^f(\phi_f^g(x)) = \phi_e^f(xf) = (xf)e = xe$.

Hence, $\langle \mathcal{M}, \Phi \rangle$ is indeed an inverse system of semigroups.

Since every m-domain M_e is a right-cancellative monoid (see Lemma 2.7), the result of the construction described in this example is very similar to the result of a construction on PP-monoids which can be found in [7].

Definition 3.3. The inverse system $\langle \mathcal{M}, \Phi \rangle$ from Example 3.2 will be denoted $\operatorname{sys} R$. We will call it the *inverse system (of semigroups) of the m-domain ring R.*

Given an inverse system of semigroups over a lower semilattice, the following definition provides a semigroup which contains all the semigroups from the inverse system as subsemigroups.

Definition 3.4. Let $\langle S, \wedge \rangle$ be a lower semilattice and let $\langle \mathcal{A}, \mathcal{H} \rangle$ be an inverse system of pairwise disjoint semigroups over $\langle S, \wedge \rangle$.

On the union $A = \bigcup_{s \in S} A_s$ of all the semigroups, we define an operation $\bullet$ in the following way. If $x \in A_s$ and $y \in A_t$, and $\cdot_{s \wedge t}$ denotes the multiplication on the semigroup $A_{s \wedge t}$, then

$$x \bullet y := h^s_{s \wedge t}(x) \cdot_{s \wedge t} h^t_{s \wedge t}(y). \tag{7}$$

Then we write $A = \mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$ and call the semigroup $\langle A, \bullet \rangle$ (the operation $\bullet$ is known to be associative, see e.g. [9]) a *strong semilattice of semigroups.*

In order to keep the notation simple, we will write $\mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$ for both the set A and the semigroup $\langle A, \bullet \rangle$.

In general, the strong semilattice of semigroups obtained from an inverse system of monoids is itself not a monoid. However, under certain conditions it is, and its identity can be obtained from the inverse system of monoids. We will deal with this situation in Section 7, where we define strong semilattices of so-called D-monoids in a way that ensures that a strong semilattice of D-monoids is itself a D-monoid.

Since every strong semilattice of semigroups is itself a semigroup and, as we will see in Section 6, every strong semilattic of D-semigroups is itself a D-semigroup and every strong semilattice of D-monoids is itself a D-monoid, we will avoid the usual term *strong semilattice of monoids* for a strong semilattice of semigroups which happen to be monoids (even if the strong semilattice is obtained from an inverse system of monoids), because such a strong semilattice might not be a monoid itself. We will speak of strong semilattices of semigroups with identities instead.

4. Decomposition of the multiplicative semigroup of an m-domain ring

In this section we investigate further the inverse system of semigroups of an m-domain ring described in Example 3.2. Recall that a *right-cancellative* semigroup is a semigroup A in which $xa = ya$ implies $x = y$ for all $x, y, a \in A$. The results in this section which deal with inverse systems of right-cancellative semigroups hold also for inverse systems af left-cancellative semigroups.

The next theorem shows that the multiplicative semigroup of an m-domain ring is a strong semilattice of semigroups induced by the inverse system $\operatorname{sys} R$ from Example 3.2.

Theorem 4.1. (a) *Let $\langle R, +, \cdot\rangle$ be an m-domain ring. Then* $\langle R, \cdot\rangle = \mathfrak{S}(\operatorname{sys} R)$.

(b) *Let $\langle \mathcal{A}, \mathcal{H}\rangle$ be an inverse system of disjoint right-cancellative semigroups with identities over a lower semilattice S. If $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle = \langle R, \cdot\rangle$ for some m-domain ring R, then $\langle \mathcal{A}, \mathcal{H}\rangle = \operatorname{sys} R$.*

Proof. (a) As described in Example 3.2, $\operatorname{sys} R = \langle \mathcal{M}, \Phi\rangle$, where $\mathcal{M}$ is the set of m-domains indexed by idempotents and Φ is the set of homomorphisms defined in Equation (6). We construct the strong semilattice of semigroups induced by the inverse system $\langle \mathcal{M}, \Phi\rangle$ using Equation (7). Since the union of all the m-domains is R by Lemma 2.7, this strong semilattice will have the form $\langle R, \bullet\rangle$. It remains to prove that the multiplication $\bullet$ coincides with the original ring multiplication.

Suppose that $x \in M_e$ and $y \in M_f$ for some elements $x, y \in R$ and idempotents $e, f \in E$. By Lemma 2.7, this means

$$x^\circ = e \text{ and } y^\circ = f \tag{8}$$

Now by Equations (7), (2), (6), centrality of idempotents, Equation (8) and Proposition 2.4(a) we have

$$\begin{aligned} x \bullet y &= \phi^e_{e\wedge f}(x) \cdot_{e\wedge f} \phi^f_{e\wedge f}(y) \\ &= \phi^e_{ef}(x) \cdot_{ef} \phi^f_{ef}(y) \\ &= xef \cdot yef \\ &= xx^\circ yy^\circ \\ &= x \cdot y. \end{aligned}$$

Thus $\langle R, \cdot\rangle = \mathfrak{S}\langle \mathcal{M}, \Phi\rangle$.

(b) Let $\langle \mathcal{A}, \mathcal{H}\rangle$ be an inverse system of disjoint right-cancellative semigroups with identities over a lower semilattice S and suppose $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle = \langle R, \cdot\rangle$ for some m-domain ring R.

In order to simplify the notation, we will work with the partial order induced on the family $\mathcal{A}$ by the semilattice S instead of referring to the semilattice S itself. We will denote this induced partial order on $\mathcal{A}$ by $\preceq$ (i.e., $A_s \preceq A_t$ iff $s \leq t$). The corresponding meet operation will be denoted by $\curlywedge$. We will index the homomorphisms from $\mathcal{H}$ by their domain and range, i.e., h^B_A denotes the semigroup homomorphism from B to A for semigroups $A, B \in \mathcal{A}$ with $A \preceq B$. The multiplication on a semigroup $A \in \mathcal{A}$ will be denoted by $\underset{A}{\circ}$.

