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Research Article

On the Generalized Taxicab Apollonian Sets

Süheyla EKMEKÇİ [1]

Abstract
In this work, the concept of Apollonian set is explored in the framework of the generalized taxicab plane and named as the generalized taxicab Apollonian sets. It is determined that these sets do not conform to the properties of generalized taxicab circles; rather, the closed simple rectilinear figures are composed of line segments. By examining various configurations based on the positions of given points, the generalized taxicab Apollonian sets are systematically classified and characterized.

Keywords and 2020 Mathematics Subject Classification
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MSC: 51B20, 51F99, 51K99

[1] *Department of Mathematics and Computer Science, Faculty of Science, Eskişehir Osmangazi University, Eskişehir, Türkiye*
[1] ✉sekmekci@ogu.edu.tr
Corresponding author: Süheyla EKMEKÇİ
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1. Introduction

The distance between two points can be measured in various ways according to different distance functions. Some of the most popular distance functions comprise the Euclidean distance, the maximum distance, the taxicab distance, and the alpha distance. Euclidean distance is a measure of the shortest distance between two points. The taxicab distance, maximum distance, alpha distance are non-Euclidean distances, and the analytical planes equipped with these distance functions are the non-Euclidean planes. These non-Euclidean geometries have been studied in different aspects by many researchers [1–9]. Taxicab distance measures the sum of Euclidean lengths of line segments parallel to the coordinate axes between two points. Wallen redefined the taxicab distance to eliminate potentially misleading symmetry [10]. The generalized taxicab distance between two points is the sum of the constant multiples of the Euclidean lengths of the horizontal and vertical line segments parallel to the coordinate axes.

This is the reason why the weighted taxicab distance is also known as the generalized taxicab distance, providing an alternative approach to computing distances in non-Euclidean geometry. In recent years, metric geometry based on the generalized taxicab distance has been studied and developed [11–14].

The Apollonian set comprises points in which the ratio of their distances from two points is a constant k. The Apollonian set in the Euclidean plane takes the form of a circle, except in the instances where k equals 0 or 1. The Apollonian sets are regarded as the degenerate cases for $k = 0$ or $k = 1$. The Apollonian set can be defined in various metric spaces. The Apollonian sets are studied and characterized in taxicab and maximum planes [15, 16]. And it is obtained that the Apollonian sets in taxicab and maximum planes are not a taxicab and maximum circles, respectively.

In this work, the Apollonian sets are examined in the generalized taxicab plane and named as the generalized taxicab Apollonian sets. It is determined that the generalized taxicab Apollonian set is not a generalized taxicab circle, but the simple closed rectilinear figure. Also, the generalized taxicab Apollonian sets are classified and characterized. In the sequel, some concepts used throughout this work are mentioned.

2. Preliminaries

Definition 1. *The generalized taxicab distance between points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in the analytical plane is*

$$d_G(A,B) = a|x_2 - x_1| + b|y_2 - y_1|,$$

where the real numbers $a, b > 0$ [10].

From the definition, it is seen that the generalized taxicab distance between the points A_1 and A_2 is equal to the sum of the positive multiples a and b of the Euclidean lengths of the sides parallel to the coordinates axes in the right triangle with the hypotenuse A_1A_2. Indeed, d_G is a family of distance functions depending on the positive numbers a and b. In the special case of $a = b = 1$, d_G is the taxicab distance. Throughout this paper, it will be assumed that a and b are constant values given at the beginning, unless otherwise stated. The generalized taxicab plane is the analytical plane equipped with the generalized taxicab distance and symbolized by $\mathbb{R}^2_G$. It is almost the same as the Euclidean plane except the distance function.

The classification of lines in the generalized taxicab plane, similar to [6], is as follows:

Definition 2. *Let m be the slope of the line l in the generalized taxicab plane. The line l is called the steep line, the gradual line and the separator line in the cases of $|m| > \frac{a}{b}$, $|m| < \frac{a}{b}$ and $|m| = \frac{a}{b}$, respectively. In special cases that the line l is parallel to the x-axis or y-axis, l is named as the horizontal line or the vertical line, respectively [11].*

Every Euclidean translation preserves the generalized taxicab distance. So, it is an isometry in $\mathbb{R}^2_G$. Reflections in the coordinate axes and the separator lines through the origin are isometries in the generalized taxicab plane. The set of axes of isometric reflections is

$$\{x = 0, y = 0, y = \frac{a}{b}x, y = -\frac{a}{b}x\}.$$

Also, the set of isometric reflections in matrix form is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{b}{a} \\ \frac{a}{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{b}{a} \\ -\frac{a}{b} & 0 \end{bmatrix} \right\}.$$

And matrices in the following set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{b}{a} \\ \frac{a}{b} & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{b}{a} \\ -\frac{a}{b} & 0 \end{bmatrix} \right\}$$

represent rotations about the origin that preserve the generalized taxicab distance [11–13].

Definition 3. *The minimum distance set (the shortest distance set) of the points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in the generalized taxicab plane is the set*

$$\{X \mid d_G(A_1, X) + d_G(X, A_2) = d_G(A_1, A_2)\}.$$

This set is a rectangular region determined by the vertical and horizontal lines through the points A_1 and A_2. In the case that the points A_1 and A_2 are on the horizontal or vertical line, it is the line segment A_1A_2 (Figure 1).

