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Existence and Uniqueness of Solutions for Nonlinear Fractional Differential Equations with $\mho$-Caputo Fractional Differential Equations

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Abstract
This paper examines the existence, uniqueness, and Ulam-Hyers stability of solutions to nonlinear $\mho$-fractional differential equations with boundary conditions with a $\mho$-Caputo fractional derivative. The acquired results for the suggested problem are validated using a novel technique and minimum assumptions about the function f. The analysis reduces the problem to a similar integral equation and uses Banach and Sadovskii fixed point theorems to reach the desired findings. Finally, the inquiry is demonstrated by illustrative example to validate the theoretical findings.

Keywords: Banach Contraction mapping, Caputo fractional derivative ,$\mho$-Caputo fractional derivative, Fixed point theorem, Fractional differential equations, Stability analysis, Ulam-Hyers stability
2010 AMS: 34A08, 26A33, 47H10, 34K07, 34K10, 34K20

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1. Introduction

Fractional calculus has been recognized to be a useful tool for modeling many processes in economics, physics, and engineering. Since fractional derivatives (FDs) are a useful tool for characterizing memory and inherited qualities of various materials and processes, fractional-order models have actually been shown to be more applicable for a number of real-world scenarios than integer-order models. Applications where this theory is useful include material theory, transport processes, wave propagation, signal theory, economics, control theory and mechanics,. For more detail (see [1]-[5] and the references therein).That is the primary benefit of fractional differential equations (FDEs) over standard integer-order models. Basic difficulties include fractional derivatives, including Riemann-Liouville [2], Caputo [3], Hilfer [4], and Hadamard [6].

In recent years, there has been a lot of interest in FDEs, particularly boundary value issues for nonlinear FDEs, which may be used to represent and describe non-homogeneous physical processes that occur in their form. Almeida introduced the $\mho$-Caputo derivative in [2] to study FDEs in general. This is different type of FD seen in the literature. We may derive numerous well-known FDs for certain choices of $\mho$, such as the Caputo and Caputo-Hadamard FDs, which depend on a kernel. This technique also appears logical when seen through a variety of applications. Using a carefully selected ”trial” function $\mho$, the $\mho$-Caputo FDs provides some control over approximating the phenomena under research. Zhang [7] used various fixed point theorems to illustrate the existence and uniqueness of outcomes for nonlinear fractional existences (P.V.Bs) using Caputo type FDs. Researchers have found unique solutions to boundary value issues for FDEs (see [8]-[13]) and other references). The importance of fractional boundary value concerns originates from the fact that they cover a wide range of dynamical systems as examples.

On the other hand, introduced the stability problem of functional equation solutions (of group homomorphisms) in a presentation at Wisconsin University in 1940 [14]. Hyers [15] provided the first answer to the topic in Banach spaces in 1941. Ulam-type stability has piqued the curiosity of numerous academics since then. Researchers became attracted to the study of stability for FDEs due to the extensive extension of the fractional calculus for more detail (see [12], [15]-[17]).

Several approaches to study FDEs have been proposed in the literature recently, based on multi-valued mappings and boundary value problems. For instance, authors [18] focused on the Caputo fractional differential inclusions with boundary conditions in a more general case, for convex-compact mappings providing critical conditions for existence and uniqueness. Based on this, Mohammadi et al. [19] studied the existence of solutions of *phi*-Caputo fractional differential inclusions by using Multi-Valued Contractions.. This method provides further evidence of the role played by contraction principles in solving non-linear inclusions.

Moreover, Kayvanloo et al. New topological techniques were introduced in [20] to prove the existence of a solution for solvability in infinite systems of Caputo-Hadamard fractional differential equations. Additionally, in [21] Mohammadi et al. provided the significance of weak Wardowski mappings and elucidates our understanding of generalized g-Caputo fractional inclusions.