We need to prove that $\mathcal{A}$ is the family of m-domains of R and that $\mathcal{H}$ is the family of all the homomorphisms ϕ^f_e defined in Equation (6).

Let $\langle \mathcal{M}, \Phi \rangle = \operatorname{sys} R$, i.e., $\mathcal{M}$ is the set of m-domains $\langle M_e, \cdot \rangle$ indexed by idempotents, and $\Phi = \{\phi_e^f \mid e, f \in E \text{ and } e \leq f\}$, where E is the set of idempotents.

Since $\mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle = \langle R, \cdot \rangle$, obviously $\mathcal{A}$ is a partition of R. Moreover, for $x \in A$ and $y \in B$, Equation (7) translates to

$$xy = h^A_{A \curlywedge B}(x) \underset{A \curlywedge B}{\circ} h^B_{A \curlywedge B}(y) \tag{9}$$

(in particular, $xy \in A \curlywedge B$).

Obviously, the identities of the semigroups from $\mathcal{A}$ are idempotents of the ring R, since the multiplication on a semigroup $A \in \mathcal{A}$ is a restriction of the ring multiplication to the set A.

We will prove that

(1) for every $a \in R$ and every $A \in \mathcal{A}$, we have $a \in A$ iff $a^\circ \in A$,
(2) every semigroup $A \in \mathcal{A}$ is the m-domain corresponding to its identity,
(3) $\mathcal{M} = \mathcal{A}$,
(4) $\mathcal{H} = \Phi$.

(1) For a semigroup $A \in \mathcal{A}$, let e be the identity of A and let $a \in A$ be an arbitrary element of A. Let $B \in \mathcal{A}$ be the semigroup containing the element a°.

Since $ae = ea = a$, we have $a^\circ e = a^\circ$ by Definition 2.5(b). Therefore $a^\circ e \in B$, because $a^\circ \in B$. But since $\langle R, \cdot \rangle = \mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle$, Equation (9) yields that $a^\circ e \in B \curlywedge A$. So $B \preceq A$.

On the other hand, by Definition 2.5(a), we have $a^\circ a = a \in A$. But now Equation (9) yields $a^\circ a \in B \curlywedge A$. So $B \curlywedge A = A$, i.e., $A \preceq B$.

Hence, $A = B$. Since A was arbitrary and B was chosen to be the semigroup containing a° for an arbitrary element $a \in A$, we conclude that a and a° are always contained in the same semigroup. I.e., for $a \in R$ and $A \in \mathcal{A}$, $a \in A$ if and only if $a^\circ \in A$.

(2) If $a \in A$, then not only $a^\circ \in A$, but a° must be the identity e of A, because by right-cancellativity of A, from $a^\circ a = a = ea$ (see Definition 2.5(a)) follows $a^\circ = e$.

Conversely, let e be the identity of A and $a \in R$ and $a^\circ = e$. Then by the previous paragraph (1), we have $a \in A$.

This proves that $A = M_e$, because the m-domain is defined as $M_e = \{x \in R \mid x^\circ = e\}$ (see Lemma 2.7). Since the multiplication $\underset{A}{\circ}$ is a restriction of the ring multiplication, A and M_e are equal not only as sets, but also as semigroups.

(3) It follows from the previous step that $\mathcal{A} \subseteq \mathcal{M}$. Since both $\mathcal{A}$ and $\mathcal{M}$ are partitions of the ring R, this immediately implies $\mathcal{A} = \mathcal{M}$.

(4) Let $h_A^B \in \mathcal{H}$. In view of the previous items, the domain and range of h_A^B are m-domains. So $h_A^B : M_f \to M_e$ for some idempotents $e, f \in E$. We will prove that $e \leq f$ and $h_A^B = \phi_e^f$.

First, observe that e is the only idempotent in M_e: If $g \in M_e$ is idempotent, then $eg = geg$ by centrality of idempotents (see Remark 2.6). So by cancellativity, $e = ge$. But $ge = g$, because e is the identity of M_e. Hence, $e = g$.

Let $x \in M_f$; then

$$\begin{aligned} h_A^B(x) &= e \underset{A}{\circ} h_A^B(x) \text{ (since } e \text{ is the identity of } M_e) \\ &= h_A^A(e) \underset{A}{\circ} h_A^B(x) \text{ (because } h_A^A \text{ is the identity map)} \\ &= ex \text{ (by Equation (9))}. \end{aligned}$$

By the first part of this theorem, $ex \in M_{e\wedge f}$, since $e \in M_e$ and $x \in M_f$. But since the range of h_A^B is M_e, we also have $ex \in M_e$. This yields $M_e = M_{e\wedge f}$. So $e = e \wedge f$, i.e., $e \leq f$. Now obviously $h_A^B(x) = ex = \phi_e^f(x)$ by Equation (6).

So indeed $\langle \mathcal{A}, \mathcal{H} \rangle = \operatorname{sys} R$. □

Fountain obtained a result on a strong semilattice decomposition of right PP-monoids with central idempotents, see [7, Theorem 1]. His result is similar to Theorem 4.1(a) and in particular Corollary 4.2(a), and there are similar ideas in the proofs. However, the author preferred a short, direct and independent proof of Theorem 4.1(a). This also establishes some notation which will be necessary in the proof of Theorem 8.1.

It follows from Theorem 4.1 that the multiplicative semigroups of m-domain rings R and R' are equal if $\operatorname{sys} R = \operatorname{sys} R'$ (the converse is also true by Example 3.2).

