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RESEARCH ARTICLE

Comparative Numerical Evaluation of Some Runge-Kutta Methods for Solving First Order Systems of ODEs

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ABSTRACT

In this study, a comparative analysis of two Runge-Kutta methods; fourth-order Runge-Kutta method and Butcher's Fifth Order Runge-Kutta method are presented and used to solve systems of first-order linear Ordinary Differential Equations (ODEs). The main interest of this work is to test the accuracy, convergence rate and computational efficiency of these methods by using different numerical problems of ODEs. Empirical conclusions are drawn after close observation of the results presented by the two methods, which further highlights their limitations and enabling researchers to make informed decisions in choosing the appropriate technique for specific systems of ODEs problems.

Keywords: Runge-Kutta Methods, First Order ODEs, Numerical Methods, Convergence rate, Numerical Analysis

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1. INTRODUCTION

The importance of numerical techniques in providing solutions to problems of ordinary differential equations (ODEs) cannot be overemphasized [1]. Hence, the demand for faster and accurate methods is required by researchers. This paper provides an informed decision to numerical analyst when a choice is to be made between fourth order Runge-Kutta method and Butcher's Fifth Order Runge-Kutta method to solve systems of ODEs of the form;

$$\frac{du}{dr} = f(r,u) \tag{1}$$

Numerous investigations have been done regarding the RK4 and BRK5. [2] conducted an assessment of numerical performance of some runge-kutta methods and new iteration method on first order differential problems. [3] studied a comparison of Runge-Kutta and Butcher's methods for stiff differential equations. Similarly, [4] carried out a comparison of explicit Runge Kutta, implicit Runge-Kutta, and Rosenbrock methods for stiff differential equations. Although considerable research has been dedicated to first-order ordinary differential equation systems, there remains a need for further investigation into the application of the RK4 and BRK5 methods for these systems. Despite the established use of these techniques, the literature [5,6] reveals a gap in comprehensive comparative studies that systematically evaluate their performance across a range of problems. Previous works have primarily focused on single methods or limited comparisons under specific conditions, often neglecting the broader implications of method selection in practice. This study addresses this gap by providing a thorough analysis of RK4 and BRK5, focusing on their accuracy, convergence, and computational efficiency.
The novelty of this research lies not only in its comparative nature but also in its systematic approach to evaluating these methods across diverse problem sets. By conducting extensive numerical experiments, this study aims to elucidate the conditions under which each method

excels or falls short. The insights gained from this evaluation are intended to inform practitioners and researchers, guiding them in the selection of the most appropriate numerical methods for their specific applications. Current studies highlight a significant gap in the comparative quantitative evaluation of these methods. While both RK4 and BRK5 have been explored individually, recent literature lacks a thorough comparative analysis. This study aims to address this gap by performing a detailed quantitative assessment of RK4 and BRK5, focusing on their efficiency, accuracy, and computational impact in solving first-order differential equation systems. The impetus for this research is the importance of refining numerical methods for these systems and their broad application potential. A review of existing literature shows that while there is considerable emphasis on the individual merits and limitations of RK4 and BRK5, a direct comparative study of these methods is missing. This research is therefore essential to provide a comprehensive comparison of RK4 and BRK5, offering valuable insights into their relative effectiveness and guiding their optimal use in various fields.

2. MATERIALS AND METHODS

In this section, a first-order differential equation in the form (1) is considered and used to formulate the selected Runge-Kutta methods as follows [7-9]:

2.1. Runge-Kutta Fourth Order (RK4) Method

The scheme is most acquainted because it is efficient, accurate, steady and easy to program. This method is noteworthy by their order in the logic that they agree with Taylor's series solution. The RK4 method is mostly used for solving initial value problems (IVPs) for ordinary differential equation and the general expression can be denotes as follows;

$$u_{t+1} = u_t + h\sigma(r,u,h) \tag{2}$$

where,

$$\left. h\sigma(r,u,h) = \sum_{n=1}^{g} A_n k_n \right\} \tag{3}$$

Thus,

$$k_1 = f(r_t, u_t) \tag{4}$$

$$k_2 = f\left(r_t + \frac{h}{2}, u_t + \frac{h}{2}k_1\right) \tag{5}$$

$$k_3 = f\left(r_t + \frac{h}{2}, u_t + \frac{h}{2}k_2\right) \tag{6}$$

$$k_4 = f(r_t + h, u_t + hk_3) \tag{7}$$

$$u_{t+1} = u_t + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{8}$$

Equation (8) is the RK4 iterative method.

