Düzce University
Journal of Science & Technology

Research Article

On Gaussian Quadra Fibona-Pell Sequence and A Quaternion Sequence Formed by the Terms of This Sequence

Faruk KAPLAN [a], Arzu ÖZKOÇ ÖZTÜRK [a,*]

[a] *Department of Mathematics, Faculty of Arts and Science , Düzce University, Düzce, TURKEY*
* *Corresponding author's e-mail address: arzuozkoc@duzce.edu.tr*
DOI: 10.29130/dubited.1555372

ABSTRACT

In this study, the Gaussian quadra Fibona-Pell sequence is proposed and examined. The quadra Fibona-Pell sequence is first extended to define the Gaussian quadra Fibona-Pell sequence. Then the generating function, Binet-like formula, and some identities are represented. In addition, some formulas related to the Gaussian quadra Fibona-Pell sequence and some matrices containing terms of the sequence are studied. Finally we define a quaternion sequence formed by the terms of Gaussian quadra Fibona-Pell sequence.

Keywords: *Fourth-order recurrence relation, Gaussian numbers, Quaternion, Quadra Fibona-Pell sequence.*

Gauss Quadra Fibona-Pell Dizisi ve Bu Dizinin Terimleri Tarafından Oluşturulan Bir Kuaterniyon Dizisi Üzerine

ÖZET

Bu çalışmada, Gauss quadra Fibona-Pell dizisi önerilmekte ve incelenmektedir. İlk olarak, quadra Fibona-Pell dizisi genişletilerek Gauss quadra Fibona-Pell dizisi tanımlanmıştır. Daha sonra, bu dizinin üreteç fonksiyonu, Binet-benzeri formülü ve bazı özdeşlikler sunulmuştur. Ayrıca, Gauss quadra Fibona-Pell dizisi ile ilgili bazı formüller ve dizinin terimlerini içeren bazı matrisler de incelenmiştir. Son olarak, Gauss quadra Fibona-Pell dizisinin terimlerinden oluşan bir kuaterniyon dizisi tanımlanmıştır.

Anahtar kelimeler: *Dördüncü mertebeden yineleme bağıntısı, Gauss sayıları, Kuaterniyon, Dörtlü Fibona-Pell dizisi.*

Geliş: 24/09/2024, Düzeltme: 09/10/2024, Kabul: 18/10/2024

I. INTRODUCTION

When it comes to number sequences, Fibonacci and others naturally come to mind. The Fibonacci number sequence and various sequences, such as Lucas, Pell, Jacobsthal, and Horadam similar to this number sequence, have found applications in many branches, not only in mathematics. When the literature is examined, there are many studies on number sequences. For detailed information, you review Koshy's book [1].

There are also Gaussian forms of some number sequences with recurrence relation in the literature. Here, too, the terms forming these number sequences are complex numbers. Again, many works in the literature involve Gaussian forms of number sequences. z is a Gaussian integer such that $z = a + ib,$ with $i^2 = -1$, where a and b are arbitrary integers. Gauss [2], in 1832, such numbers were first published and also mentioned the properties of the set of complex integers.

Horadam [3], in 1963, introduced the concept the complex Fibonacci numbers called the Gaussian Fibonacci numbers as follows:

$GF_n = GF_{n-1} + GF_{n-2}$, for $n \geq 2$

where $GF_0 = i$ and $GF_1 = 1$ are initial values.

Note that $F_n + iF_{n-1} = GF_n$ for all $n > 0$ and $GF_{-n} = F_{-n} + iF_{-n-1}$ where F_n , $n-th$ Fibonacci number. The table below gives the first few values of Gaussian Fibonacci numbers with positive and negative subscripts.

Table 1. Gaussian Fibonacci numbers with positive and negative subscripts.

n	GF_n	GF_{-n}
0	i	i
1	1	$1-i$
2	$1+i$	$-1+2i$
3	$2+i$	$2-3i$
4	$3+2i$	$-3+5i$
5	$5+3i$	$5-8i$
6	$8+5i$	$-8+13i$
7	$13+8i$	$13-21i$
8	$21+13i$	$-21+34i$
.	.	.
.	.	.
.	.	.

And then, Jordan [4], expanded the knowledge on the subject for Fibonacci sequences. If γ and δ are the roots of the characteristic equation of the Gaussian Fibonacci sequence, Binet-like formula of the Gaussian Fibonacci sequence can be given as follows:

$$GF_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} + i\frac{\gamma\delta^n - \delta\gamma^n}{\gamma - \delta}$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$. Generating function of GF_n is

$$GF(t) = \frac{i + t - it}{1 - t - t^2}. \tag{1.1}$$

Halıcı and Öz [5], in 2016, introduced the Gaussian Pell numbers as follow:

$$GP_n = 2GP_{n-1} + GP_{n-2}, \text{ for } n \geq 2$$

where $GP_0 = i$ and $GP_1 = 1$.

Note that $GP_n = P_n + iP_{n-1}$ and $GP_{-n} = P_{-n} + iP_{-n-1}$ where P_n , $n-th$ Pell number. The table below gives the first few values of Gaussian Pell numbers with positive and negative subscripts.