As a special case of Theorem 4.1, we obtain the following.

Corollary 4.2. (a) *Let $\langle R, +, \cdot, 1 \rangle$ be a reduced Rickart ring. Then* $\langle R, \cdot \rangle = \mathfrak{S}(\operatorname{sys} R)$.

(b) *Let $\langle \mathcal{A}, \mathcal{H} \rangle$ be an inverse system of disjoint right-cancellative semigroups over lower semilattice S. If $\mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle = \langle R, \cdot \rangle$ for some reduced Rickart ring R, then $\langle \mathcal{A}, \mathcal{H} \rangle = \operatorname{sys} R$.*

Proof. (a) Immediate from Theorem 4.1(a).

(b) Suppose that $\mathfrak{S}\langle\mathcal{A},\mathcal{H}\rangle = \langle R,\cdot\rangle$ for some reduced Rickart ring R. Since every reduced Rickart ring is also an m-domain ring, it suffices to prove that the semigroups from the inverse system $\langle\mathcal{A},\mathcal{H}\rangle$ have identities. We use the fact that every reduced Rickart ring is unital.

Since $\mathcal{A}$ is a partition of R, the element 1 of the ring is contained in some semigroup from $\mathcal{A}$. Let T be this semigroup. For every element x of the ring, if $x \in A$, then $x \cdot 1 \in A \curlywedge T$ by Equation 9. But obviously $x \cdot 1 = x \in A$. Therefore, $A = A \curlywedge T$. Since this is the case for all $A \in \mathcal{A}$, the semigroup T must be the greatest element of $\mathcal{A}$.

Therefore, for every $A \in \mathcal{A}$, there is a homomorphism h_A^T from T to A. Now consider the element $h_A^T(1) \in A$. For every element $a \in A$, since h_A^A is the identity map (see Definition 3.4), we have $h_A^T(1) \underset{A}{\circ} a = h_A^T(1) \underset{A}{\circ} h_A^A(a) = 1 \cdot a$ by Equation (9). In the same way, $a \underset{A}{\circ} h_A^T(1) = a$. So the element $h_A^T(1)$ is the identity of A. □

It is already known from Fountain's result in [7] that the semigroup $\mathfrak{S}\langle\mathcal{A},\mathcal{H}\rangle$ mentioned in Corollary 4.2(b) is a left PP monoid with central idempotents.

It follows from Corollary 4.2 that the multiplicative semigroups of reduced Rickart rings R and R' are equal if $\operatorname{sys} R = \operatorname{sys} R'$ and that $\mathfrak{S}(\operatorname{sys} R)$ is the unique representation of the semigroup $\langle R,\cdot\rangle$ as a strong semilattice of right-cancellative semigroups.

5. D-semigroups, D-monoids and D-rings

The question arises if Theorem 4.1 can be modified to include also the operation $^\circ$. I.e., we want to find out what happens if instead of the inverse system of m-domains (seen as semigroups $\langle M_e,\cdot\rangle$) we deal with the inverse system of "enriched" m-domains (seen as algebras of the kind $\langle M_e,\cdot,^\circ\rangle$ or $\langle M_e,\cdot,^\circ,e\rangle$). This section will provide the necessary tools for the following sections, in which that question will be answered. We need to work with algebras which have not only multiplication, but also a unary operation similar to the operation $^\circ$. D-semigroups are such algebras, and therefore, we recall their definition and basic properties in this section.

Definition 5.1. [15] A semigroup A is said to be a *D-semigroup* if there exists some subset U of its set of idempotents E such that, for all $a \in A$, there exists a smallest $e \in U$ with the property that $ea = a$ (smallest in the sense of the standard partial order of idempotents, see Equation (1)).

D-semigroups can also be characterized in the following way.

Proposition 5.2. [15] *A semigroup A is a D-semigroup if and only if it can be equipped with a unary operation $^\circ$ satisfying the following for all $a, b \in A$:*

(a) $a^\circ a = a$,

(b) $(a^\circ)^\circ = a^\circ$,

(c) $(ab)^\circ a^\circ = a^\circ (ab)^\circ = (ab)^\circ$.

In a D-semigroup A, the set U from Definition 5.1 is the range of the operation $^\circ$ from Proposition 5.2. For each element $a \in A$, the element a° from Proposition 5.2 is the smallest $e \in U$ with the property that $ea = a$. Hence the set U from Definition 5.1 uniquely determines the operation $^\circ$ from Proposition 5.2 and vice versa.

We will treat D-semigroups as algebras of the kind $\langle A, \cdot, ^\circ \rangle$ (where $\cdot$ denotes the multiplication on the semigroup).

Note that, since $a^\circ a^\circ = (a^\circ)^\circ a^\circ = a^\circ$ by Proposition 5.2(b) and Proposition 5.2(a), the element a° is idempotent for every $a \in A$.

Definition 5.3. [15] A D-semigroup $\langle A, \cdot, ^\circ \rangle$ is called D-semiadequate if $a^\circ b^\circ = b^\circ a^\circ$ for all $a, b \in A$.

The notion of *D-abundant* D-semigroup is also defined in [15], and in the same article, the following was proved to be equivalent to the original definition of a D-abundant D-semigroup.

Definition 5.4. A D-semigroup $\langle A, \cdot, ^\circ \rangle$ is said to be *D-abundant* if, for all $x, y \in A^1$ and all $a \in A$, $xa = ya$ implies $xa^\circ = ya^\circ$ (where A^1 denotes the monoid created from the semigroup A by adding a new element which acts like an identity).

In this article, we will refer to D-semigroups with identity $\langle A, \cdot, ^\circ, 1 \rangle$ as *D-monoids*. We will deal with the following special subclasses of D-monoids.