2.2. Butcher's Fifth Order Runge-Kutta (BRK5) Method

Butcher's fifth-order method builds on the RK4 method to deliver even greater accuracy. It achieves this by adding extra stages and coefficients, which enhances its order of convergence and leads to more precise results. This technique has been extensively studied and has shown improved accuracy across different problem scenarios [10-14].

with initial criteria $u(r_0) = u_0$, the method focuses on calculating u_{t+1}.

$$u_{t+1} = u_t + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right) \tag{9}$$

Equation (9) is BRK5 iterative Method.

where,

$$k_1 = f\left(r_t, u_t\right)$$

$$k_2 = f\left(r_t + \frac{h}{4}, u_t + \frac{h}{4}k_1\right)$$

$$k_3 = f\left(r_t + \frac{h}{4}, u_t + \frac{1}{8}hk_1 + \frac{1}{8}hk_2\right)$$

$$k_4 = f\left(r_t + \frac{h}{2}, u_t - \frac{1}{2}hk_2 + \frac{1}{8}hk_3\right)$$

$$k_5 = f\left(r_t + \frac{3}{4}h, u_t + \frac{3}{16}hk_1 + \frac{9}{16}hk_4\right)$$

$$k_6 = f\left(r_t + h, u_t - \frac{3}{7}hk_1 + \frac{2}{7}hk_2 + \frac{12}{7}hk_3 - \frac{12}{7}hk_4 + \frac{8}{7}hk_5\right)$$

3. ERROR ANALYSIS

The errors in numerical solution of ODEs are of two types, namely: Truncation errors and Round-off errors. Errors exhibited when a mathematical procedure is stopped or approximated at a finite point rather than being carried out at infinity is referred as truncation error while the error caused by computer interference is known as round-off error [15,16]. Largely, the stability in arithmetic systems for solving ODEs has to do with how errors move over the calculations [17]. For linear stability study, one defines the stability function $R(z)$ of the process applied to the test question $y' = \lambda y$, where $z = \lambda h$. For the RK4 method, the stability function is given by:

$$R(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}$$

The stability region is the set of values of z for which $|R(z)| \le 1$. For RK4, this region is higher than in lower order methods and means that RK4 is stable for a large range of h. An arithmetic system is convergent if $h \to 0$ (step size goes to zero), implying that the numerical solution should tend to the exact solution. [18-19].

4. NUMERICAL ALGORITHMS

4.1. Runge-Kutta Fourth Order Algorithm

Step 1: Prompt the function $f(r,u)$ in way such that $f(r,u) \in [c,d]$.

Step 2: Provide the preliminary estimate for r_0 and u_0.

Step 3: Select the desired step size $h = \dfrac{d-c}{n}$, where n is the number of steps.

Step 4: Insert c, d, r_0, u_0, N.

Step 5: for x from 1 to N do.

Compute k_1, k_2, k_3 *and* k_4 as denoted in the RK4 method.

Step 6: Set $r_{t+l} \to R_t$, then.

Compute $u_{t+1} = u_t + \dfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Step 7: Output r_0 *and* u_0

Step 8: Terminate the procedure if $r_t \ge d$ Such that $\|u_{t+1} - u_t\| < \varepsilon$

4.2. Butcher's Fifth Order Runge-Kutta Algorithm

Step 1: Prompt the function $f(r,u)$ in way such that $f(r,u) \in [c,d]$.

Step 2: Provide the preliminary estimate for r_0 and u_0.

Step 3: Indicate the desired step size $h = \dfrac{d-c}{n}$, where n is number of steps.

Step 4: Insert c, d, r_0, u_0, N.

Step 5: for x from 1 to N do.

Compute k_1, k_2, k_3, k_4, k_5 and k_6 as denoted in the BRK5 method.

Step 6: Set $r_{t+l} \to R_t$, then.

Compute $u_{t+1} = u_t + \dfrac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$.

Step 7: Output r_0 and u_0.