Table 2. Gaussian Pell numbers with positive and negative subscripts.

n	GP_n	GP_{-n}
0	i	i
1	1	$1-2i$
2	$2+i$	$-1+5i$
3	$5+2i$	$5-12i$
4	$12+5i$	$-12+29i$
5	$29+12i$	$29-70i$
6	$70+29i$	$-70+169i$
7	$169+70i$	$169-408i$
8	$408+169i$	$-408+985i$
.	.	.
.	.	.
.	.	.

For this new sequence, [5] has the following. If Ψ and Ω are the roots of the characteristic equation of the Gaussian Pell sequence, the Binet-like formula of the Gaussian Pell sequence can be given as follows:

$$GP_n = \frac{\Psi^n - \Omega^n}{\Psi - \Omega} + i\frac{\Psi\Omega^n - \Omega\Psi^n}{\Psi - \Omega}$$

where $\Psi = 1 + \sqrt{2}$ and $\Omega = 1 - \sqrt{2}$. Generating function of GP_n is

$$GP(t) = \frac{i + t - 2it}{1 - 2t - t^2}. \tag{1.2}$$

Looking at the literature, there are studies similar to these, such as [6–8], and more. As for the present day, in [9], the author introduced the Gaussian Mersenne numbers and also gave the matrix form and various identities. The same author, in [10], studied the Gaussian numbers for Padovan and Pell-Padovan numbers, which are integer sequences with a recurrence relation of the third order.

In [11], the author gave linear summation for the Gaussian generalized Pentanacci numbers he introduced. In [12], also the author focuses on the Gauss form of (p, q) −Jacobsthal and (p, q) −Jacobsthal Lucas numbers. Some identities are presented for the relevant new sequences. In [13], the authors obtained various identities for Gauss Leonardo numbers such as the Binet formula, Cassini identity, and generating function. Earliest, we remind some properties of about quadra Fibona-Pell sequence. The information in this section is taken from [14]. The quadra Fibona-Pell numbers are defined by fourth-order recurrence relation as

$$\mathrm{W}_n = 3\mathrm{W}_{n-1} - 3\mathrm{W}_{n-3} - \mathrm{W}_{n-4}, \text{ for } n \geq 4 \tag{1.3}$$

with the beginning conditions $\mathrm{W}_0 = \mathrm{W}_1 = 0$, $\mathrm{W}_2 = 1$, and $\mathrm{W}_3 = 3$.

The characteristic equation of (1.3) and the roots of this equation are

$$x^4 - 3x^3 + 3x + 1 = 0$$

and

$$\gamma = \frac{1+\sqrt{5}}{2}, \delta = \frac{1-\sqrt{5}}{2}, \Psi = 1+\sqrt{2}, \text{ and } \Omega = 1-\sqrt{2}.$$

(1.3) can be extended to negative subscripts as follows:

$$\mathrm{W}_n = 3\mathrm{W}_{n-1} - 3\mathrm{W}_{n-3} - \mathrm{W}_{n-4}, \text{ for all } n \geq 4.$$

Note that for convenience throughout the paper, we use the abbreviation qFP for quadra Fibona-Pell. The first few qFP numbers with positive subscripts and negative subscript are given in the table below:

Table 3. qFP numbers with positive and negative subscripts.

$\boldsymbol{n}$	$\boldsymbol{W_n}$	$\boldsymbol{W_{-n}}$
0	0	0
1	0	0
2	1	−1
3	3	3
4	9	−9
5	24	24
6	62	−62
7	156	156
8	387	−387
.	.	.
.	.	.
.	.	.

One of the different applications of the qFP sequence can be seen in the article [15].

II. GAUSSIAN QUADRA FIBONA-PELL NUMBERS

In this part of the study, inspired by some of the works mentioned in the references, we introduce the iteration relation of the Gaussian qFP sequence, which is used to construct the Binet formula and derive some identities, etc.

Definition 2.1. Gaussian qFP numbers can define GW_n recursively:

$$\mathrm{GW}_n = 3\mathrm{GW}_{n-1} - 3G\mathrm{W}_{n-3} - \mathrm{GW}_{n-4}, \; n \geq 4 \tag{2.1}$$

with the initial conditions

$\mathrm{GW}_0 = 0$

$\mathrm{GW}_1 = 0$

$\mathrm{GW}_2 = 1$

$\mathrm{GW}_3 = 3 + i.$

Moreover, take note of that

$\mathrm{GW}_n = \mathrm{W}_n + i\mathrm{W}_{n-1}.$

The first few Gaussian qFP numbers with positive subscripts and negative subscript are given in the table below:

Table 4. Gaussian qFP numbers with positive and negative subscripts.

n	GW_n	GW_{-n}
0	0	0
1	0	$-i$
2	1	$-1+3i$
3	$3+i$	$3-9i$
4	$9+3i$	$-9+24i$
5	$24+9i$	$24-62i$
6	$62+24i$	$-62+156i$
7	$156+62i$	$156-387i$
8	$387+156i$	$-387+951i$
.	.	.
.	.	.
.	.	.