Definition 5.5. A D-semigroup or D-monoid is said to be *right-cancellative* if the underlying semigroup is right-cancellative.

We call a D-monoid $\langle A, \cdot, ^\circ, 1 \rangle$ *$^\circ$-trivial* if $a^\circ = 1$ for all $a \in A$.

Every monoid can be turned into a $^\circ$-trivial D-monoid in the following very straight-forward way:

Proposition 5.6. *Let $\langle A, \cdot, 1 \rangle$ be a monoid, and let $a^\circ := 1$ for all $a \in A$. Then $\langle A, \cdot, ^\circ, 1 \rangle$ is a D-monoid.*

Proof. Easy calculations show that all the conditions of Definition 5.1 are satisfied: $a^\circ a = 1a = a$, $(a^\circ)^\circ = 1 = 1^\circ$ and $(ab)^\circ a^\circ = a^\circ (ab)^\circ = (ab)^\circ = 1$. □

Proposition 5.7. *Let $\langle A, \cdot, 1\rangle$ be a right-cancellative monoid with a unary operation $^\circ$. Then the following are equivalent.*

(a) *The algebra $\langle A, \cdot, ^\circ, 1\rangle$ is a D-monoid.*

(b) *The operation $^\circ$ satisfies the conditions* (a) *and* (b) *from Proposition 5.2.*

(c) *$a^\circ = 1$ for all $a \in A$.*

Proof. (a) $\implies$ (b) is obvious.

(b) $\implies$ (c) follows by right-cancellativity from $a^\circ a = 1 \cdot a$.

(c) $\implies$ (a) is immediate from Proposition 5.6. □

The next lemma establishes the relationships between right-cancellative, $^\circ$-trivial and D-abundant D-monoids.

Lemma 5.8. *A D-monoid $\langle A, \cdot, ^\circ, 1\rangle$ is right-cancellative if and only if it is both D-abundant and $^\circ$-trivial.*

Proof. Let $\langle A, \cdot, ^\circ, 1\rangle$ be a D-monoid.

First, assume that $\langle A, \cdot, ^\circ, 1\rangle$ is both D-abundant and $^\circ$-trivial. Let $a, x, y \in A$ such that $xa = ya$. Then $xa^\circ = ya^\circ$, since $\langle A, \cdot, ^\circ, 1\rangle$ is D-abundant (see Definition 5.4). But since it is also $^\circ$-trivial (see Definition 5.5), $a^\circ = 1$, so $x = y$. Hence, $\langle A, \cdot, ^\circ, 1\rangle$ is right-cancellative.

For the opposite direction, assume that $\langle A, \cdot, ^\circ, 1\rangle$ is right-cancellative. Then it is obviously D-abundant, because for all $x, y, a \in A$, $xa = ya$ implies $x = y$. Thus also $xa^\circ = ya^\circ$, and $^\circ$-triviality follows from Proposition 5.7. □

A ring is said to be a *D-ring* if its multiplicative semigroup is a D-semigroup (of course, every ring which has the multiplicative identity can be turned into a $^\circ$-trivial D-ring by chosing $a^\circ = 1$ for all elements a). It is immediate from Definition 5.1 and the definition of a C-ring (see [6]) that the multiplicative semigroup of a C-ring is a D-semigroup such that the set U from Definition 5.1 is the set of all central idempotents. Hence, C-rings are examples of D-rings.

We will treat D-rings as algebras of the kind $\langle R, +, \cdot, ^\circ, 1\rangle$ (like enriched reduced Rickart rings) or $\langle R, +, \cdot, ^\circ\rangle$ (like enriched m-domain rings). A D-ring is said to be *D-abundant* if its multiplicative semigroup is D-abundant.

The purpose of the following proposition is to clarify the relation between D-rings and enriched m-domain rings by showing that every enriched m-domain ring is a D-abundant D-ring in a unique way.

Proposition 5.9. *Let $\langle R, +, \cdot, ^\circ\rangle$ be an enriched m-domain ring.*

(a) *Then* $\langle R, +, \cdot, ^\circ \rangle$ *is a D-semiadequate D-abundant D-ring. The set* U *from Definition* 5.1 *which corresponds to the operation* $^\circ$ *is the set of all idempotents* E*.*

(b) *If an algebra* $\langle R, +, \cdot, ^+ \rangle$ *(i.e., the same ring with another unary operation* $^+$*) is a D-abundant D-ring, too, then the operations* $^\circ$ *and* $^+$ *coincide.*

Proof. (a) To prove that $\langle R, +, \cdot, ^\circ \rangle$ is a D-ring, we will verify that $\langle R, \cdot, ^\circ \rangle$ satisfies the conditions for being a D-semigroup given in Proposition 5.2. Obviously, the first condition of Proposition 5.2 holds, because it is identical to the first item of Definition 2.5 (recall that idempotents are central by Remark 2.6). The third condition of Proposition 5.2 follows immediately from the third item of Definition 2.5 and from the fact that a° is a central idempotent for every $a \in R$. For the second condition of Proposition 5.2, let $a \in R$. By Lemma 2.7, there exists some m-domain M_e such that $a \in M_e$, and $a^\circ = e$. Since $e \in M_e$, too, Lemma 2.7 also yields that $e^\circ = e$. Therefore, $(a^\circ)^\circ = e^\circ = e = a^\circ$. So $\langle R, +, \cdot, ^\circ \rangle$ is a D-ring.

To prove that $\langle R, \cdot, ^\circ \rangle$ is also D-abundant, let $x, y, a \in R$ be such that $xa = ya$. Then $(x - y)a = 0$.

So by Definition 2.5(c) and Equation (5), $(x - y)^\circ a^\circ = ((x - y)a)^\circ = 0^\circ = 0$. Hence, by Definition 2.5(a), $(x - y)a^\circ = (x - y)(x - y)^\circ a^\circ = 0$. Therefore, $xa = ya$ implies $xa^\circ = ya^\circ$, so by Definition 5.4, $\langle R, +, \cdot, ^\circ \rangle$ is D-abundant.