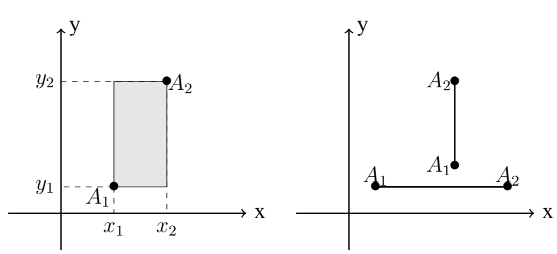

Fig. 1. The minimum distance set

Definition 4. *In the generalized taxicab plane, the generalized taxicab midset of the points A_1 and A_2 is defined by the set*

$$\{X \mid d_G(A_1, X) = d_G(X, A_2)\}.$$

It is well known that the Euclidean midset is a line passing through the midpoint of the points A_1 and A_2, and perpendicular to the line A_1A_2. Besides, the generalized taxicab midset has different shapes depending on the points A_1 and A_2 as follows:

i. If the points A_1 and A_2 are on a horizontal or vertical line, the generalized taxicab midset is the line passing through the midpoint of them and perpendicular to the line A_1A_2 as in the Euclidean case (Figure 2 (b)).

ii. If the points A_1 and A_2 are on a separator line, the generalized taxicab midset consists of two regions and a line segment connecting these regions. The line segment is the intersection of the minimum distance set of given points and the other separator line passing through the midpoint of them. And, regions are formed by the horizontal and vertical lines passing through the points A_1 and A_2 such that the intersection of regions and the minimum distance set is the endpoints of the line segment (Figure 2 (a)).

iii. If the points A_1 and A_2 are on a gradual line or a steep line, the generalized taxicab midset consists of two rays and a line segment connecting these rays. The line segment is the intersection of the minimum distance set of the given points and the separator line passing through the midpoint of them and having the opposite sign of the slope of line A_1A_2. And rays are parallel to the coordinate axis and have opposite directions such that the initial points of the rays are the endpoints of the line segment (Figure 2 (c), Figure 2 (d)).

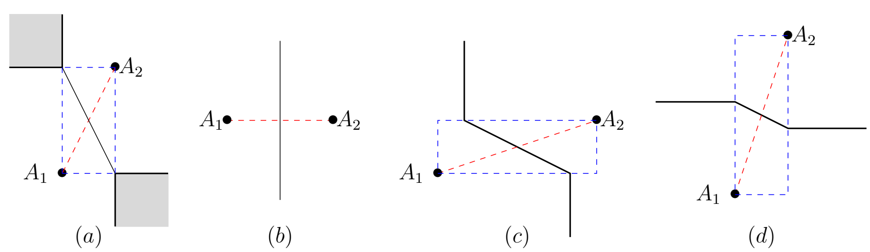

Fig. 2. (a) (b) (c) (d) The generalized taxicab midsets

3. The generalized taxicab Apollonian sets

In this section, it is explored that the Apollonian set defined by the ratio of generalized taxicab distances to two fixed points is constant. To put it more clearly, two fixed points A_1 and A_2 in the generalized taxicab plane and $k \in [0,\infty)$, the Apollonian set is

$$\mathscr{A}(A_1,A_2;k)_G = \left\{ X \in \mathbb{R}^2_G \mid \frac{d_G(A_1,X)}{d_G(A_2,X)} = k \right\}.$$

In Euclidean plane, the Apollonian set is a circle except for $k \in \{0,1,\infty\}$ and in the case of $k = 1$, it is the midset of the given points such that it is the perpendicular bisector of the line passing through the given points. It is immediately seen from the definition of the generalized taxicab Apollonian set that the followings hold:

i. When the constant k is equal to 1, the generalized taxicab Apollonian set is the generalized taxicab midset of two fixed points.

ii. When k approaches zero, the points in the generalized taxicab Apollonian set get closer to the point A_1.

iii. When k approaches ∞, the points in the generalized taxicab Apollonian set get closer to the point A_2.

Note that the generalized taxicab Apollonian set is never a generalized taxicab circle, unlike the Euclidean case.

When the positions of the points and the constant k are altered, the generalized taxicab Apollonian sets have interesting shapes. In this study, the conditions related to the constant k are used to classify the generalized taxicab Apollonian sets.

The following lemma states that in the definition of the Apollonian set, it is sufficient to focus only on the value k in the interval $(1,\infty)$.

Theorem 5. *Let any two distinct points be A_1 and A_2 in the generalized taxicab plane and $k > 0$. Then $\mathscr{A}(A_1,A_2;k)_G$ is equal to $\mathscr{A}(A_2,A_1;\frac{1}{k})_G$.*

In special cases, the Apollonian sets $\mathscr{A}(A_1,A_2;0)_G$ and $\mathscr{A}(A_1,A_2;\infty)_G$ are equal to $\mathscr{A}(A_2,A_1;\infty)_G$ and $\mathscr{A}(A_2,A_1;0)_G$, respectively.

Theorem 6. *Let A_1 and A_2 be any two points in the generalized taxicab plane and $k > 0$. If the map φ is an isometry in the generalized taxicab plane, then*

$$\varphi(\mathscr{A}(A_1,A_2;k)_G) = \mathscr{A}(\varphi(A_1),\varphi(A_2);k)_G.$$

Since the generalized taxicab isometries preserve the properties of Apollonian sets, throughout this study, choosing one of two points at the origin and the other in the first quadrant of the coordinate plane will simplify the analysis and will not affect the generality.

Theorem 7. *Let A_1 and A_2 be any two points in the generalized taxicab plane and $k > 0$. If φ is the rotation by π-angle at the midpoint M of the points A_1 and A_2, then*

$$\varphi(\mathscr{A}(A_1,A_2;k)_G) = \mathscr{A}(A_1,A_2;\frac{1}{k})_G.$$

The proofs of the above three theorems are similar to the proofs of the propositions and lemmas given in [14–16].

Now, the generalized Apollonian set, defined by two points and a positive constant, is examined based on the position of the line passing through these points and is presented as follows.

The following theorem outlines the characteristic properties of the generalized taxicab Apollonian set in the case where two fixed points lie on the same axis:

Theorem 8. *Let A_1 and A_2 be two distinct points on the same coordinate axis in the generalized taxicab plane and $k \in (1,\infty)$. The generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the following properties :*

i. It consists of four line segments. The slopes of the two line segments are $\pm\frac{a(1+k)}{b(1-k)}$ and the other two line segments are on separator lines.

ii. It has four vertices such that two of them lie on the coordinate axis passing through the point A_2 and the other two are on the vertical or horizontal line passing through the point A_2.

iii. It is symmetric about the line A_1A_2.

iv. When the opposite sides are extended to the lines, the intersection points of them are on the line passing through the point A_1 and perpendicular to the line A_1A_2.

v. The point A_2 divides the diagonals internally in the ratios $\frac{k-1}{k+1}$ and 1.