The characteristic equation of (2.1) is

$$x^4 - 3x^3 + 3x + 1 = 0. \tag{2.2}$$

Then, the roots of (2.2) are

$$\Omega = 1 - \sqrt{2}, \Psi = 1 + \sqrt{2}, \gamma = \frac{1+\sqrt{5}}{2}, \delta = \frac{1-\sqrt{5}}{2}. \tag{2.3}$$

Immediately note that γ and δ belongs to the Gaussian Fibonacci numbers also Ψ and Ω belongs to the Gaussian Pell numbers for the characteristic equations. Our next result is the generating function of the sequence in our focus.

Theorem 2.1. The generating function of Gaussian quadra Fibona-Pell sequence GW_n is as follow:

$$GW(t) = \frac{\mathrm{t}^2 + i\mathrm{t}^3}{\mathrm{t}^4 + 3\mathrm{t}^3 - 3\mathrm{t} + 1}.$$

Proof. The generating function of the related sequence is a function such that whose formal power series expansion at $t = 0$ has the form

$$GW(t) = \sum_{r=0}^{\infty} GW_r\ t^r = GW_0\ + GW_1\ t + GW_2\ t^2 + GW_3\ t^3 + GW_4\, t^4 + \cdots$$

Therefore, from the power series,

$$GW(t) = \sum_{r=0}^{\infty} GW_r\ t^r = GW_0\ + GW_1\ t + GW_2\ t^2 + GW_3\ t^3 + GW_4\, t^4 + \cdots$$

$$-3tGW(t) = -3t\sum_{r=0}^{\infty} GW_r\ t^r = -3GW_0\ t - 3GW_1\ t^2 - 3GW_2\ t^3 - 3GW_3\ t^4 - 3GW_4\, t^5 + \cdots$$

$$3t^3GW(t) = 3t^3\sum_{r=0}^{\infty} GW_r\ t^r = 3GW_0\ t^3 + 3GW_1\ t^4 + 3GW_2\ t^5 + 3GW_3\ t^6 + 3GW_4\, t^7 + \cdots$$

$$t^4GW(t) = t^4\sum_{r=0}^{\infty} GW_r\ t^r = GW_0\ t^4 + GW_1\ t^5 + GW_2\ t^6 + GW_3\ t^7 + GW_4\, t^8 + \cdots.$$

If both sides of the equation are added together

$$GW(t) - 3tGW(t) + 3t^3GW(t) + t^4GW(t) = GW_0\ + GW_1\ t + GW_2\ t^2 + GW_3\ t^3 + (-3GW_0\ t - 3GW_1\ t^2 - 3GW_2\ t^3) + 3GW_0\ t^3.$$

Hence, if necessary arrangements are made, we have

$$GW(t) = \frac{GW_0\ + t(GW_1\ - 3GW_0\,) + t^2(GW_2\ - 3GW_1\ ) + t^3(GW_3\ - 3GW_2\ + 3GW_0}{1 - 3t + 3t^3 + t^4}.$$

Here, when the initial conditions are written, we have

$$GW(t) = \frac{\mathrm{t}^2 + i\mathrm{t}^3}{\mathrm{t}^4 + 3\mathrm{t}^3 - 3\mathrm{t} + 1}. \tag{2.4}$$

The present result gives the Binet formula for Gaussian qFP sequence.

Theorem 2.2. For $n \geq 0$, the Binet formula for a related sequence is

$$G\mathrm{W}_n = \left(\frac{\Psi^n - \Omega^n}{\Psi - \Omega} + i\frac{\Psi\Omega^n - \Omega\Psi^n}{\Psi - \Omega}\right) - \left(\frac{\gamma^n - \delta^n}{\gamma - \delta} + i\frac{\gamma\delta^n - \delta\gamma^n}{\gamma - \delta}\right). \tag{2.5}$$

Proof. We know that from (2.4)

$$GW(t) = \frac{t^2 + it^3}{t^4 + 3t^3 - 3t + 1}.$$

Adjust this a bit, we have

$$GW(t) = \frac{i + t - it}{1 - t - t^2} - \frac{i + t - 2it}{1 - 2t - t^2}.$$

If the generating functions of Gaussian Pell and Gaussian Fibonacci sequences are taken into consideration, that is, it follows that from (1.1) and (1.2)

$$GW(t) = GP(t) - GF(t).$$

The following result gives the sum of the first n terms of GW_n .

Theorem 2.3. The sum of the first n terms of GW_n is

$$\sum_{r=1}^{n} GW_r = \frac{GW_n + 4GW_{n-1} + 4GW_{n-2} + GW_{n-3} + i + 1}{2}$$

for n $\geq$ 3.