D-semiadequateness follows immedidately from centrality of idempotents a° and b° (for arbitrary a and b).

Obviously, the set U from Definition 5.1 must be the set of all idempotents, because the operation $^\circ$ maps every idempotent on itself (see Remark 2.6).

(b) For uniqueness of the operation $^\circ$, suppose $^+$ is another unary operation defined on R such that $\langle R, +, \cdot, ^+ \rangle$ is a D-abundant D-ring. Then

$$a^+ a = a = a^\circ a \tag{10}$$

for every $a \in R$ by Proposition 5.2(a). This yields $a^+ = a^+ a^+ = a^\circ a^+$ by Definition 5.4. In the same way, Equation (10) also yields $a^\circ = a^\circ a^\circ = a^+ a^\circ$. So $a^+ = a^\circ a^+ = a^+ a^\circ = a^\circ$ by centrality of a°. □

From here on, given an enriched m-domain ring $\langle R, +, \cdot, ^\circ \rangle$, the D-semigroup $\langle R, \cdot, ^\circ \rangle$ will be called the *D-semigroup reduct of* R, and for an enriched reduced

Rickart ring $\langle R, +, \cdot, ^\circ, 1\rangle$, the D-monoid $\langle R, \cdot, ^\circ, 1\rangle$ will be called the *D-monoid reduct of R*.

6. Inverse systems: a continuation

In Theorem 4.1 we saw that, first, the multiplicative semigroup $\langle R, \cdot\rangle$ of an m-domain ring is the strong semilattice induced by its inverse system of semigroups, and, second, if the strong semilattice induced by a given inverse system $\langle \mathcal{A}, \mathcal{H}\rangle$ of right-cancellative semigroups with identity happens to be the multiplicative semigroup of some m-domain ring, then $\langle \mathcal{A}, \mathcal{H}\rangle$ is the inverse system of semigroups of the ring.

We are going to prove similar results about the D-semigroup reduct of an enriched m-domain ring and about the D-monoid reduct of an enriched reduced Rickart ring in Section 8. Therefore in this section we turn the m-domains into $^\circ$-trivial D-monoids and prove that the semigroup homomorphisms defined in Equation (6) between the m-domains are also D-monoid homomorphisms. Hence we obtain an inverse system of D-monoids in an enriched m-domain ring.

Proposition 6.1. *Let R be an enriched m-domain ring and let E be its semilattice of idempotents. For every $e \in E$, let $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$ be the algebra of type* $(2, 1, 0)$ *defined by*

$$a \cdot_e b := a \cdot b \tag{11}$$

(see also Example 3.2*) and*

$$a^\circ_e := e. \tag{12}$$

Then $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$ is a right-cancellative D-monoid.

Proof. By Proposition 5.9, $\langle R, +, \cdot, ^\circ\rangle$ is a D-abundant D-ring. The operation $\cdot_e$ is the restriction of the ring multiplication $\cdot$ to the m-domain M_e, and the operation ${}^\circ_e$ is the restriction of the operation $^\circ$ from Definition 2.5 (recall that M_e is closed under $\cdot$ by Lemma 2.7 and under $^\circ$ by Remark 2.9). So $\langle M_e, \cdot_e, {}^\circ_e\rangle$ is a sub-D-semigroup of $\langle R, \cdot, ^\circ\rangle$. Therefore, it must be D-abundant, too.

Moreover, since M_e has the identity e (by Lemma 2.7), it is a D-monoid. It is evident from Equation (12) that the D-monoid $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$ is $^\circ$-trivial. Now Lemma 5.8 yields that it is right-cancellative. □

Proposition 6.2. *Let R be an enriched m-domain ring, let E be its semilattice of idempotents and for every pair of idempotents e, f with $e \le f$, let $\phi^f_e : M_f \to M_e$ be the map given by Equation* (6) *(i.e., $\phi^f_e(x) = xe$). Let*

$$\mathcal{M}^\circ_1 = \{\langle M_e, \cdot_e, {}^\circ_e, e\rangle \mid e \in E\} \tag{13}$$

with $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$ *as in Proposition* 6.1 *and let*

$$\Phi = \{\phi_e^f \mid e, f \in E \text{ and } e \le f\} \tag{14}$$

as in Example 3.2. Then $\langle \mathcal{M}_1^\circ, \Phi\rangle$ *is an inverse system of right-cancellative D-monoids.*

Proof. First let us prove that ϕ_e^f is a D-monoid homomorphism between the D-monoids $\langle M_f, \cdot_f, {}^\circ_f, f\rangle$ and $\langle M_e, \cdot_e, {}^\circ_e, e\rangle$.

We have already proved (see Example 3.2) that the map ϕ_e^f is a semigroup homomorphism between $\langle M_f, \cdot_f\rangle$ and $\langle M_e, \cdot_e\rangle$. It also preserves the identity, because by Equation (6), $\phi_e^f(f) = fe = e$, since $e \le f$. Since it preserves the identity, it must also preserve the unary operation, because the D-monoids are $^\circ$-trivial: By Equation (12) $\phi_e^f(x_f^\circ) = \phi_e^f(f) = e = (\phi_e^f(x))^\circ_e$. Hence, ϕ_e^f is a D-monoid homomorphism.