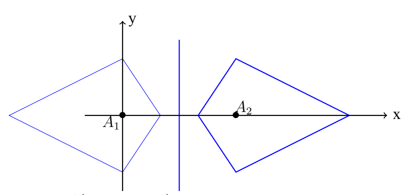

Fig. 3. $\mathscr{A}(A_1,A_2;k)_G$ (on the right) and $\mathscr{A}(A_2,A_1;k)_G$ (on the left) when A_1 and A_2 on the same coordinate axis

Proof. Suppose that the points A_1 and A_2 are on the x-axis. Let the coordinates of the points A_1 and A_2 be $(0,0)$ and $(x_0,0)$ where $x_0 > 0$. The generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the equality

$$\frac{a|x|+b|y|}{a|x-x_0|+b|y|} = k, \tag{1}$$

where $k > 1$.

Where the value x is negative, there is no point that satisfies the equality (1).

If $0 \leq x < x_0$, the equality (1) means that the line segment with the equation $a(1+k)x + b(-1+k)y = kax_0$ joining the points $V_1 = (\frac{k}{k+1}x_0, 0)$ and $V_4 = (x_0, \frac{ax_0}{b(1-k)})$ when $y < 0$, and the line segment with the equation $a(1+k)x + b(1-k)y = kax_0$ joining the points $V_1 = (\frac{k}{k+1}x_0, 0)$ and $V_2 = (x_0, \frac{ax_0}{b(k-1)})$ when $y \geq 0$. In the case where $x \geq x_0$, the equality (1) implies that the line segment with the equation $a(1-k)x + b(-1+k)y = -kax_0$ joining the points $V_3 = (\frac{k}{k-1}x_0, 0)$ and $V_4 = (x_0, \frac{ax_0}{b(1-k)})$ when $y < 0$, and the line segment with the equation $a(1-k)x + b(1-k)y = -kax_0$ joining the points $V_3 = (\frac{k}{k-1}x_0, 0)$ and

$V_2 = (x_0, \frac{ax_0}{b(k-1)})$ when $y \geq 0$. Thus, the generalized taxicab Apollonian set $\mathscr{A}(A_1, A_2; k)_G$, corresponding to equation (1) consists of the line segments V_iV_j $i = 1, 3,$ $j = 2,$ 4 (Figure 3).

The intersection points of the opposite sides V_1V_2 and V_3V_4 ; V_1V_4 and V_2V_3 in $\mathscr{A}(A_1, A_2; k)_G$ are the points $(0, \pm\frac{akx_0}{b(1-k)})$. It is observed that these intersection points and A_1 are collinear (Figure 4).

The reflection Ω in the line A_1A_2 leaves the vertices V_1 and V_3 fixed, while mapping the vertices V_2 and V_4 onto each other. Also, the sides V_1V_2 and V_2V_3 are mapped to the sides V_1V_4 and V_3V_4 under Ω. Thus, $\mathscr{A}(A_1, A_2; k)_G$ is symmetric about the line A_1A_2.

It is easily to see that the point A_2 divides the diagonal V_1V_3 in the ratio $\frac{|V_1A_2|}{|A_2V_3|} = \frac{d_G(V_1,A_2)}{d_G(A_2,V_3)} = \frac{k-1}{k+1}$, and the diagonal V_2V_4 in the ratio $\frac{|V_2A_2|}{|A_2V_4|} = \frac{d_G(V_2,A_2)}{d_G(A_2,V_4)} = 1$.

In the case that the points A_1 and A_2 are on the y-axis, it can immediately be seen that $\mathscr{A}(A_1, A_2; k)_G$ has the same properties by using the reflection in the separator line $y = \frac{a}{b}x$ and Theorem 6. ■

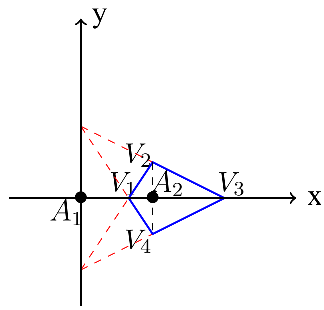

Fig. 4. $\mathscr{A}(A_1, A_2; k)_G$ when A_1 and A_2 on the same coordinate axis

It is known that translations preserve the properties of the generalized taxicab Apollonian sets according to Theorem 6. In the case where two fixed points lie on a line parallel to the coordinate axis, the following results can be immediately obtained by applying a suitable translation to the points in Theorem 7.

Corollary 9. *Let A_1 and A_2 be two distinct points on the same horizontal or vertical line in the generalized taxicab plane and $k \in (1, \infty)$. The generalized taxicab Apollonian set $\mathscr{A}(A_1, A_2; k)_G$ has the following properties:*

i. It consists of four line segments. The slopes of the two line segments are $\pm\frac{a(1+k)}{b(1-k)}$, and the other two line segments are on separator lines.

ii. It has four vertices such that two of them lie on the line A_1A_2 and the other two are on the line passing through the point A_2 and perpendicular to the line A_1A_2.

iii. It is symmetric about the line A_1A_2.

iv. The intersection points of the opposite sides are on the line passing through the point A_1 and perpendicular to the line A_1A_2.

In the following theorem, the generalized taxicab Apollonian set is examined, and its properties are presented when two fixed points lie on the same separator line.

Theorem 10. *Let A_1 and A_2 be two distinct points on the same separator line in the generalized taxicab plane and $k \in (1, \infty)$. The generalized taxicab Apollonian set $\mathscr{A}(A_1, A_2; k)_G$ has the following properties:*

i. It consists of four line segments. The slopes of the two line segments are $m\frac{k+1}{k-1}$ and $m\frac{k-1}{k+1}$, and the other two line segments are parallel to the separator line with the slope $-m$ where m is the slope of the line A_1A_2.

ii. Its vertices are on the horizontal and vertical lines passing through the point A_2.

iii. The completions of the non-parallel opposite sides intersect at the point A_1.

iv. $\mathscr{A}(A_1, A_2; k)_G$ is symmetric about the line A_1A_2.

v. The point A_2 divides the diagonals internally in the ratio $\frac{k-1}{k+1}$.