Benchohra, Hamani and Ntouyas in [8] investigated the existence of solutions of the following existence for Caputo FDEs.

$$^{C}D^{\eth}_{+0}y(t) = f(t, y(t)), \qquad x \in [0,T], \qquad 0 < \eth \leq 1,$$

with boundary condition

$$ay(0) + by(T) = c,$$

where $^{C}D^{\eth}_{+0}$ is the Caputo derivative with $0 < \eth < T$, a,b are real constant such that $a + b \neq 0$, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [22] researcher examined the existence and uniqueness of solutions for the following ℧-Caputo FDEs with boundary conditions.

$$\begin{cases} ^{C}D^{\beta,\mho(x)}_{0^+}u(x) = f(x, u(x)), \qquad x \in [0,T], \\ u(0) = u'(0) = 0, \quad u(T) = u_T. \end{cases}$$

Here $^{C}D^{\beta,\mho(x)}_{0^+}$ is the ℧-Caputo derivative with $2 < \beta < 3$, $T > 0$, $u \in C^1[0,1]$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivated by the works above, in this paper, we study the existence, uniqueness and stability of solutions for the following ℧-Caputo FDEs of arbitrary order with fractional boundary conditions shown below:

$$^{C}D^{\eth,\mho}_{0^+}u(x) = f(x, u(x)), \quad x \in J = [1,T], \tag{1.1}$$

with the boundary condition

$$\begin{cases} u(0) = \Omega_1, \\ Au(0) + Bu(T) = \Omega_2, \end{cases} \tag{1.2}$$

where $A, B.\Omega_1$ and Ω_2 are constant, and $^{C}D^{\eth,\mho}_{0^+}$, is ℧-Caputo FDs of order $1 < \eth \leq 2$ with $f : J \times \mathbb{R} \to \mathbb{R}$, is continuous function.

2. Preliminaries

In this section, we give some notations and definitions.

Definition 2.1. *(℧-Riemann-Liouville fractional integral) [23].*
Let $\eth > 0$, $f \in L^1(J,\mathbb{R})$, and $\mho \in C^n(J,\mathbb{R})$ such that $\mho'(\Im) > 0$ for all $x \in J$. The ℧-Riemann-Liouville fractional integral at order $\eth$ of the function f is given by by

$$I_{0^+}^{\eth} f(x) := \frac{1}{\Gamma(\eth)} \int_{0^+}^{x} [\mho(x) - \mho(\Im)]^{\eth-1} f(\Im)\mho'(\Im) d\Im, \tag{2.1}$$

Remark 2.2. *Note that if $\mho(x) = x$ and $\mho(x) = \log(x)$, then the equation (2.1) is reduced to the Riemann-Liouville and Hadamard fractional integrals, respectively.*

Definition 2.3. *(℧-Caputo fractional derivative)[23].*
Let $\eth > 0$, $f \in L^1(J,\mathbb{R})$, and $\mho \in C^n(J,\mathbb{R})$ such that $\mho'(\Im) > 0$ for all $x \in J$. The ℧-Caputo FDs at order $\eth$ of the function f is given by

$$^{C}D_{0^+}^{\eth,\mho} f(x) := \frac{1}{\Gamma(n-\eth)} \int_{0^+}^{x} [\mho(x) - \mho(\Im)]^{n-\eth-1} \mho'(\Im)\delta^{[n]} d\Im, \tag{2.2}$$

where $n-1 < \eth < n$,$n = [a]+1$, $\delta^{[n]}(\Im) = (\frac{1}{\mho'(\Im)}\frac{d}{d\Im})^n f(\Im)$, and $[\eth]$ denotes the integer part of the real number $\eth$, and Γ is the gamma function.