In Example 3.2 we already saw that for every $e \in E$, the map ϕ_e^e is the identity map, and that $\phi_e^f \phi_f^g = \phi_e^g$ for all $e, f, g \in E$ with $e \le f \le g$. We conclude that $\langle E, \mathcal{M}, \Phi\rangle$ is an inverse system of D-monoids. □

Definition 6.3. The inverse system of D-monoids $\langle \mathcal{M}_1^\circ, \Phi\rangle$ from Proposition 6.2 will be called the *inverse system of D-monoids of the enriched m-domain ring R* and we will denote it by $\operatorname{sys}_1^\circ R$. When we want to treat the m-domains just as D-semigroups instead of D-monoids, then we will speak of the *inverse system of D-semigroups of the enriched m-domain ring R* and write $\operatorname{sys}^\circ R$ to refer to the inverse system $\langle \mathcal{M}^\circ, \Phi\rangle$, where $\mathcal{M}^\circ = \{\langle M_e, \cdot_e, {}^\circ_e\rangle \mid e \in E\}$.

7. Strong semilattices of D-monoids

In this section, we will settle the terminology concerning strong semilattices of D-semigroups and D-monoids.

The following definition is a variation of the left/right dual construction of the one in [7, Theorem 1]. It is more general in that it does not assume the D-monoids to be right-cancellative, whereas in [7], the monoids are required to be left-cancellative (so the dual would require them to be right-cancellative).

Definition 7.1. (a) Let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of pairwise disjoint D-semigroups over a lower semilattice S. On the union of all the D-semigroups $A = \bigcup_{s\in S} A_s$, we define a binary operation $\bullet$ as in Equation (7) and a unary operation in the following way: For $x \in A_s$,

$$x^\bullet := x_s^\circ. \tag{15}$$

Then we write $A = \mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ and call the algebra $\langle A, \bullet, ^\bullet\rangle$ a *strong semilattice of D-semigroups (induced by the inverse system $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$).*

(b) If $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ is even an inverse system of D-monoids over a semilattice S which has the greatest element $\top$, then we can define also a constant $\mathbf{1}$ by

$$\mathbf{1} \quad := \quad 1_\top. \tag{16}$$

We write $A = \mathfrak{S}_1^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ and call the algebra $\langle A, \bullet, ^\bullet, \mathbf{1}\rangle$ a *strong semilattice of D-monoids.*

Obviously, if $\langle A, \bullet, ^\bullet\rangle$ is a strong semilattice of D-semigroups, then $\langle A, \bullet\rangle$ is a strong semilattice of semigroups.

The purpose of the next Proposition is to justify the terminology by giving a positive answer to the naturally arising question whether a strong semilattice of D-semigroups (D-monoids) is itself also a D-semigroup (D-monoid).

Proposition 7.2. (a) *Every strong semilattice of D-semigroups is a D-semigroup.*

(b) *Every strong semilattice of D-monoids is a D-monoid.*

Proof. (a) Let S, $\mathcal{A}^\circ$ and $\mathcal{H}$ be as in Definition 7.1, and let $\bullet$ and $^\bullet$ be the operations defined on the union $A = \bigcup_{s\in S} A_s$ as in Definition 7.1(a).

Since $\langle A, \bullet\rangle$ is a strong semilattice of semigroups, it is clear that it is a semigroup. So we only need to prove that the identities from Proposition 5.2 are satsified.

Let $x \in A_s$, $y \in A_t$ and $z \in A_u$.

By Equation (7), Equation (15), Definition 3.1 (recall that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ is an inverse system) and Proposition 5.2(a):

$$x^\bullet \bullet x = h_s^s(x_s^\circ) \cdot_s h_s^s(x) = x_s^\circ \cdot_s x = x$$

By Equation (15) and Proposition 5.2(b):

$$(x^\bullet)^\bullet = (x_s^\circ)_s^\circ = x_s^\circ = x^\bullet.$$

The third identity follows similarly from Proposition 5.2(c):

$$\begin{aligned}
&(x \bullet y)^\bullet \bullet x^\bullet \\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\bullet \bullet x_s^\circ && \text{(by Equations (7) and (15))}\\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\circ_{s\wedge t} \cdot_{s\wedge t} h^s_{s\wedge t}(x_s^\circ) && \text{(by Equations (15) and (7))}\\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\circ_{s\wedge t} \cdot_{s\wedge t} (h^s_{s\wedge t}(x))^\circ_s && (h^s_{s\wedge t} \text{ is a D-semigroup morphism})\\
&= (h^s_{s\wedge t}(x) \cdot_{s\wedge t} h^t_{s\wedge t}(y))^\circ_{s\wedge t} && \text{(by Proposition 5.2(c))}\\
&= (x \bullet y)^\bullet && \text{(by Equations (7) and (15)),}
\end{aligned}$$

and similarly for the multiplication from the other side.

(b) Suppose the semilattice S has a top element $\top$ and $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ is an inverse system of D-monoids over S. By the previous item, $\langle A, \bullet, {}^\bullet\rangle$ is a D-semigroup.

The element $\mathbf{1}$ is the neutral element with respect to the operation $\bullet$, because 1_s is the neutral element for $\cdot_s$:

$$x \bullet \mathbf{1} = h^s_{s\wedge\top}(x) \cdot_{s\wedge\top} h^\top_{s\wedge\top}(1_\top) = h^s_s(x) \cdot_s h^\top_s(1_\top) = h^s_s(x) \cdot_s 1_s = x,$$

by Equation (7) and (16), since $\top$ is the top element of S and $h^\top_s$ is a monoid homomorphism. (The identity $\mathbf{1} \bullet x = x$ is proved in the same way.) Hence, $\langle A, \bullet, \mathbf{1}\rangle$ is a monoid. □

Observe that the operation $^\circ$ is used neither in Equation (16) nor in the proof of Proposition 7.2(b). So given an inverse system of monoids (instead of D-monoids) over a lower semilattice which has the greatest element, we can also define a constant $\mathbf{1}$ by Equation (16), and $\mathbf{1}$ is the identity of the strong semilattice of semigroups $\langle A, \bullet\rangle$.