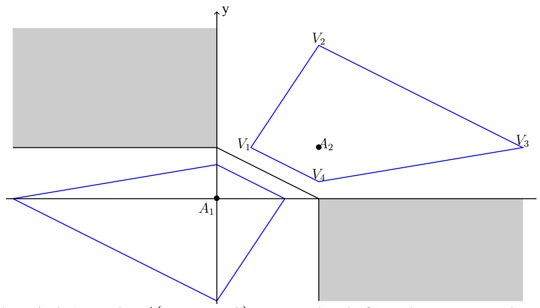

Fig. 5. $\mathscr{A}(A_1,A_2;k)_G$ (on the right) and $\mathscr{A}(A_2,A_1;k)_G$ (on the left) when A_1 and A_2 on the same separator line

Proof. Suppose that the points A_1 and A_2 are on the separator line $y=\frac{a}{b}x$. Let the abscissas of the points A_1 and A_2 be 0 and x_0 where $x_0>0$. The generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the equality

$$\frac{a|x|+b|y|}{a|x-x_0|+b\left|y-\frac{a}{b}x_0\right|}=k, \tag{2}$$

where $k>1$. Focusing on all possible cases in the equality (2) and solving for the absolute values appropriately, $\mathscr{A}(A_1,A_2;k)_G$ is determined.

In the regions where the value x is negative, the equality (2) reduces to an equation that has no solution.

In the regions where the value x is equal to or greater than zero and is less than x_0, (2) states two line segments such that one of them is the line segment with the equation

$$ax+by=\frac{2k}{k+1}ax_0 \tag{3}$$

joining $V_1=(\frac{k-1}{k+1}x_0,\frac{a}{b}x_0)$ and $V_4=(x_0,\frac{a(k-1)}{b(k+1)}x_0)$ when $0\le y<\frac{a}{b}x_0$ and the other is the line segment with the equation

$$a(k+1)x+b(1-k)y=0 \tag{4}$$

joining V_1 and $V_2=(x_0,\frac{a(k+1)}{b(k-1)}x_0)$ when $y\ge\frac{a}{b}x_0$. In the regions where the value x is equal to or greater than x_0, the points satisfying the equality (2) are on the line segment with the equation

$$a(1-k)x+b(1+k)y=0 \tag{5}$$

joining $V_3=(\frac{k+1}{k-1}x_0,\frac{a}{b}x_0)$ and V_4 when $0\le y<\frac{a}{b}x_0$ and on the line segment with the equation

$$ax+by=\frac{2k}{k-1}ax_0 \tag{6}$$

joining V_2 and V_3 when $y\ge\frac{a}{b}x_0$ (Figure 5).

It is seen that all the vertices V_i are on the horizontal and vertical lines through A_2. Also, the sides V_1V_4 and V_2V_3 are parallel and on the separator lines with the opposite sign of the slope of line A_1A_2. The completions of two non-parallel sides V_1V_2 and V_3V_4 intersect at the point A_1. While the reflection in the line A_1A_2 maps the points V_1, V_2 and the side V_1V_2 to the points V_4, V_3 and the side V_3V_4, it leaves the sides V_1V_4 and V_2V_3 fixed. Thus $\mathscr{A}(A_1,A_2;k)_G$ is symmetric about the separator line A_1A_2. Since $\frac{d_G(V_1,A_2)}{d_G(A_2,V_3)}=\frac{k-1}{k+1}=\frac{d_G(V_2,A_2)}{d_G(A_2,V_4)}$, the point A_2 divides the diagonals V_1V_3 and V_2V_4 in the ratio $\frac{k-1}{k+1}$.

When the points A_1 and A_2 are on the separator line $y=-\frac{a}{b}x$, it is readily achieved that the set $\mathscr{A}(A_1,A_2;k)_G$ possesses the analogous properties by using the reflection in x-axis. ∎

The following Theorem 11 and Theorem 13 describe the properties of the generalized taxicab Apollonian sets when two given points are on a gradual line. These theorems highlight conditions based on the relationship between the value k and the slope of the line passing through two given points and reflect how varying conditions affect the structure and properties of the generalized taxicab Apollonian set.

Theorem 11. *Let A_1 and A_2 be two distinct points on a gradual line in the generalized taxicab plane and $k\in(1,\infty)$. If $k<\frac{a}{b|m|}$, where m is the slope of the line segment A_1A_2, then the generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the following properties:*

i. It consists of six line segments. The slopes of three line segments are $\pm sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$, *and the other three are parallel to the separator lines. It has two parallel sides.*

ii. Four of the six vertices are on the horizontal and vertical lines passing through the point A_2, *and the other two are on the horizontal line through the point* A_1.

iii. The completions of the sides with the slopes $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$, *and the separator line passing through the point* A_2 *are concurrent.*

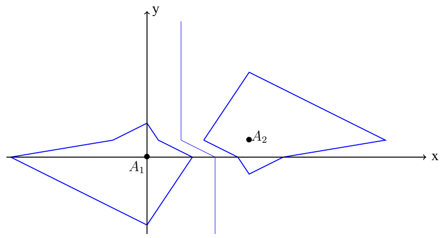

Fig. 6. The Apollonian sets when A_1 and A_2 on the same gradual line

Proof. Suppose that the coordinates of the points A_1 and A_2 are $(0,0)$ and (x_0,y_0), where $x_0,\ y_0 > 0$. Since the line A_1A_2 is a gradual line, $y_0 < \frac{a}{b}x_0$. The generalized taxicab Apollonian set has the equality

$$\frac{a|x|+b|y|}{a|x-x_0|+b|y-y_0|} = k, \tag{7}$$

where $k > 1$. By concentrating on all the cases in equality (7) and suitably analyzing the absolute values, one gets $\mathscr{A}(A_1,A_2;k)_G$ as follows:
Consider the regions where $0 \le x < x_0$. When $y < 0$, the equality (7) becomes the equation