Remark 2.4. *Note that if $\mho(x) = x$ and $\mho(x) = \log(x)$, then the equation (2.2) is reduced to the Caputo and Caputo-Hadamard FDs, respectively.*

Remark 2.5. *If $\eth \in]0,1[$ then, we have*

$$^{C}D_{0^+}^{\eth,\mho} f(x) = I_{0^+}^{1-\eth,\mho}\left(\frac{f'(x)}{\mho'(x)}\right) = \frac{1}{\Gamma(\eth)} \int_{0^+}^{x} [\mho(x) - \mho(\Im)]^{\eth-1} f'(\Im) d\Im.$$

Definition 2.6. *(Ulam-Hyers stable) [24].*
The equation (1.1) is called ***Ulam-Hyers stable*** *if there $\exists$ a constant $q > 0$ such that for each $\varepsilon > 0$, when $u \in C(J,\mathbb{R})$ is any solution of the inequality*

$$|^{C}D_{1+}^{\eth,\mho(x)} u(x) - f(x,u(x))| \leq \varepsilon, \quad x \in J, \tag{2.3}$$

then there $\exists$ another solution $w \in C(J,\mathbb{R})$ of the equation (1.1) satisfied

$$|u(x) - w(x)| \leq q\varepsilon, \quad x \in J.$$

Definition 2.7. *[24] The equation (1.1) is said to be* ***Ulam-Hyers-Rassias stable*** *with respect to $\rho \in C(J,\mathbb{R})$ and $b > 0$ is any constant such that for each $\varepsilon > 0$ and for each solution $u \in C(J,\mathbb{R})$ of the inequality*

$$|^{C}D_{1+}^{\eth,\mho(x)} u(x) - f(x,u(x))| \leq \varepsilon\rho(x), \quad x \in J,$$

then there $\exists$ a solution $w \in C(J,\mathbb{R})$ of the equation (1.1) satisfied

$$|u(x) - w(x)| \leq b\varepsilon\rho(x), \quad x \in J.$$

Proposition 2.8. *Let $\eth > 0$. If $f \in C^n(J,\mathbb{R})$, then we have*

1) $^{C}D_{0^+}^{\eth,\mho} I_{0^+}^{\eth,\mho} f(x) = f(x)$

2) $I_{0^+}^{\eth,\mho C}D_{0^+}^{\eth,\mho} f(x) = f(x) - \sum_{r=0}^{n-1} \frac{f_{\mho}^{[r]}(0)}{r!}(\mho(x) - \mho(0)^r$

3) $I_{0^+}^{\eth,\mho}$ *is linear and bounded from $(J,\mathbb{R})$ to $(J,\mathbb{R})$.*

Theorem 2.9. *(The Banach Fixed Point Theorem)*
Let $J \subset \mathbb{R}$, be a closed, not necessarily bounded, interval and $H : J \to \mathbb{R}$ a function with $H(J) \subset J$ and which satisfies for a fixed k, $0 \le k < 1$,then the inequality

$$|H(x) - H(y)| \le k|x - y|$$

for all $x, y \in J$. Then there $\exists$ exactly one fixed point of H, i.e. a $\xi \in J$, with

$$H(\xi) = \xi.$$

In order to state Sadovskii's theorem, we define the following concepts.

Definition 2.10. Let Q be a bounded set in metric space (X,d). The Kuratowski measure of non compactness, $\mu(Q)$, is defined as:

$$\mu(Q) = \inf\{\varepsilon : Q covered\ by\ finitely\ many\ sets\ such\ that\ the\ diameter\ of\ each\ set\ is \le \varepsilon\}.$$

Definition 2.11. *Let $\theta : \Omega(\theta) \subseteq X \to X$ be a bounded and continuous operator on a Banach space X. Then θ is called a condensing map if $\mu(\theta(w)) < \mu(w)$ for all bounded sets $w \subset \Omega(\theta)$, where μ denotes the Kuratowski measure of non compactness.*

Lemma 2.12. *The map $P_1 + P_2$ is a k-set contraction with $0 \le k < 1$, and thence also condensing, if*

1. *$P_1, P_2 : \Omega \subseteq \Xi \to \Xi$ are operators on the Banach space Ξ,*
2. *P_1 is k-contractive, that is $||P_1 x - P_1 y|| \le k||x - y||$ for all $x, y \in \Omega$ and fixed $k \in [0, 1)$*
3. *P_2 is compact*

Lemma 2.13. *(Sadovskii's fixed point theorem)*
Assume that w be a convex, bounded and closed subset of a Banach space Ξ, and let $\theta : w \to w$ be a condensing map. Then θ has a fixed point.