Proof. We know that the recurrence relation of the relevant sequence is

$$GW_n = 3GW_{n-1} - 3GW_{n-3} - GW_{n-4}$$

$$GW_{n-3} + GW_{n-4} = 3GW_{n-1} - 2GW_{n-3} - GW_n$$

If we make a little edit and write open for n, we have

$$GW_1 + GW_0 = 3GW_3 - 2GW_1 - GW_4$$

$$GW_2 + GW_1 = 3GW_4 - 2GW_2 - GW_5$$

$$GW_3 + GW_2 = 3GW_5 - 2GW_3 - GW_6$$

$$GW_{n-4} + GW_{n-5} = 3GW_{n-2} - 2GW_{n-4} - GW_{n-1}$$

$$\vdots \qquad \vdots$$

$$GW_{n-3} + GW_{n-4} = 3GW_{n-1} - 2GW_{n-3} - GW_n$$

Here, if the collection is done, both of side

$$\begin{aligned} GW_{n-3} + GW_0 + 2(GW_1 + GW_2 + \cdots + GW_{n-4} &= 3(GW_3 + GW_4 + \cdots GW_n) \\ &-2(GW_1 + GW_2 + \cdots GW_{n-3}) \\ &-(GW_4 + GW_5 + \cdots GW_n). \end{aligned}$$

The two results for the relevant sequence are as follows:

Theorem 2.4. For n $\geq$ 0, the following recurrence relations are valid:

$$\boldsymbol{a}.\ \mathrm{GW}_{2n} = 9\mathrm{GW}_{2n-2} - 20\mathrm{GW}_{2n-4} + 9\mathrm{GW}_{2n-6} - \mathrm{GW}_{2n-8}.$$

$$\boldsymbol{b}.\ \mathrm{GW}_{2n+1} = 9\mathrm{GW}_{2n-1} - 20\mathrm{GW}_{2n-3} + 95 - \mathrm{GW}_{2n-7}.$$

Proof. $\boldsymbol{a}$. In the proof, we start from the recurrence relation, $GW_n = 3GW_{n-1} - 3GW_{n-3} - GW_{n-4}$.

Here, if we put n instead of $2n$ and use (1.3), then

$$\begin{aligned}
\mathrm{GW}_{2n} = 3\mathrm{GW}_{2n-1} - 3\mathrm{GW}_{2n-3} - \mathrm{GW}_{2n-4} &= 3(3\mathrm{GW}_{2n-2} - 3\mathrm{GW}_{2n-4} - 9\mathrm{GW}_{2n-5} \\
&\quad -3(3\mathrm{GW}_{2n-4} - 3\mathrm{GW}_{2n-6} - \mathrm{GW}_{2n-7}) \\
&\quad -3(3\mathrm{GW}_{2n-5} - 3\mathrm{GW}_{2n-7} - \mathrm{GW}_{2n-8}) \\
&= 9\mathrm{GW}_{2n-2} - 18\mathrm{GW}_{2n-4} - 6\mathrm{GW}_{2n-5} + \mathrm{GW}_{2n-6} \\
&\quad +6\mathrm{GW}_{2n-7} + \mathrm{GW}_{2n-8} \\
&= 9\mathrm{GW}_{2n-2} - 20\mathrm{GW}_{2n-4} + 9\mathrm{GW}_{2n-6} - \mathrm{GW}_{2n-8}.
\end{aligned}$$

b. The proof here is done by following the previous proof steps.

The exponential generating function for $G\mathrm{W}_n$ is below.

Theorem 2.5. Exponential generating function for the $G\mathrm{W}_n$ is

$$\sum_{n=0}^{\infty} G\mathrm{W}_n \frac{x^n}{n!} = \left(\frac{\mathrm{e}^{\Psi x} - \mathrm{e}^{\Omega x}}{\Psi - \Omega} + i\frac{\Psi\mathrm{e}^{\Omega x} - \Omega\mathrm{e}^{\Psi x}}{\Psi - \Omega}\right) - \left(\frac{\mathrm{e}^{\gamma x} - \mathrm{e}^{\delta x}}{\gamma - \delta} + i\frac{\gamma\mathrm{e}^{\delta x} - \delta\mathrm{e}^{\gamma x}}{\gamma - \delta}\ \right)$$

for $\geq$ 0 .

Proof Motivating from related Binet formula

$$\sum_{n=0}^{\infty} G\mathrm{W}_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}\left[\left(\frac{\Psi^n - \Omega^n}{\Psi - \Omega} + i\frac{\Psi\Omega^n - \Omega\Psi^n}{\Psi - \Omega}\right) - \left(\frac{\gamma^n - \delta^n}{\gamma - \delta} + i\frac{\gamma\delta^n - \delta\gamma^n}{\gamma - \delta}\ \right)\right]\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{\Psi - \Omega}\left((\Psi^n - \Omega^n) + i(\Psi\Omega^n - \Omega\Psi^n)\right) - \frac{1}{\gamma - \delta}\quad \left((1 - i\delta)\gamma^n - i(1 - i\gamma)\delta^n\right.\right.$$

$$= \frac{1}{\Psi - \Omega}\left[\sum_{n=0}^{\infty}\frac{(\Psi x)^n}{n!} - \sum_{n=0}^{\infty}\frac{(\Omega x)^n}{n!} + i\left(\Psi\sum_{n=0}^{\infty}\frac{(\Omega x)^n}{n!} - \Omega\sum_{n=0}^{\infty}\frac{(\Psi x)^n}{n!} - \right)\right]$$

$$-\frac{1}{\gamma-\delta}\left[(1-i\delta)\sum_{n=0}^{\infty}\frac{(\gamma x)^n}{n!}-(1-i\gamma)\sum_{n=0}^{\infty}\frac{(\delta x)^n}{n!}\right].$$

We derive the matrix representation for $G\mathrm{W}_n$ below.