$$a(1+k)x + sgn(m)b(-1+k)y = k(ax_0 + sgn(m)by_0). \tag{8}$$

Since $k < \frac{a}{bm}$, the intersection of the line in equation (8) and the region is the line segment joining the points $V_5 = (x_0, \frac{-ax_0+sgn(m)kby_0}{sgn(m)b(k-1)})$ and $V_6 = (\frac{k(ax_0+sgn(m)by_0)}{a(k+1)}, 0)$ where $\frac{-ax_0+sgn(m)kby_0}{sgn(m)b(k-1)} < 0$ and $0 < \frac{k(ax_0+sgn(m)by_0)}{a(k+1)} < x_0$. When $0 \le y < y_0$, the line with the equation

$$ax + sgn(m)by = \frac{k}{k+1}(ax_0 + sgn(m)by_0) \tag{9}$$

meets the region along the line segment joining the points $V_1 = (\frac{kax_0+sgn(-m)by_0}{a(k+1)}, y_0)$ and V_6 where $0 < \frac{kax_0+sgn(-m)by_0}{a(k+1)} < x_0$. For $y \ge y_0$, the side of the generalized taxicab Apollonian set in this region is formed by the line segment with the equation

$$a(1+k)x + sgn(-m)b(k-1)y = k(ax_0 + sgn(-m)by_0) \tag{10}$$

joining points V_1 and $V_2 = (x_0, \frac{ax_0+sgn(m)kby_0}{sgn(m)b(k-1)})$ where $\frac{ax_0+sgn(m)kby_0}{sgn(m)b(k-1)} > y_0$. Take into account the regions where $x \ge x_0$. When y is negative, the line with the equation

$$a(1-k)x + sgn(m)b(-1+k)y = k(-ax_0 + sgn(m)by_0) \tag{11}$$

intersects the region along the line segment whose endpoints are points $V_4 = (\frac{k(ax_0+sgn(-m)by_0)}{a(k-1)}, 0)$ and V_5. Since $k < \frac{a}{bm}$, $\frac{k(ax_0+sgn(-m)by_0)}{a(k-1)} > x_0$ and $\frac{-ax_0+sgn(m)kby_0}{sgn(m)b(k-1)} < 0$. When $0 \le y < y_0$, the line with the equation

$$a(k-1)x + sgn(-m)b(k+1)y = k(ax_0 + sgn(-m)by_0) \tag{12}$$

meets the designated region along the line segment whose endpoints are $V_3 = (\frac{kax_0+sgn(m)by_0}{a(k-1)}, y_0)$ and V_4. Since $ax_0 + by_0 > 0$, it is clear that $\frac{kax_0+sgn(-m)by_0}{a(k-1)} > x_0$. Considering the case of $y \ge y_0$, the line segment with the equation

$$ax + sgn(m)by = \frac{k}{k-1}(ax_0 + sgn(m)by_0) \tag{13}$$

joining the points V_2 and V_3 is the sixth side of $\mathscr{A}(A_1,A_2;k)_G$.

It is seen immediately that the completions of the sides V_1V_2 and V_3V_4 and the separator line passing through the point A_2 intersect at the point $(\frac{ax_0+sgn(-m)by_0}{2a},\frac{sgn(-m)ax_0+by_0}{2b})$. So, these lines are concurrent. Also, it is obvious from equations (9) and (13) that the sides V_1V_6 and V_2V_3 are parallel (Figure 6). ■

Considering the points in the hypothesis of Theorem 6, the intersection points of the separator lines passing through the points $A_1=(0,0)$ and $A_2=(x_0,y_0)$ are $A_{12}=(\frac{ax_0+by_0}{2a},\frac{ax_0+by_0}{2b})$ and $A_{21}=(\frac{ax_0-by_0}{2a},\frac{-ax_0+by_0}{2b})$.

The points A_{21} and A_2 are on the separator line. The set $\mathscr{A}(A_{21},A_2;k)_G$ is the union of the following line segments: The line segment is defined by

$$ax+sgn(m)by=\frac{k}{k+1}(ax_0+sgn(m)by_0),\frac{kax_0+sgn(-m)by_0}{a(k+1)}\leq x\leq x_0 \tag{14}$$

joining the vertices $K_1=(\frac{kax_0+sgn(-m)by_0}{a(k+1)},y_0)$ and $K_4=(x_0,\frac{-ax_0+sgn(m)kby_0}{sgn(m)b(k-1)})$, the line segment

$$a(1+k)x+sgn(-m)b(k-1)y=k(ax_0+sgn(-m)by_0),\ \frac{kax_0+sgn(-m)by_0}{a(k+1)}\leq x\leq x_0 \tag{15}$$

joining the vertices K_1 and $K_2=(x_0,\frac{ax_0+sgn(m)kby_0}{sgn(m)b(k-1)})$, the line segment

$$ax+sgn(m)by=\frac{k}{k-1}(ax_0+sgn(m)by_0),\ x_0\leq x\leq\frac{kax_0+sgn(m)by_0}{a(k-1)} \tag{16}$$

joining the points K_2 and $K_3=(\frac{kax_0+sgn(m)by_0}{a(k-1)},y_0)$ and the line segment

$$a(k-1)x+sgn(-m)b(k+1)y=k(ax_0+sgn(-m)by_0),\ x_0\leq x\leq\frac{kax_0+sgn(m)by_0}{a(k-1)} \tag{17}$$

joining the points K_3 and K_4 where m is the slope of the line segment A_1A_2.

It is immediately evident that the vertices V_1, V_2 and V_3 of $\mathscr{A}(A_1,A_2;k)_G$ in Theorem 6 coincide with the vertices K_1, K_2 and K_3 of $\mathscr{A}(A_{21},A_2;k)_G$. Therefore, the sides $K_1K_2$and K_2K_3 of $\mathscr{A}(A_{21},A_2;k)_G$ are also the sides of $\mathscr{A}(A_1,A_2;k)_G$.