Step 8: Terminate the procedure if $r_t \ge d$ Such that $\|u_{t+1} - u_t\| < \varepsilon$

5. NUMERICAL RESULTS AND ANALYSIS

In this section, the two methods were compared by using first-order ODE problems to test their numerical efficiency. All computations are done using Python software and errors are represented graphically.

Test 1: $$f'(s) = k(s)$$ $$k'(s) = 2f(s) - k(s)$$

Subject to the initial conditions:

$f(0) = 1,\ k(0) = -1.$

Analytical Solution: $$f(s) = \frac{e^s}{3} + \frac{2e^{-2s}}{3}$$ $$k(s) = \frac{e^s}{3} - \frac{4e^{-2s}}{3}$$

Test 2: $$f'(s) - 2f + 3k = 0$$ $$k'(s) + 2f - k = 0$$

Bound by the initial conditions:

$f(0) = 8,\ k(0) = 3.$

Analytical Solution: $$f(s) = 3e^{4s} + 5e^{-s}$$ $$k(s) = -2e^{4s} + 5e^{-s}$$

Test 3: $$f'(s) = f(s) + k(s)$$ $$k'(s) = -f(s) + k(s)$$

Bound by the initial conditions: $f(0) = 0,\ k(0) = 1.$

Analytical Solution: $f(s) = -\frac{1}{2} + \frac{e^{2s}}{2}$

$$k(s) = \frac{e^{2s}}{2} + \frac{1}{2}$$

Test 4: $$f'(s) = f(s) + k(s)$$ $$k'(s) = 2f(s) + 2k(s)$$

Bound by the initial conditions: $f(0) = 5,\ k(0) = 0.$

Analytical Solution: $f(s) = 3e^{-2s} + 2e^{4s}$

$$k(s) = -2e^{-s} + 2e^{4s}$$

Test 5: $$f'(s) = f(s)$$ $$k'(s) = f(s) - k(s)$$

Bound by the initial conditions: $f(0) = 1,$ $k(0) = 2.$

Analytical Solution: $f(s) = e^{s}$

$$k(s) = \frac{1}{2}e^{s} + \frac{3}{2}e^{-s}$$

The computational comparisons are presented in various figures and tables. The numerical solutions of the two methods are denoted as $RK4_f$ and $BRK5_f$ for $f'(s)$ solutions and $RK4_k$ and $BRK5_k$ for $k'(s)$ solutions.

Table 1: $RK4_f$ and $BRK5_f$ Analytical and Numerical Solutions for Test 1

S	Analytical Solution	$RK4_f$ Numerical Solution	$BRK5_f$ Numerical Solution
0.0	1.000	1.000	-1.000
0.1	0.914	0.914	-0.723
0.2	0.854	0.854	-0.486
0.3	0.815	0.815	-0.281
0.4	0.796	0.796	-0.101
0.5	0.794	0.794	0.059
0.6	0.808	0.794	0.205
0.7	0.835	0.835	0.342
0.8	0.876	0.876	0.472
0.9	0.930	0.930	0.599
1.0	0.996	0.996	0.725

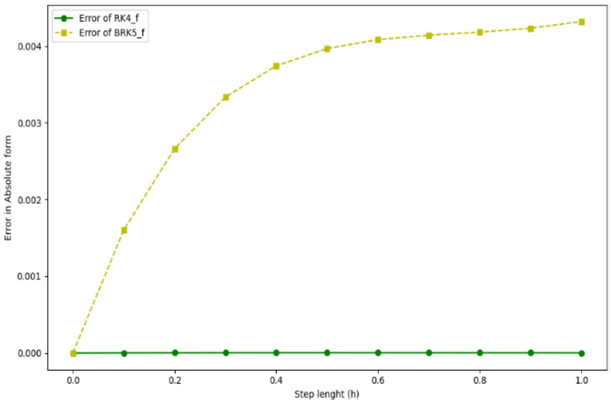

Figure 1: Error Plot for Numerical Test 1(f)

Table 2: $RK4_k$ and $BRK5_k$ Analytical Numerical Solutions for Test 1

s	Analytical Solution	$RK4_k$ Numerical Solution	$BRK5_k$ Numerical Solution
0.0	-1.000	-1.000	-1.000
0.1	-0.723	-0.723	-0.720
0.2	-0.486	-0.486	-0.481
0.3	-0.281	-0.281	-0.275
0.4	-0.101	-0.101	-0.095
0.5	0.059	0.059	0.065
0.6	0.205	0.205	0.211
0.7	0.342	0.342	0.347
0.8	0.472	0.472	0.477
0.9	0.599	0.599	0.603
1.0	0.725	0.725	0.728