8. Decomposition of the D-semigroup reduct of an enriched m-domain ring

In this section, we look at the strong semilattice of D-semigroups (D-monoids) induced by the inverse system $\mathrm{sys}^\circ R$ ($\mathrm{sys}^\circ_1 R$) of a given enriched m-domain ring (reduced Rickart ring) R, and we see that it is the D-semigroup $\langle R, \cdot, {}^\circ\rangle$ (and for a reduced Rickart ring, the D-monoid $\langle R, \cdot, {}^\circ, 1\rangle$). So the D-semigroup reduct of an enriched m-domain ring is a strong semilattice of right-cancellative D-semigroups, and the D-monoid reduct of a reduced Rickart ring is a strong semilattice of right-cancellative D-monoids.

The next theorem is the D-semigroup version of Theorem 4.1. It proves that the D-semigroup reduct $\langle R, \cdot, {}^\circ\rangle$ of an enriched m-domain ring is a strong semilattice of right-cancellative D-semigroups.

Theorem 8.1. (a) *Let $\langle R, +, \cdot, {}^\circ\rangle$ be an enriched m-domain ring. Then $\langle R, \cdot, {}^\circ\rangle = \mathfrak{S}^\circ(\mathrm{sys}^\circ R)$.*

(b) *Let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of pairwise disjoint right-cancellative D-semigroups with identities over some lower semilattice. Suppose $\mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle R, \cdot, {}^\circ\rangle$ for some enriched m-domain ring $\langle R, +, \cdot, {}^\circ\rangle$. Then $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \mathrm{sys}^\circ R$.*

Proof. Let R be an enriched m-domain ring and let $\langle \mathcal{M}^\circ, \Phi\rangle = \mathrm{sys}^\circ R$. So $\mathcal{M}^\circ = \{\langle M_e, \cdot_e, {}^\circ_e \mid e \in E\rangle\}$ (see Definition 6.3) and Φ is the family defined in Equation (14), i.e., $\Phi = \{\phi_e^f \mid e, f \in E \text{ and } e \leq f\}$, where ϕ_e^f are the maps defined in Equation (6). Let $\langle R, \bullet, ^\bullet\rangle = \mathfrak{S}^\circ\langle \mathcal{M}^\circ, \Phi\rangle$.

(a) We need to prove that $\langle R, \bullet, ^\bullet\rangle = \langle R, \cdot, ^\circ\rangle$. To prove that the binary operation $\bullet$ coincides with $\cdot$, let us forget about the other operations for a moment. Consider the family of semigroups $\mathcal{M}$ obtained from $\mathcal{M}^\circ$ by replacing each element $\langle M_e, \cdot_e, {}^\circ_e\rangle$ by its reduct $\langle M_e, \cdot_e\rangle$. By Definition 3.3, $\langle \mathcal{M}, \Phi\rangle = \operatorname{sys} R$. So by Theorem 4.1, $\mathfrak{S}\langle \mathcal{M}, \Phi\rangle = \langle R, \cdot\rangle$. But $\mathfrak{S}\langle \mathcal{M}, \Phi\rangle = \langle R, \bullet\rangle$. So the operation $\bullet$ indeed coincides with the ring multiplication $\cdot$. The unary operations $^\circ$ and $^\bullet$ obviously coincide, too, because $x^\bullet = x_e^\circ = e = x^\circ$ by Equations (15), Equation (12) and Lemma 2.7.

(b) Now let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of right-cancellative D-semigroups with identities over a lower semilattice such that $\mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle R, \cdot, ^\circ\rangle$. We have to prove that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \operatorname{sys}^\circ R$, i.e., that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle \mathcal{M}^\circ, \Phi\rangle$ (with $\langle \mathcal{M}^\circ, \Phi\rangle$ as defined in Definition 6.3).

Let $\mathcal{M}$ and $\mathcal{A}$ be the families of semigroups obtained from $\mathcal{M}^\circ$ and $\mathcal{A}^\circ$ by replacing all the elements by their semigroup reducts. So $\mathfrak{S}\langle \mathcal{A}, \mathcal{H}\rangle = \langle R, \cdot\rangle$, and since the semigroups from $\mathcal{A}$ have identities, we can use Theorem 4.1(b), obtaining that $\langle \mathcal{A}, \mathcal{H}\rangle = \operatorname{sys} R$, i.e., $\mathcal{M} = \mathcal{A}$ and $\Phi = \mathcal{H}$. In particular, the underlying sets of the members of $\mathcal{A}$ are the m-domains of the ring R, and the multiplications $\cdot_e$ on them are just restrictions of the ring multiplication.

It remains to prove that the unary operations on each m-domain M_e also coincide. Let $\mathcal{A}^\circ = \{\langle M_e, \cdot_e, {}^+_e\rangle \mid e \in E\}$. By Proposition 5.2(a), for every element a of the m-domain M_e, we have $a_e^+ a = a$, because $\langle M_e, \cdot_e, {}^+_e\rangle$ is a D-semigroup. Since by Proposition 6.1 the m-domain M_e with the usual unary operation ${}^\circ_e$ is also a D-semigroup, we also have $a_e^\circ a = a$. So $a_e^+ a = a_e^\circ a$. But by right-cancellativity of the semigroup $\langle M_e, \cdot_e\rangle$, this implies $a_e^+ = a_e^\circ$. So $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \operatorname{sys}^\circ R$. □

For an inverse system $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ of D-semigroups with identities, let us denote by $\mathcal{A}_1^\circ$ the family obtained from $\mathcal{A}^\circ$ by including the identities of the D-semigroups into their signatures. Then $\langle \mathcal{A}_1^\circ, \mathcal{H}\rangle$ is even an inverse system of D-monoids.