If the sides V_3V_4 of $\mathscr{A}(A_1,A_2;k)_G$ and K_3K_4 of $\mathscr{A}(A_{21},A_2;k)_G$ are extended to line, they coincide. Since $x_0<\frac{k(ax_0+sgn(-m)by_0)}{a(k-1)}$, the side K_3K_4 includes the side V_3V_4. Any point X on the side K_3K_4 that is not on the line segment V_3V_4 is in the region bounded by $\mathscr{A}(A_1,A_2;k)_G$, since $\frac{d_G(A_1,X)}{d_G(A_2,X)}\geq k$. Similarly, the side V_1V_6 is included by the side K_1K_4. The points on the side K_1K_4 that are not on the side V_1V_6 are in the region enclosed by the set $\mathscr{A}(A_1,A_2;k)_G$. The set

$$\left\{(x,y):\frac{a\left|x-\frac{ax_0-by_0}{2a}\right|+b\left|y-\frac{-ax_0+by_0}{2b}\right|}{a|x-x_0|+b|y-y_0|}\geq k\right\}$$

defines the region bounded by $\mathscr{A}(A_{21},A_2;k)_G$ and it is included by the region bounded by $\mathscr{A}(A_1,A_2;k)_G$ (Figure 7).

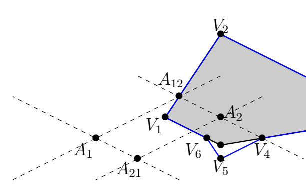

Fig. 7. The region bounded by $\mathscr{A}(A_{21},A_2;k)_G$ in $\mathscr{A}(A_1,A_2;k)_G$

For the points A_{12} and A_2 on the same separator line, the set $\mathscr{A}(A_{12},A_2;k)_G$, consists of the union of the line segments in the following. These are the line segment

$$a(1+k)x+sgn(m)b(-1+k)y=k(ax_0+sgn(m)by_0),\ \frac{kax_0+sgn(m)by_0}{a(k+1)}\leq x\leq x_0 \tag{18}$$

joining the vertices $K_1' = (\frac{kax_0+sgn(-m)by_0}{a(k+1)}, y_0)$ and $K_4' = (x_0, \frac{sgn(-m)ax_0+kby_0}{b(k-1)})$, the line segment

$$a(1+k)x + sgn(-m)b(1+k)y = k(ax_0 + sgn(-m)by_0),\ \frac{kax_0+sgn(m)by_0}{a(k+1)} \leq x \leq x_0 \tag{19}$$

joining the vertices K_1' and $K_2' = (x_0, \frac{sgn(m)ax_0+kby_0}{b(k+1)})$, the line segment

$$a(-1+k)x + sgn(m)b(1+k)y = k(ax_0 + sgn(m)by_0),\ x_0 \leq x \leq \frac{kax_0+sgn(-m)by_0}{a(k-1)}$$

joining the vertices $K_2' =$ and $K_3' = (\frac{kax_0+sgn(-m)by_0}{a(k-1)}, y_0)$ and the line segment

$$a(1-k)x + sgn(m)b(-1+k)y = k(-ax_0 + sgn(m)by_0),\ x_0 \leq x \leq \frac{kax_0+sgn(-m)by_0}{a(k-1)}$$

joining the vertices K_3' and K_4'.

It is seen that the vertex V_5 of $\mathscr{A}(A_1,A_2;k)_G$ in Theorem 6 and the vertex K_4' of $\mathscr{A}(A_{12},A_2;k)_G$ coincide. If the sides V_4V_5 of $\mathscr{A}(A_1,A_2;k)_G$ and $K_3'K_4'$ of $\mathscr{A}(A_{12},A_2;k)_G$ are extended to line, they coincide. Since $\frac{k(ax_0+sgn(-m)by_0)}{a(k-1)} < \frac{kax_0+sgn(-m)by_0}{a(k-1)}$, the side $K_3'K_4'$ includes the side V_4V_5. For any point $X=(x,y)$ on the side $K_3'K_4'$ that is not on the line segment V_4V_5 satisfies in the following

$$a(1-k)x + sgn(m)b(-1+k)y = k(-ax_0 + sgn(m)by_0),\ \frac{k(ax_0+sgn(-m)by_0)}{a(k-1)} < x \leq \frac{kax_0+sgn(-m)by_0}{a(k-1)}\ 0 < y \leq y_0.$$

Since $\frac{d_G(A_1,X)}{d_G(A_2,X)} = k(\frac{ax+by}{ax-by}) > k$, the point X is in the region enclosed by $\mathscr{A}(A_1,A_2;k)_G$. Similarly, for any point X on the sides $K_1'K_4'$, $K_1'K_2'$ and $K_2'K_3'$, it is obtained that $\frac{d_G(A_1,X)}{d_G(A_2,X)} \geq k$. So every point in the set $\mathscr{A}(A_{12},A_2;k)_G$ is in the region defined by $\mathscr{A}(A_1,A_2;k)_G$. Thus, it is concluded that the region

$$\left\{(x,y) : \frac{a\left|x - \frac{ax_0+by_0}{2a}\right| + b\left|y - \frac{ax_0+by_0}{2b}\right|}{a|x-x_0| + b|y-y_0|} \geq k\right\},$$

bounded by $\mathscr{A}(A_{12},A_2;k)_G$, is contained in the region bounded by $\mathscr{A}(A_1,A_2;k)_G$ (Figure 8).