III. MATRIX REPRESENTED OF GAUSSIAN QUADRA FIBONA-PELL NUMBERS

Matrices and matrix representations find a place in many fields and can also be associated with the recurrence relation. For $n \geq 0$.

$$\begin{pmatrix} G\mathrm{W}_{n+4} \\ G\mathrm{W}_{n+3} \\ G\mathrm{W}_{n+2} \\ G\mathrm{W}_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G\mathrm{W}_{n+4} \\ G\mathrm{W}_{n+3} \\ G\mathrm{W}_{n+2} \\ G\mathrm{W}_{n+1} \end{pmatrix}$$

where the square matrix S of order 4 is;

$$S = \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 3.1. Let $n \geq 0$ be integer . Then

$$\begin{pmatrix} G\mathrm{W}_{n+4} \\ G\mathrm{W}_{n+3} \\ G\mathrm{W}_{n+2} \\ G\mathrm{W}_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} G\mathrm{W}_3 \\ G\mathrm{W}_2 \\ G\mathrm{W}_1 \\ G\mathrm{W}_0 \end{pmatrix}.$$

Proof It can be proven using mathematical induction. If $n \geq 0$, the result is clear. Suppose that the statement is true for $n = m - 1$. Then,

$$\begin{pmatrix} G\mathrm{W}_{m+3} \\ G\mathrm{W}_{m+2} \\ G\mathrm{W}_{m+1} \\ G\mathrm{W}_{m} \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} G\mathrm{W}_3 \\ G\mathrm{W}_2 \\ G\mathrm{W}_1 \\ G\mathrm{W}_0 \end{pmatrix}.$$

We want to show that this is true for $n = m$. This means that

$$\begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{m} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{m+3} \\ GW_{m+2} \\ GW_{m+1} \\ GW_{m} \end{pmatrix}$$

$$= \begin{pmatrix} GW_{m+4} \\ GW_{m+3} \\ GW_{m+2} \\ GW_{m+1} \end{pmatrix}$$

Thus the proof is concluded.

IV. ON GAUSSIAN QUADRA FIBONA-PELL QUATERNIONS

The quaternion structure, discovered by Hamilton in 1843, attracted much attention in mathematics. A quaternion is represented in mathematics in the form:

$\mathbb{H} = \{q = a_0\,1 + a_1\,i + a_2\,j + a_3\,k : a_0, a_1, a_2, a_3 \epsilon \mathbb{R}\}$.

which satisfies the following multiplication rules:

$i^2 = j^2 = k^2 = -1$ (4.1)

$jik = -1$

$ij = k$

$jk = i.$

Point out that $\{1, i, j, k\}$ does not provide the commutative property in the products between the base elements, but the set $\mathbb{H}$ holds the associative property of multiplication. In the literature, the symbol $\mathbb{H}$ was adopted for the set of quaternions based on Hamilton. Here are some basic arithmetic operations provided by the set $\mathbb{H}$. A quaternion can also be written as

$$q = a_0\, 1 + a_1\, i + a_2\, j + a_3\, k$$

where q can be written separately as the scalar part of the quaternion:

$$S_q = a_0\, 1$$

and as the vector part

$$V_q = a_1\, i + a_2\, j + a_3\, k.$$

Therefore, the quaternion q can be expressed both scalar and vector as

$$q = S_q + V_q = a_1\, i + a_2\, j + a_3\, k.$$

Searching the literature for quaternions and number sequences, we realize that the author at [3] introduced the Fibonacci quaternion and presented recurrence relation for relevant sequence. Since then, the growing interest in this field is evident from the existence of many works [16–19]. For example, in [20], the authors worked on sequences of quaternions with polynomial coefficients. Also in [21] the authors combined quaternion-type structures with different integer sequences. An example of an octonion, which is a quaternion formed using eight bases, can be found in [22]. In [23], the same authors studied the binomial transform of Horadam quaternions. In [24], the authors introduced quaternions whose coefficients are Gaussian Fibonacci numbers and presented various identities. In [25], authors worked on quaternions whose coefficients are Gaussian Lucas numbers. It is seen from many similar studies such as [26–28] that the interest in this field has not decreased. Especially, one of the interesting points of quaternion-Gaussian numbers is that quaternion-Gaussian numbers are used to investigate graphical models in [29]. Motivated by these studies, we identify quaternions with qFP coefficients and examine some identities.

Describe the quaternions with Gaussian quadra Fibona-Pell coefficients as:

$$QGW_n = GW_n + GW_{n+1}\, i + GW_{n+2}\, j + GW_{n+3}\, k \tag{4.2}$$

where initial conditions of (4.2) as follows:

$$QGW_0 = 3k$$

$QGW_1 = i + 10k$

$QGW_2 = 3i + 27k$

$QGW_3 = 10i + 71k.$

We can write the following recurrence relation with Gaussian quadra Fibona-Pell sequence

$$QGW_n = 3QGW_{n-1} - 3QGW_{n-3} - QGW_{n-4}, \text{ for } n \geq 0 \quad (4.3)$$

The characteristic equation of (4.3) is

$$x^4 - 3x^3 + 3x + 1 = 0. \quad (4.4)$$

The roots of the characteristic equation of (4.4) are $\Psi = 1 + \sqrt{2},\ \Omega = 1 - \sqrt{2},\ \gamma = \frac{1+\sqrt{5}}{2},\ \delta = \frac{1-\sqrt{5}}{2}$.