The following corollary shows that in the case of an enriched reduced Rickart ring, Theorem 8.1 can be modified to include the multiplicative identities not only into the inverse system, but also into the strong semilattice construction. In particular, it says that the D-monoid reduct $\langle R, \cdot, ^\circ, 1\rangle$ of an enriched reduced Rickart ring is a strong semilattice of right-cancellative D-monoids.

Corollary 8.2. (a) *If R is an enriched reduced Rickart ring, then* $\langle R, \cdot, ^\circ, 1\rangle = \mathfrak{S}^\circ_1(\mathrm{sys}^\circ_1 R)$.

(b) *Let $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ be an inverse system of pairwise disjoint right-cancellative D-semigroups with identities over some lower semilattice, and let $\mathcal{A}^\circ_1$ denote the family of D-monoids obtained from the family of D-semigroups $\mathcal{A}^\circ$ by including the identities into the signatures. Suppose $\mathfrak{S}^\circ\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \langle R, \cdot, ^\circ\rangle$ for some enriched reduced Rickart ring R. Then $\langle \mathcal{A}^\circ_1, \mathcal{H}\rangle = \mathrm{sys}^\circ_1 R$.*

Proof. (a) If R is not only an enriched m-domain ring, but even an enriched reduced Rickart ring, then its lattice of idempotents E has the greatest element 1. Hence, we can apply the second part of Definition 7.1 to obtain a strong semilattice of D-monoids $\mathfrak{S}^\circ_1\langle \mathcal{M}^\circ_1, \Phi\rangle$. Let $\langle R, \bullet, ^\bullet, \mathbf{1}\rangle = \mathfrak{S}^\circ_1\langle \mathcal{M}^\circ_1, \Phi\rangle$.

It is clear from Theorems 4.1(a) and 8.1(a) that the operation $\bullet$ is the ring multiplication and that $^\bullet$ is the operation from Lemma 2.4.

By Definition 7.1, $\mathbf{1}$ is the identity element with respect to the operation $\cdot_\top$ of the D-monoid $\langle M_\top, \cdot_\top, ^\circ_\top, \top\rangle$ corresponding to the top element $\top$ of the lattice E. Since $\top = 1$, obviously $\mathbf{1} = 1$.

(b) By Theorem 8.1(b), we already know that $\langle \mathcal{A}^\circ, \mathcal{H}\rangle = \mathrm{sys}^\circ R$. So $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ is an inverse system over a semilattice which actually is (isomorphic to) the lattice of idempotents E, and $\mathcal{A}^\circ = \mathcal{M}^\circ$. Therefore, the constant $\mathbf{1}$ obtained from $\langle \mathcal{A}^\circ, \mathcal{H}\rangle$ by Equation (16) is the identity of the m-domain M_1, that is, $\mathbf{1} = 1$. Hence, $\langle \mathcal{A}^\circ_1, \mathcal{H}\rangle = \mathrm{sys}^\circ_1 R$. □

Acknowledgement. I thank my supervisor Professor Jānis Cīrulis for his helpful and detailed advice and patience. I would also like to thank the anonymous referees for their time and work spent on the peer review of this paper.

Disclosure statement. The author reports there are no competing interests to declare.

References

[1] G. F. Birkenmeier, J. K. Park and S. T. Rizvi, Extensions of Rings and Modules, Springer, New York, 2013.

[2] J. Cīrulis, *Extending the star order to Rickart rings*, Linear Multilinear Algebra, 64(8) (2015), 1498-1508.

[3] J. Cīrulis and I. Cremer, *Notes on reduced Rickart rings, I. Representation and equational axiomatizations*, Beitr. Algebra Geom., 59(2) (2018), 375-389.

[4] J. Cīrulis and I. Cremer, *Correction to Notes on reduced Rickart rings, I. Representation and equational axiomatizations*, Beitr. Algebra Geom., 61(3) (2020), 579-580.

[5] W. H. Cornish, *The variety of commutative Rickart rings*, Nanta Math., 5(2) (1972), 43-51.

[6] W. H. Cornish, *Boolean orthogonalities and minimal prime ideals*, Comm. Algebra, 3(10) (1975), 859-900.

[7] J. Fountain, *Right PP monoids with central idempotents*, Semigroup Forum, 13(3) (1976/77), 229-237.

[8] J. A. Fraser and W. K. Nicholson, *Reduced PP-rings*, Math. Japon., 34(5) (1989), 715-725.

[9] J. M. Howie, Fundamentals of Semigroup Theory, London Math. Soc. Monogr. (N.S.), 12 Oxford Sci. Publ., 1995.

[10] M. F. Janowitz, *A note on Rickart rings and semi-Boolean algebras*, Algebra Universalis, 6(1) (1976), 9-12.

[11] C. J. Penning, *Minimal duplicator rings*, Nederl. Akad. Wetensch. Proc. Ser. A 66 Indag. Math., 25 (1963), 295-312.

[12] J. Płonka, *On a method of construction of abstract algebras*, Fund. Math., 61 (1967), 183-189.

[13] J. J. Rotman, Advanced Modern Algebra, Grad. Stud. Math., 114, 2010.

[14] T. P. Speed, *A note on commutative Baer rings*, J. Austral. Math. Soc., 14(3) (1972), 257-263.

[15] T. Stokes, *Domain and range operations in semigroups and rings*, Comm. Algebra, 43(9) (2015), 3979-4007.

[16] N. V. Subrahmanyam, *Structure theory for a generalised Boolean ring*, Math. Ann., 141 (1960), 297-310.

[17] I. Sussman, *Ideal structure and semigroup domain decomposition of associate rings*, Math. Ann., 140(2) (1960), 87-93.

Insa Cremer
Riga Technical University
Institute of Applied Mathematics
Tel.: +371-25444156
e-mail: insa.kremere@gmail.com