Lemma 2.14. *For any $u(x) \in C(J, \mathbb{R})$, $n - 1 < \eth \le n$, then the existence (1.1)-(1.2) has a solution*

$$\begin{aligned} u(x) =& \Omega_1 + \Omega_3 [\mho(x) - \mho(0)] - \frac{[\mho(x) - \mho(0)]}{[\mho(T) - \mho(0)]} \int_0^T \mho'(\Im) \frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \\ &+ \int_0^x \mho'(\Im) \frac{[\mho(x) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \end{aligned}$$

where $\Omega_3 = \frac{(\Omega_2 - (A+B)\Omega_1)}{B[\mho(T) - \mho(0)]}$ and $\mho(T) \ne \mho(0)$.

Proof. Applying the Ʊ-fractional integral of order ð from 1 to x on both sides of Ʊ-Caputo FDEs in (1.1) Which can be rewritten as follows:

$$u(x) = c_0 + c_1 [\mho(x) - \mho(0)] + \int_0^x \mho'(\Im) \frac{[\mho(x) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

Now, we will apply the boundary conditions(1.2) to find c_0 and c_1,

$$u(x) = \Omega_1 + c_1 [\mho(x) - \mho(0)] + \int_0^x \mho'(\Im) \frac{[\mho(x) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im.$$

To find c_1.

$$u(T) = \Omega_1 + c_{n-1} [\mho(T) - \mho(0)] + \int_0^T \mho'(\Im) \frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

$$Au(0) + Bu(T) = A\Omega_1 + B\Omega_1 + Bc_{n-1} [\mho(T) - \mho(0)] + B \int_0^T \mho'(\Im) \frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

$$c_{n-1} = \frac{\Omega_2 - (A+B)\Omega_1}{B\left[\mho(T) - \mho(0)\right]} - \frac{1}{\left[\mho(T) - \mho(0)\right]} \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

$$\begin{aligned} u(x) =& \Omega_1 + \left[\mho(x) - \mho(0)\right] \frac{(\Omega_2 - (A+B)\Omega_1)}{B\left[\mho(T) - \mho(0)\right]} - \frac{\left[\mho(x) - \mho(0)\right]}{\left[\mho(T) - \mho(0)\right]} \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \\ &+ \int_0^x \mho'(\Im) \frac{\left[\mho(x) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im, \end{aligned}$$

and this complete the poof. □

3. Main Results

In the sequel, we denote by $\zeta := C(J, \mathbb{R})$ the Banach space of all continuous functions equipped with the norm

$$||u|| = \sup\{|u(x)|; x \in J\}.$$

To prove the main results, we need the following assumptions:
(**H1**) There $\exists$ a constant $L > 0$, such that $\left|f(x, u(x))\right| \leq L|u|$, for all $x \in J$ and all $u \in \mathbb{R}$
(**H2**) There $\exists$ a constant $k_1 > 0$, such that $\left|f(x, u(x)) - fx, w(x))\right| \leq k_1|u - w|$. For all $x \in J$ and all $u, w \in \mathbb{R}$.