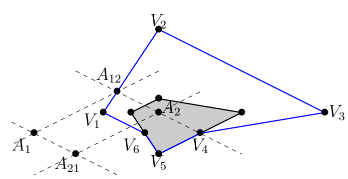

Fig. 8. The region bounded by $\mathscr{A}(A_{12},A_2;k)_G$ in $\mathscr{A}(A_1,A_2;k)_G$

Based on the results obtained, the following corollary can be stated:

Corollary 12. *Let A_1 and A_2 be two distinct points on a gradual line in the generalized taxicab plane and $k \in (1,\infty)$ with $k < \frac{a}{b|m|}$. If the intersection points of the separator lines passing through the points A_1, A_2 are A_{12} and A_{21}, then the region bounded by $\mathscr{A}(A_1,A_2;k)_G$ is the union of the regions bounded by $\mathscr{A}(A_{12},A_2;k)_G$ and $\mathscr{A}(A_{21},A_2;k)_G$ (Figure 9).*

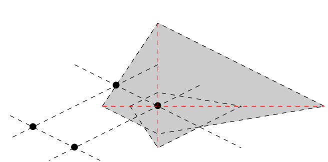

Fig. 9. The union of $\mathscr{A}(A_{21},A_2;k)_G$ and $\mathscr{A}(A_{12},A_2;k)_G$

Theorem 13. *Let A_1 and A_2 be two distinct points on a gradual line in the generalized taxicab plane and $k>1,\ k\in\mathbb{R}^+$. If $k\geq\frac{a}{b|m|}$, where m is the slope of the line segment A_1A_2, then the generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the following properties:*

i. *It consists of four line segments. The slopes of three line segments are $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$ and $sgn(-m)\frac{a}{b}$.*

ii. *Its vertices are on the horizontal and vertical lines passing through the point A_2.*

iii. *The completions of the sides with slopes $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$ and the separator line passing through the point A_2 are concurrent.*

iv. *Two sides are parallel.*

v. *$\mathscr{A}(A_1,A_2;k)_G$ is symmetric about the separator line with the slope $sgn(m)\frac{a}{b}$ passing through point A_2.*

vi. *The lengths of the diagonals and a side in $\mathscr{A}(A_1,A_2;k)_G$ are equal.*

vii. *The point A_2 divides the diagonals internally in the ratios $\frac{k-1}{k+1}$ and $\frac{k+1}{k-1}$.*

Proof. Suppose that the coordinates of the points A_1 and A_2 are $(0,0)$ and (x_0,y_0), where x_0 and $y_0>0$. Since the line A_1A_2 is a gradual line, $y_0<\frac{a}{b}x_0$. The generalized taxicab Apollonian set has the equality (7). It is useful to state that its vertices V_i $i=1,2,3$ are the same as the vertices obtained in the proof of Theorem 11. The equality (7) becomes an equation with no solution in the regions where the value of x is negative. Consider the region where $0\leq x<x_0$. In the part $y<0$ of this region, the line described by equation (8) does not meet the region, since $k\geq\frac{a}{bm}$. In the part $0\leq y<y_0$, the line with equation (9) meets the region along the line segment that forms a side of $\mathscr{A}(A_1,A_2;k)_G$ and its endpoints are the points V_1 and $V_4=(x_0,\frac{sgn(-m)ax_0+kby_0}{(k+1)b})$ where $0\leq\frac{kax_0+sgn(-m)by_0}{a(k+1)}<x_0$ and $0\leq\frac{sgn(-m)ax_0+kby_0}{(k+1)b}<y_0$. In the remaining part $y\geq y_0$ of the region, the line segment with the equation (10) joining points V_1 and V_2 is another side of the generalized taxicab Apollonian set where $\frac{sgn(m)ax_0+kby_0}{b(k-1)}>y_0$. Consider the regions where $x\geq x_0$. Since the constant k is greater than or equal to the value $\frac{a}{bm}$, the line with the equation (11) does not intersect the region where y is negative. The line in (12) meets the region where $0\leq y<y_0$ along the line segment whose endpoints are V_3 and V_4. It is clear that $\frac{kax_0+sgn(m)by_0}{a(k-1)}>x_0$. In the region where $y\geq y_0$, the line segment with equation (13) joining the points V_2 and V_3 is a side of $\mathscr{A}(A_1,A_2;k)_G$.

It is immediately apparent that the completions of the sides V_1V_2 and V_3V_4 and the separator line with the slope $sgn(m)\frac{a}{b}$ passing through the point A_2 intersect at the point $(\frac{ax_0+sgn(-m)by_0}{2a},\frac{sgn(-m)ax_0+by_0}{2b})$.

Let Ω be the reflection in the separator line with the slope $sgn(m)\frac{a}{b}$ passing through point A_2. While Ω maps the vertices V_1, V_2, and the side V_1V_2 to the vertices V_4, V_3 and the side V_3V_4, respectively, it leaves the sides V_1V_4 and V_2V_3 fixed. Thus, $\mathscr{A}(A_1,A_2;k)_G$ is symmetric about the line $y-y_0=sgn(m)\frac{a}{b}(x-x_0)$.

Also, it is obvious that the sides V_1V_4 and V_2V_3 are parallel. Besides, the generalized taxicab lengths of the diagonals V_1V_3 and V_2V_4 and the side V_1V_2 are equal, and this length is equal to $\frac{2k}{(k-1)(k+1)}|ax_0+sgn(m)by_0|$. Since $\frac{d_G(V_1,A_2)}{d_G(A_2,V_3)}=\frac{k-1}{k+1}$ and $\frac{d_G(V_2,A_2)}{d_G(A_2,V_4)}=\frac{k+1}{k-1}$, the point A_2 divides the diagonals V_1V_3 and V_2V_4 internally in the ratios $\frac{k-1}{k+1}$ and $\frac{k+1}{k-1}$, respectively (Figure 10). ∎

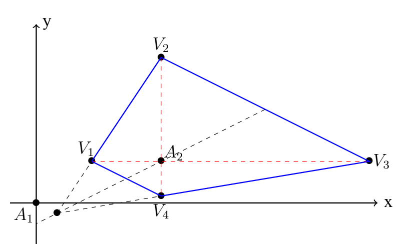

Fig. 10. The Apollonian set when A_1 and A_2 on the same gradual line for $k\geq\frac{a}{b|m|}$

It is known from Theorem 5 that if the positive value k is less than 1, then the roles of the points A_1 and A_2 interchange in the set $\mathscr{A}(A_1,A_2;k)_G$.