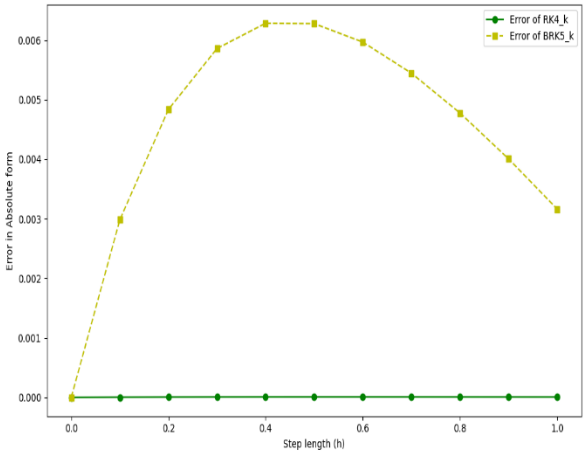

Figure 2: Error Plot for Numerical Test 1(k)

Table 3: $RK4_f$ and $BRK45_f$ Analytical Numerical Solutions for Test 2

S	Analytical Solution	$RK4_f$ Numerical Solution	$BRK5_f$ Numerical Solution
0.0	8.000	8.000	8.000
0.1	8.999	8.999	9.015
0.2	10.770	10.769	10.820
0.3	13.664	13.662	13.781
0.4	18.210	18.207	18.448
0.5	25.199	25.193	25.648
0.6	35.813	35.801	36.624
0.7	51.816	51.795	53.236
0.8	75.844	75.808	78.276
0.9	111.827	111.7670	115.924
1.0	165.633	165.533	172.444

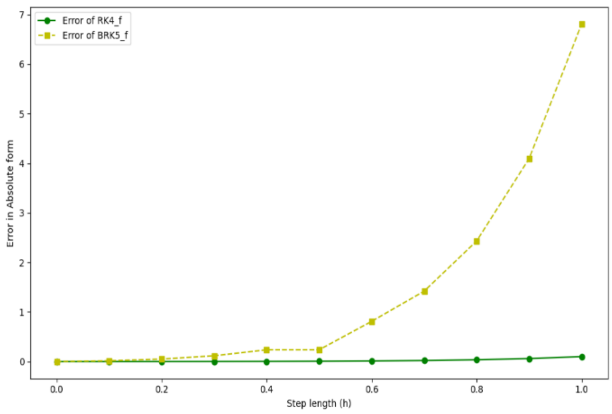

Figure 3: Error Plot for Numerical Test 2(f)

Table 4: $RK4_k$ and $BRK5_k$ Analytical Numerical Solutions for Test 2

s	Analytical Solution	$RK4_k$ Numerical Solution	$BRK5_k$ Numerical Solution
0.0	3.000	3.000	3.000
0.1	1.540	1.540	1.525
0.2	-0.357	-0.356	-0.398
0.3	-2.936	-2.934	-3.023
0.4	-6.554	-6.552	-6.724
0.5	-11.745	-11.740	-12.058
0.6	-19.302	-19.294	-19.857
0.7	-30.406	-30.392	-31.368
0.8	-46.818	-46.794	-48.455
0.9	-71.163	-71.123	-73.910
1.0	-107.356	-107.290	-111.913

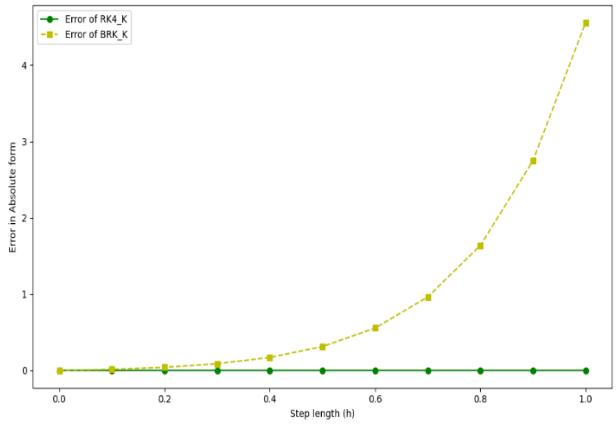

Figure 4: Error Plot for Numerical Test 2(k)