First, start by finding the generating function of the sequence.

Theorem 4.1. The generating function of QGW_n is

$$QGW(t) = \frac{QGW_0 + t(QGW_1 - 3QGW_0) + t^2(QGW_2 - 3QGW_1) + t^3(QGW_3 - 3QGW_2 + 3QGW_0)}{1 - 3t + 3t^3 + t^4}$$

where $QGW_0 = 3k, QGW_1 = i + 10k, QGW_2 = 3i + 27k,$ and $QGW_3 = 10i + 71k.$

Proof In the following equation, where formal power series expansion for QGW_r is given, all the information necessary for the generating function is obtained.

$$QGW(t) = \sum_{r=0}^{\infty} QGW_r\, t^r = QGW_0 + QGW_1\, t + QGW_2\, t^2 + QGW_3\, t^3 + \cdots + QGW_r + \cdots.$$

Therefore, from the power series,

$$QGW(t) = \sum_{r=0}^{\infty} QGW_r\ t^r = QGW_0\ + QGW_1\ t + QGW_2\ t^2 + QGW_3\ t^3 + QGW_4\, t^4 + \cdots$$

$$-3tQGW(t) = -3t\sum_{r=0}^{\infty} QGW_r\ t^r = -3QGW_0\ t - 3QGW_1\ t^2 - 3QGW_2\ t^3 - 3QGW_3\ t^4 + \cdots$$

$$3t^3QGW(t) = 3t^3\sum_{r=0}^{\infty} QGW_r\ t^r = 3QGW_0\ t^3 + 3QGW_1\ t^4 + 3QGW_2\ t^5 + 3QGW_3\ t^6 + \cdots$$

$$t^4QGW(t) = t^4\sum_{r=0}^{\infty} QGW_r\ t^r = QGW_0\ t^4 + QGW_1\ t^5 + QGW_2\ t^6 + QGW_3\ t^7 + QGW_4\, t^8 + \cdots.$$

Hence, if necessary arrangements are made, we get

$$QGW(t) = \frac{QGW_0\ + t(QGW_1\ - 3QGW_0\,) + \mathrm{t}^2(QGW_2\ - 3QGW_1\,) + \mathrm{t}^3(QGW_3 - 3QGW_2 + 3QGW_0\,)}{1 - 3\mathrm{t} + 3\mathrm{t}^3 + \mathrm{t}^4}$$

so desired is achieved.

Theorem 4.2. The Binet formula for the relevant quaternion sequence QGW_n is

$$QGW_n\ = \frac{1}{\Psi - \Omega}(\Psi^n A - \Omega^n \mathrm{B} + \Psi\Omega^n C - \Omega\Psi^n D) - \frac{1}{\gamma - \delta}\quad (\gamma^n E - \delta^n \mathrm{F} + \gamma\delta^n G - \delta\gamma^n H)$$

where $\mathrm{A}\ = 1 + \Psi i + \Psi^2 j + \Psi^3 k,\ \mathrm{B}\ =\ 1 + \Omega i + \Omega^2 j + \Omega^3 k,\ C = i - \Omega + \Omega^2 k - \Omega^3 j,$

$D = i - \Psi + \Psi^2 k - \Psi^3 j, E = 1 + \gamma i + \gamma^2 j + \gamma^3 k, F = 1 + \delta i + \delta^2 j + \delta k,$

$G = i - \delta + \delta^2 k - \delta^3 j, H = i - \gamma + \gamma^2 k - \gamma^3 j.$

Proof. Binet's formula for the Gaussian quadra Fibona-Pell number sequence in (2.5) was also derived. If we substitute this formula into the quadra Fibona-Pell quaternion, we have

$$QGW_n = GW_n + GW_{n+1}\, i + GW_{n+2}\, j + GW_{n+3}\, k$$

$$= \left(\frac{\Psi^n - \Omega^n}{\Psi - \Omega} + i\frac{\Psi\Omega^n - \Omega\Psi^n}{\Psi - \Omega}\right) - \left(\frac{\gamma^n - \delta^n}{\gamma - \delta} + i\frac{\gamma\delta^n - \delta\gamma^n}{\gamma - \delta}\right)$$

$$+ \left(\frac{\Psi^{n+1} - \Omega^{n+1}}{\Psi - \Omega} + i\frac{\Psi\Omega^{n+1} - \Omega\Psi^{n+1}}{\Psi - \Omega}\right)i - \left(\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} + i\frac{\gamma\delta^{n+1} - \delta\gamma^{n+1}}{\gamma - \delta}\right)j$$

$$+ \left(\frac{\Psi^{n+2} - \Omega^{n+2}}{\Psi - \Omega} + i\frac{\Psi\Omega^{n+2} - \Omega\Psi^{n+2}}{\Psi - \Omega}\right)j - \left(\frac{\gamma^{n+2} - \delta^{n+2}}{\gamma - \delta} + i\frac{\gamma\delta^{n+2} - \delta\gamma^{n+2}}{\gamma - \delta}\right)j$$