3.1 Existence the result for problem (1.1)

Here we apply Sadovskii's fixed point to derive the existence result for the problem (1.1)

Theorem 3.1. *Assume $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (**H1**)-(**H2**). Then the existence (1.1)-(1.2) has at least one solution in J.*

Proof. We define the integer r, let $H_r = \{u \in \zeta : ||u|| \leq r\}$ be a closed bounded and convex subset of ζ, where r is a fixed constant. It is sufficient to show that Φ has a fixed point. We define an operator $\Phi : \zeta \to \zeta$ in a similar way in light of Lemma 2.12.:

$$\begin{aligned} \Phi(u(x)) = & \Omega_1 + \Omega_3 \left[\mho(x) - \mho(0)\right] - \frac{\left[\mho(x) - \mho(0)\right]}{\left[\mho(T) - \mho(0)\right]} \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \\ &+ \int_0^x \mho'(\Im) \frac{\left[\mho(x) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \end{aligned}$$

for all $x \in J$. We also define the operators $\Phi_1, \Phi_2 : \zeta \to \zeta$ by

$$\Phi_1(u(x)) = \Omega_1 + \int_0^x \mho'(\Im) \frac{\left[\mho(x) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

and

$$\Phi_2(u(x)) = \Omega_3 \left[\mho(x) - \mho(0)\right] - \frac{\left[\mho(x) - \mho(0)\right]}{\left[\mho(T) - \mho(0)\right]} \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im$$

and note that,

$$\Phi(u(x)) = \Phi_1(u(x)) + \Phi_2(u(x)) \qquad for\ all\ x \in J.$$

If the sum of operators $\Phi_1 + \Phi_2$ has a fixed point, then follows that operator Φ also has one. To demonstrate that $\Phi_1 + \Phi_2$ has a fixed point, the operators Φ_1 and Φ_2 shall be proved to meet the hypothesis of Lemma 2.13. This will be accomplished in numerous steps.

Step 1: $\Phi H_r \subset H_r$ Let us select

$$r_1 \geq |\Omega_1 + \Omega_3 \left[\mho(T) - \mho(0)\right]| + |\frac{2\left[\mho(T) - \mho(0)\right]^{\eth}}{\Gamma(\eth + 1)} L|$$

$$|(\Phi u)(x)| = \Big|\Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im$$
$$+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im\Big|,$$

$$\leq \Omega_1 + \Omega_3\,[\mho(T)-\mho(0)] + \frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}L||u|| < r_1.$$

For all $u \in H_r$, which implies that $\Phi H_r \subset H_r$.

Step 2: Φ_2 is compact. Consider that the operator Φ_2 is uniformly limited in view of step 1. Let $t_1, t_2 \in J$, where $t_1 < t_2$ and $u \in H_r$. Then we acquire.

$$|(\Phi_2 u)(x_1) - (\Phi_2 u)(x_2)| = \Big|\Omega_3\,(\mho(x_1)-\mho(0)) - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im$$
$$-\Omega_3\,[\mho(x)-\mho(0)] + \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im\Big|,$$

$$\leq \Omega_3\,(\mho(x_1)-\mho(x_2)) - \frac{(\mho(x_1)-\mho(x_2))}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im,$$

$$|(\Phi_2 u)(x_1) - (\Phi_2 u)(x_2)| \leq |\Omega_3\,(\mho(x_1)-\mho(x_2))| + |\frac{(\mho(x_1)-\mho(x_2))}{[\mho(T)-\mho(0)]}\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|L||u||$$

which is independent of u and tends to zero as $x_2 \to x_1$. Thus, Φ_2 is equicontinuous. Hence, by the Arzelá-Ascoli theorem, $\Phi_2(H_r)$ is a relatively compact set.

Step 3: Φ_1 is k-contractive. Let $u_1, u_2 \in H_r$. Then, we have

$$||(\Phi_1 u_1)(x) - (\Phi_1 u_2)(x)|| = \Big|\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u_1(\Im)-f(\Im,u_2(\Im))d\Im\Big|,$$

$$\leq \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1||u_1-u_2||,$$

set $\lambda = \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1$ then we obtain

$$||(\Phi_1 u_1)(x) - (\Phi_1 u_2)(x)|| \leq \lambda||u_1-u_2||.$$

Since $\lambda < 1$ Then, Φ_1 is a contractive mapping.