Table 5: $RK4_f$ and $BRK45_f$ Analytical Numerical Solutions for Test 3

S	Analytical Solution	$RK4_f$ Numerical Solution	$BRK5_f$ Numerical Solution
0.0	0.000	0.000	0.000
0.1	0.110	0.110	0.109
0.2	0.245	0.242	0.239
0.3	0.4110	0.398	0.394
0.4	0.612	0.580	0.574
0.5	0.859	0.790	0.781

0.6	1.160	1.028	1.018
0.7	1.527	1.297	1.284
0.8	1.976	1.596	1.581
0.9	2.524	1.926	1.909
1.0	3.194	2.287	2.269

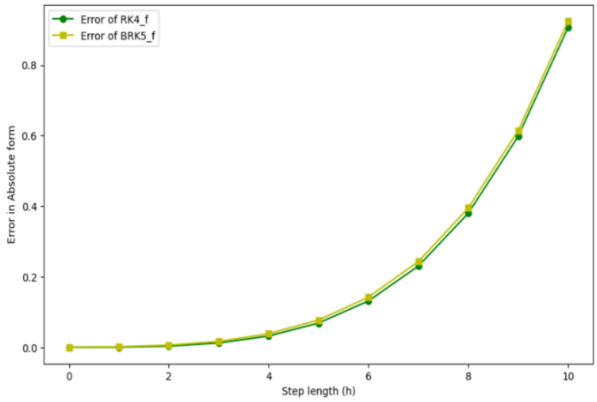

Figure 5: Error Plot for Numerical Test 3(f)

Table 6: $RK4_k$ and $BRK5_k$ Analytical Numerical Solutions for Test 3

S	Analytical Solution $RK4_k$	$RK4_k$ Numerical Solution	$BRK5_k$ Numerical Solution
0.0	1.000	1.000	1.000
0.1	1.110	1.099	1.098
0.2	1.245	1.197	1.194
0.3	1.411	1.289	1.286
0.4	1.612	1.374	1.369
0.5	1.859	1.446	1.442
0.6	2.160	1.503	1.498
0.7	2.527	1.540	1.535
0.8	2.976	1.550	1.546
0.9	3.524	1.528	1.526
1.0	4.194	1.468	1.468

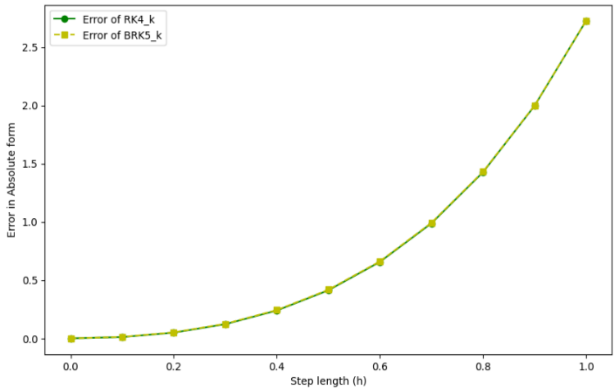

Figure 6: Error Plot for Numerical Test 3(k)

Table 7: $RK4_f$ and $BRK45_f$ Analytical Numerical Solutions for Test 4

S	Analytical Solution	$RK4_f$ Numerical Solution	$BRK5_f$ Numerical Solution
0.0	5.000	5.000	5.000
0.1	5.698	5.583	5.560
0.2	6.907	6.370	6.310
0.3	8.862	7.432	7.311
0.4	11.917	8.866	8.650
0.5	16.597	10.802	10.439
0.6	23.692	13.415	12.830
0.7	34.379	16.942	16.025
0.8	50.413	21.702	20.296
0.9	74.416	28.129	26.003
1.0	110.299	36.803	33.631