$$+ \left(\frac{\Psi^{n+3} - \Omega^{n+3}}{\Psi - \Omega} + i\frac{\Psi\Omega^{n+3} - \Omega\Psi^{n+3}}{\Psi - \Omega}\right)j - \left(\frac{\gamma^{n+3} - \delta^{n+3}}{\gamma - \delta} + i\frac{\gamma\delta^{n+3} - \delta\gamma^{n+3}}{\gamma - \delta}\right)j$$

$$= \frac{\Psi^n(1 + \Psi i + \Psi^2 j + \Psi^3 k) - \Omega^n(1 + \Omega i + \Omega^2 j + \Omega^3 k)}{\Psi - \Omega}$$

$$+ \frac{\Psi\Omega^n(i - \Omega + \Omega^2 k - \Omega^3 j) - \Omega\Psi^n(i - \Psi + \Psi^2 k - \Psi^3 j)}{\Psi - \Omega}$$

$$- \frac{\gamma^n(1 + \gamma i + \gamma^2 j + \gamma^3 k) - \delta^n(1 + \delta i + \delta^2 j + \delta k)}{\gamma - \delta}$$

$$+ \frac{\gamma\delta^n(i - \delta + \delta^2 k - \delta^3 j) - \delta\gamma^n(i - \gamma + \gamma^2 k - \gamma^3 j)}{\gamma - \delta}$$

where $\mathrm{A} = 1 + \Psi i + \Psi^2 j + \Psi^3 k, \mathrm{B} = 1 + \Omega i + \Omega^2 j + \Omega^3 k, C = i - \Omega + \Omega^2 k - \Omega^3 j,$

$D = i - \Psi + \Psi^2 k - \Psi^3 j, E = 1 + \gamma i + \gamma^2 j + \gamma^3 k, F = 1 + \delta i + \delta^2 j + \delta k$

$G = i - \delta + \delta^2 k - \delta^3 j, H = i - \gamma + \gamma^2 k - \gamma^3 j.$

The QGW_n^* used in the next theorem is the conjugate of QGW_n .

Theorem 4.3. The quaternions with Gaussian qFP coefficients satisfy:

$\boldsymbol{a}. QGW_{n+1} = 3QGW_n - 3QGW_{n-2} - QGW_{n-3}$ for $n \geq 3$.

$\boldsymbol{b}. QGW_n + QGW_n^* = 2GW_n$.

Proof. Start from the right side equation and use (4.3), we have

$\boldsymbol{a}.$ For n ≥ 3,

$3QGW_n - 3QGW_{n-2} - QGW_{n-3}$

$= 3(GW_n + GW_{n+1} i + GW_{n+2} j + GW_{n+3} k)$

$-3(GW_{n-2} + GW_{n-1} i + GW_n j + GW_{n+1} k)$

$-(GW_{n-3} 1 + GW_{n-2} i + GW_{n-1} j + GW_n k)$

$= (3GW_n - 3GW_{n-2} - GW_{n-3})$

$+(3GW_{n+1} - 3GW_{n-1} - GW_{n-2})i$

$+(3GW_{n+2} - 3GW_n - GW_{n-1})j$

$$+(3GW_{n+3} - 3GW_{n+1} - GW_n)k$$

$$= GW_{n+1} + GW_{n+2}\, i + GW_{n+3}\, j + GW_{n+4}\, k$$

$$= QGW_{n+1} \, .$$

b. From (4.3) and quaternion conjugate, we get

$$QGW_n + QG\,W_n^* = GW_n + GW_{n+1}\, i + GW_{n+2}\, j + GW_{n+3}\, k$$

$$+GW_n - GW_{n+1}\, i - GW_{n+2}\, j - GW_{n+3}\, k$$

$$= 2GW_n$$

V. CONCLUSION

Since quaternions find serious areas of study in many fields, from physics (such as quantum physics) to mathematics (graphical modelling), in this article, it was considered to examine quaternions in terms of the Gaussian qFP integer coefficient sequence.

Firstly, after giving detailed preliminary information and literature about the quadra Fibonacci-Pell sequence, the Gaussian Fibona-Pell sequence was introduced in the second part. Sum formulas, recurrence relation, Binet formula, some identities, and generating functions were introduced for the related new sequence. The matrix representation for the relevant sequence was presented in the third part. In the last part, the quadra Fibona-Pell quaternion sequence, which is the focus of our attention, was introduced. Therefore, in our future paper, we idea to present some known identities, such as Cassini, Catalan, and d'Ocagne, and create a binomial transform for quaternion sequences and their key characteristics.

VII. REFERENCES

[1] T. Koshy, "*Fibonacci and Lucas numbers with applications,* " John Wiley and Sons, vol. 1, 2018.

[2] C. Gauss, "*Theoria residuorum biquadraticorum,*" Commentatio prima, Typis Dieterichchianis, 1832.

[3] A. F. Horadam, "Complex Fibonacci numbers and Fibonacci quaternions," *American Mathematical Monthly*, vol. 70, pp. 289-291, 1963.

[4] J. H. Jordan, "Gaussian Fibonacci and Lucas numbers," *The Fibonacci Quarterly*, vol. 3, pp. 315-318, 1965.