In the case that the points A_1 and A_2 are on the same gradual line and $k<1$, observations show that if $\frac{b}{a}|m|<k$, then the set $\mathscr{A}(A_1,A_2;k)_G$ has the properties expressed in Theorem 11, and otherwise the properties in Theorem 13 are valid.

Assume that points A_1 and A_2 are on a steep line.

In the generalized taxicab plane, the reflection Ω in the separator line with the slope $sgn(m)\frac{a}{b}$ passing through point A_1 transforms the steep line A_1A_2 to the gradual line A_1A_2' where m is the slope of the line A_1A_2. From Theorem 6, it is known that

$$\Omega(\mathscr{A}(A_1,A_2;k)_G) = \mathscr{A}(A_1,A_2';k)_G,$$

where $\Omega(A_2) = A_2'$. So, $\mathscr{A}(A_1,A_2;k)_G$ and $\mathscr{A}(A_1,A_2';k)_G$ are symmetric about the separator line with the slope $sgn(m)\frac{a}{b}$ passing through point A_1, and they share the same properties. Since the vertices, the sides and the properties of $\mathscr{A}(A_1,A_2;k)_G$ can be immediately obtained using this reflection, theorems regarding this case will be presented without proof as follows. However, it is obvious that the slope of the gradual line A_1A_2' is $\frac{a^2}{b^2m}$. While characterizing the set $\mathscr{A}(A_1,A_2;k)_G$, the cases where the value k is greater than or equal to $\frac{b}{a}|m|$ and less than $\frac{b}{a}|m|$ will be taken into consideration.

Theorem 14. *Let A_1 and A_2 be two distinct points on a steep line in the generalized taxicab plane and $k > 1,\ k \in \mathbb{R}^+$. If $k \geq \frac{b}{a}|m|$, where m is the slope of the line segment A_1A_2, then the generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the following properties:*

i. It consists of four line segments. The slopes of three line segments are $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$ and $sgn(-m)\frac{a}{b}$.

ii. Its vertices are on the horizontal and vertical lines passing through the point A_2.

iii. The completions of the sides with the slopes $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$ and the separator line with the slope $sgn(m)\frac{a}{b}$ passing through the point A_2 are concurrent.

iv. Two of its sides are parallel.

v. $\mathscr{A}(A_1,A_2;k)_G$ is symmetric about the separator line with the slope $sgn(m)\frac{a}{b}$ passing through point A_2.

vi. The lengths of the diagonals and a side in $\mathscr{A}(A_1,A_2;k)_G$ are equal.

In the case of $k \geq \frac{b}{a}|m|$, let the vertices of the sets $\mathscr{A}(A_1,A_2;k)_G$ and $\mathscr{A}(A_1,A_2';k)_G$ be denoted by V_i and V_i', respectively, where $\Omega(A_2) = A_2'$ and $\Omega(V_i) = V_i'$, $i = 1,2,3,4$. In the sets $\mathscr{A}(A_1,A_2;k)_G$ and $\mathscr{A}(A_1,A_2';k)_G$, the completions of sides with the slopes $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$, and the separator lines with the slope $sgn(m)\frac{a}{b}$ passing through the points A_2 and A_2', respectively, meet the points B and B'. Since these intersection points lie on the separator line with the slope $sgn(-m)\frac{a}{b}$ passing through the point A_1, the points A_1, B, and B' are collinear. Also, if the sides of $\mathscr{A}(A_1,A_2;k)_G$ and $\mathscr{A}(A_1,A_2';k)_G$ are completed to lines, the intersection points $V_iV_{i+1} \cap V_i'V_{i+1}'$ $i = 1,3$, and $V_jV_{j+2} \cap V_j'V_{j+2}'$ $j = 1,2$ lie on the separator line with the slope $sgn(m)\frac{a}{b}$ passing through the point A_1.

Theorem 15. *Let A_1 and A_2 be two distinct points on a steep line in the generalized taxicab plane, and $k > 1,\ k \in \mathbb{R}^+$. If $k < \frac{b}{a}|m|$, where m is the slope of the line segment A_1A_2, then the generalized taxicab Apollonian set $\mathscr{A}(A_1,A_2;k)_G$ has the following properties:*

i. It consists of six line segments. The slopes of three line segments are $\pm sgn(m)\frac{a(k-1)}{b(k+1)}$ and $sgn(m)\frac{a(k+1)}{b(k-1)}$, while the other three are parallel to the separator lines.

ii. Four of the six vertices are on the horizontal and vertical lines passing through the point A_2, and the other two are on the coordinate axis through the point A_1.

iii. The completions of the sides with the slopes $sgn(m)\frac{a(k+1)}{b(k-1)}$, $sgn(m)\frac{a(k-1)}{b(k+1)}$, and the separator line passing through the point A_2 are concurrent.

iv. Two of its sides are parallel.

In the case that the points A_1 and A_2 are on the same steep line and $k \in (0,1)$, it is seen that if $\frac{a}{b|m|} \geq k$, then the set $\mathscr{A}(A_1,A_2;k)_G$ has the properties mentioned in Theorem 14, and if $\frac{a}{b|m|} < k$, then it has the properties in Theorem 14.

4. Conclusions

In this study, the Apollonian sets in the generalized taxicab plane are obtained using d_G-distance, based on the ratio of distances to two fixed points being a positive constant real number. The properties and structures of the generalized taxicab Apollonian sets have been thoroughly investigated. In the analysis, it has been observed that these are not the generalized taxicab circles, but rather simple, closed and rectilinear figures composed of line segments. The line on which the fixed points lie plays an important role in determining the shape and properties of the generalized taxicab Apollonian set. When the fixed points are on a coordinate axis or a horizontal line or a vertical line or a separator line, there is no need for any condition on the value k while examining the structure of the generalized taxicab Apollonian set. However, when the points are on a gradual or steep line, the analysis is conducted based on the condition imposed on the value of k. As the value k depending on the condition varies, the geometric configuration changes accordingly. The differences observed in the set are expressed and presented in the theorems. Overall, it is thought that this study would contribute to further exploration in this area of research.

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