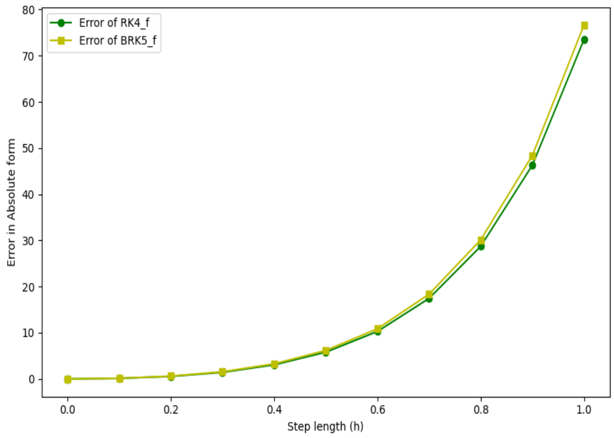

Figure 7: Error Plot for Numerical Test 4(f)

Table 8: $RK4_k$ and $BRK5_k$ Analytical Numerical Solutions for Test 4

S	Analytical Solution	$RK4_k$ Numerical Solution	$BRK5_k$ Numerical Solution
0.0	0.000	0.000	0.000
0.1	1.173	1.166	1.121
0.2	2.813	2.740	2.620
0.3	5.158	4.864	4.623
0.4	8.565	7.733	7.300
0.5	13.565	11.604	10.878
0.6	20.948	16.830	15.660
0.7	31.896	23.884	22.051
0.8	48.166	33.405	30.592
0.9	72.383	46.258	42.007
1.0	108.460	63.607	57.262

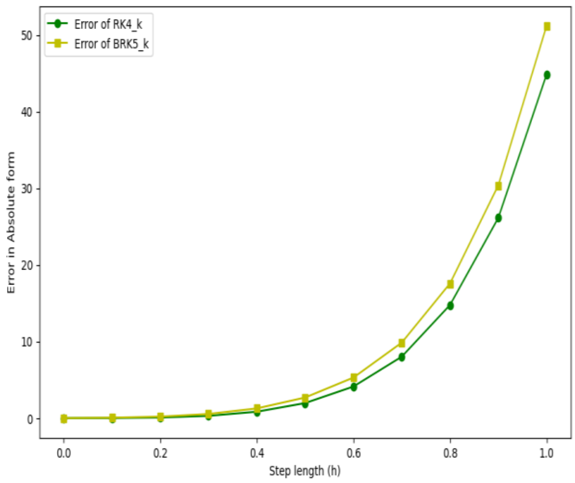

Figure 8: Error Plot for Numerical Test 4(k)

Table 9: $RK4_f$ and $BRK45_f$ Analytical Numerical Solutions for Test 5

S	Analytical Solution	$RK4_f$ Numerical Solution	$BRK5_f$ Numerical Solution
0.0	1.000	1.000	1.000
0.1	1.105	1.105	1.104
0.2	1.221	1.221	1.219
0.3	1.349	1.349	1.345
0.4	1.491	1.491	1.485
0.5	1.648	1.648	1.640
0.6	1.822	1.822	1.811
0.7	2.013	2.013	2.000
0.8	2.225	2.225	2.208
0.9	2.459	2.459	2.438
1.0	2.718	2.718	2.691

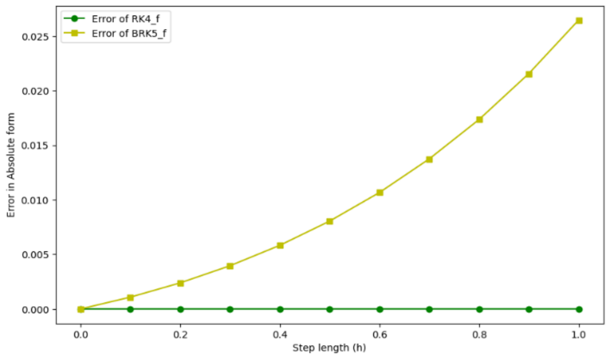

Figure 9: Error Plot for Numerical Test 5(f)

Table 10: $RK4_k$ and $BRK5_k$ Analytical Numerical Solutions for Test 5

S	Analytical Solution	$RK4_k$ Numerical Solution	$BRK5_k$ Numerical Solution
0.0	2.000	2.000	2.000
0.1	1.909	1.909	2.316
0.2	1.838	1.838	2.677
0.3	1.786	1.786	3.087
0.4	1.751	1.751	3.554
0.5	1.734	1.734	4.085
0.6	1.734	1.734	4.688
0.7	1.751	1.751	5.372
0.8	1.786	1.786	6.148
0.9	1.839	1.839	7.027
1.0	1.910	1.910	8.023