[5] S. Halıcı and S. Öz, "On some Gaussian Pell and Pell–Lucas numbers," *Ordu University Journal of Science and Technology*, vol. 6, no.1, pp. 8-18, 2016.

[6] G. Berzsenyi, "Gaussian Fibonacci numbers,*" The Fibonacci Quarterly*, vol. 15, no. 3, pp. 233-236, 1997.

[7] C. J. Harman, "Complex Fibonacci numbers," *The Fibonacci Quarterly*, vol. 19, no.1, 82-86, 1981.

[8] S. Pethe and A. F. Horadam, "Generalised Gaussian Fibonacci numbers," *Bulletin of the Australian Mathematical Society*, vol. 33, no. 1, pp. 37-48, 1986.

[9] D. Taşcı, "On Gaussian Mersenne numbers," *Journal of Science and Arts,* vol. 57, no. 4, pp. 1021-1028, 2021.

[10] D. Taşcı, "Gaussian Padovan and Gaussian Pell-Padovan Sequences," *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*, vol. 67, no. 2, pp. 82-88, 2018.

[11] Y. Soykan, "Linear summing formulas of generalised pentanacci and Gaussian generalised pentanacci numbers," *Journal of Advances in Mathematics and Computer Science*, vol. 33, no. 3, pp. 1-14, 2022.

[12] N. Saba and A. Boussayoud, "Gaussian (p,q)-Jacobsthal and Gaussian (p,q)-Jacobsthal Lucas numbers and their some interesting properties," *Journal of Mathematical Physics*, vol. 63, no. 11, pp. 1-17, 2022.

[13] K. Prasad, R. Mohanty, M. Kumari, and H. Mahato, "Some new families of generalized k-Leonardo and Gaussian Leonardo numbers," *Communication in Combinatorics and Optimization,* vol. 3, no. 9, pp. 539-553, 2024.

[14] A. Özkoç, "Some algebraic identities on quadra Fibona- Pell integer sequence," *Advances in Difference Equation*, vol. 2015, no. 148, pp. 1-10, 2015.

[15] A. Özkoç Öztürk and E. Gündüz, "Binomial transform for Quadra Fibona-Pell sequence and Quadra Fibona-Pell quaternion," *Universal Journal of Mathematics and Applications*, vol. 5, no. 4, pp. 145-155, 2022.

[16] S. Halıcı, "On Fibonacci quaternions," *Advances in Applied Clifford Algebras*, vol. 22, pp. 321-327, 2012.

[17] E. Polatlı, “A generalization of Fibonacci and Lucas quaternions,” *Advances in Applied Clifford Algebras*, vol. 26, no. 2, pp. 719-730, 2016.

[18] S. Halıcı and A. Karataş, “On a generalization for Fibonacci quaternions,” *Chaos, Solitons and Fractals*, vol. 98, pp. 178-182, 2017.

[19] E. Ozkan and M. Uysal, “O*n* quaternions with higher order Jacobsthal numbers components,” *Gazi University Journal of Science*, vol. 36, no. 1, pp. 336-347, 2023.

[20] A. Özkoç Öztürk and F. Kaplan, “Some properties of bivariate Fibonacci and Lucas quaternion polynomials,” *Facta Universitatis, Series: Mathematics and Informatics*, vol. 35, no. 1, pp. 073-087, 2020.

[21] E. Polatlı, “On Certain Properties of Quadrapell Quaternions,” *Karaelmas Fen ve Mühendislik Dergisi*, vol. 8, no. 1, pp. 305-308, 2018.

[22] C. Kızılateş and E. Polatlı, “New families of Fibonacci and Lucas octonions with q-integer components,” *Indian Journal of Pure and Applied Mathematics*, vol. 52, no. 1, pp. 231-240, 2021.

[23] F. Kaplan and A. Özkoç Öztürk, “On the binomial transforms of the Horadam quaternion sequence,” *Mathematical Methods in the Applied Sciences* , vol. 45, no. 8, pp. 12009-12022, 2022.

[24] S. Halıcı and G. Cerda Morales, “On quaternion-Gaussian Fibonacci numbers and their properties,” Analele Stiintifice ale Universitatii Ovidius Constanta, vol. 29, no. 1, pp. 71-81, 2021.

[25] S. Halıcı, “On quaternion-Gaussian Lucas numbers,” Mathematical Methods in the Applied Sciences, vol. 44, no. 9, pp. 7601-7606, 2021.

[26] H. Arslan, “Gaussian Pell and Gaussian Pell-Lucas quaternions,” *Filomat*, vol. 35, no. 5, pp. 1609-1617, 2021.

[27] A. Z. Azak, “Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions,” *Mathematics*, vol. 10, no. 24, pp. 4655, 2022.

[28] A. Ertaş and F. Yılmaz, “On quaternions with Gaussian Oresme coefficients,” *Turkish Journal of Mathematics and Computer Science*, vol. 15, no. 1, pp. 192-202, 2023.

[29] A. Sloin and A. Wiesel, “Proper quaternion Gaussian graphical models,” *IEEE Transactions on Signal Processing*, vol. 62, no. 20, pp. 5487-5496, 2014.