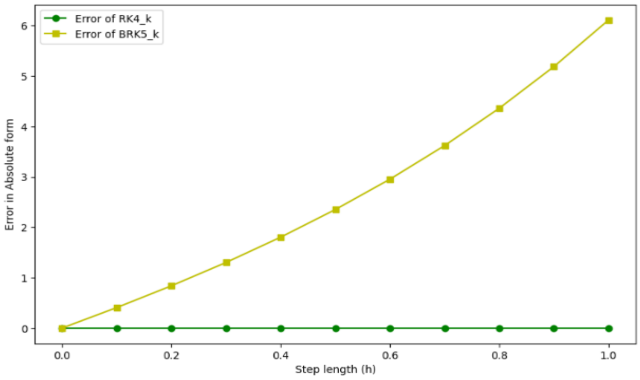

Figure 10: Error Plot for Numerical Test 5(k)

6. DISCUSSION

In this study, the acquired results are displayed in Tables 1-10 and the errors are graphically presented in Figures (1-10). The approximate solutions and errors are computed with the step size h = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 and also compared to the exact solution. In Tables 1-6, it was observed that RK4 method gives a more accurate and better results than BRK5. Furthermore, the argument is clarified by Figures 1-10 which provides more clarity on the errors exhibited by each first order ODE investigated. Thus, we can say that a numerical solution will converge to the exact solution if the step size h is decreased. The results underscore the effectiveness and accuracy of two numerical methods in solving the presented problems.

Comparison of Numerical Estimations (Tables 1-10)

i. The tables show that when employing a constant step size of 0.1, the RK4 and BRK5 methods produce numerical solutions for all three problems that closely align with the analytical results.
ii. The solutions from the RK4 and BRK5 methods exhibit some differences, indicating that both approaches offer comparable levels of approximation to the actual solution.
iii. The results highlight the efficacy of the two numerical methods in addressing the identified problems.

Error Comparison (Tables 1-10 and Figures 1-10)

i. The tables and graphical representations of errors offer a more thorough understanding of the effectiveness of each method.
ii. It is evident that the RK4 method consistently produces more accurate results and demonstrates lower error values than the other methods.
iii. The error curves for the RK4 method shown in Figures 1-10 reveal a decreasing trend towards zero, signifying convergence to the exact solution, provided the step size is kept constant. In contrast, the BRK5 method shows relatively higher error values, suggesting a markedly less accurate approximation of the true solution.

Discoveries

i.The comparative analysis clearly demonstrates that the RK4 method is more effective than the BRK5 approach in solving Problems 1-5.
ii.The RK4 technique is distinguished by its remarkable accuracy and efficiency, consistently approaching the analytical solution with minimal error. Although the BRK5 methods yield reliable results, they may display higher error rates due to their complexity, which can exacerbate round-off errors and result in numerical instability. Consequently, the RK4 method continues to be the most accurate numerical solution for these problems. Understanding these limitations is crucial for selecting the appropriate numerical methods in scientific and engineering contexts [20-21].

6. CONCLUSION

This study provides a comprehensive comparative evaluation of various Runge-Kutta methods, focusing on the fourth-order Runge-Kutta (RK4) and Butcher's fifth-order Runge-Kutta (BRK5) methods for solving first-order systems of ordinary differential equations (ODEs). The findings demonstrate that while both RK4 and BRK5 methods are effective, the RK4 method consistently outperforms the BRK5 approach in terms of accuracy and stability, particularly when maintaining a constant step size. The table and graph analysis, reveals that the RK4 method delivers reliable numerical solutions with minimal error, showcasing its robustness and efficiency across a range of problems involving first ODEs. In contrast, although the BRK5 method offers dependable results, its increased complexity can lead to higher error rates, making it less suitable for certain applications. Gaining a clear understanding of the strengths and limitations of these methods is vital for professionals in science and engineering fields. This research highlights the importance of selecting the appropriate numerical technique based on the specific requirements of the problem at hand. Future work may explore further refinements of these methods or investigate additional techniques to enhance computational efficiency and accuracy. Overall, this study contributes valuable insights into the comparative performance of Runge-Kutta methods, aiding in the informed selection of numerical methods for solving ODEs in various applications.

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