Step 4: ϕ is compressing. Lemma 2.12 states that $\Phi : H_r \to H_r$, with $\Phi = \Phi_1 + \Phi_2$, is a condensing map on H_r due to Φ_1 being continuous, a u-contraction, and Φ_2 compact. Using Lemma 2.13, we may conclude that the operator Φ has a fixed point. As a result, the boundary value problem (1.1)-(1.2) has at least one solution on J. □

3.2 Uniqueness the result for problem (1.1)

Now we apply Banach's contraction mapping principle to prove existence and uniqueness of solutions for problems (1.1)-(1.2)

Theorem 3.2. Assume $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (**H1**)-(**H2**). Let $\eta = \sup_{x\in[1,T]} f(x,0)$, if

$$|\frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}| < 1.$$

Then the existence (1.1)-(1.2) has a unique solution.

Proof. Define the operator $\Theta : \zeta \to \zeta$ as the following

$$\begin{aligned}(\Theta u)(x) =&\Omega_1 + \Omega_3 \left[\mho(x) - \mho(0)\right] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im,\end{aligned}$$

We have to show that Θ has a fixed point on G_r which it is solution of the existence (1.1)-(1.2). Firstly we show that $\Theta H_r \subset \Theta$, The operator Θ is bounded set into the bounded sets in ζ. For any $r>0$, then for each $x \in J = [1,T]$. Then, we have

$$\begin{aligned}|(\Theta u)(x)| =&\Bigg| \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u(\Im)) - f(\Im,0) + f(\Im,0))d\Im \\ &- \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u(\Im)) - f(\Im,0) + f(\Im,0))d\Im\Bigg|, \\ &\le \left| \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u(\Im)) - f(x,0) + f(x,0))d\Im\right| \\ &+ \left|\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u(\Im)) - f(x,0) + f(x,0))d\Im\right|, \\ &\le \left|\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}d\Im\right|(k_1+\eta) + \left|\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}d\Im\right|(k_1+\eta), \\ &\le |\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|(k_1+\eta) + |\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|(k_1+\eta), \\ &\le |\frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|(k_1+\eta).\end{aligned}$$

Now, let $u, u_1 \in \zeta$ and for each $x \in J$. We need to prove that Θ is contraction mapping.

$$\begin{aligned}|(\Theta u)(x) - (\Theta u_1)(x)| =& \Bigg| \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u(\Im)) - f(\Im,u_1(\Im))d\Im \\ &- \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im,u(\Im)) - f(\Im,u_1(\Im))d\Im\Bigg|, \\ &\le \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}d\Im k_1||u-u_1|| + \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-2}}{\Gamma(\eth-1)}d\Im k_1||u-u_1||, \\ &\le |\frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|k_1||u-u_1||.\end{aligned}$$

If $|\frac{2[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}| < 1$. Then, Θ is a contraction mapping. Therefore, by using The Banach contraction mapping, Θ has a unique Fixed point which is a unique solution of the existence (1.1)-(1.2). □

4. Stability Theorems

In this section, we study stability of our result.

4.1 Ulam-Hyers stability

Theorem 4.1. Assume that the assumptions (**H2**) is hold. Then the fractional differential equation (1.1) with the boundary condition (1.2) is Ulam-Hyers stable.

Proof. let $w \in C(J,\mathbb{R})$ be a solution of the inequality (2.3) i.e,

$$|^{C}D_{0^+}^{\eth,\mho(x)} w(x) - f(x,w(x))| \leq \varepsilon, \quad x \in J. \tag{4.1}$$

If we defined $u \in C(J,\mathbb{R})$ the unique solution of the existence (1.1)-(1.2)

When u and w being continuous functions on J. From lemma 2.14 we obtain

$$\begin{aligned} u(x) =&\Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \end{aligned}$$

we will take the integration of (4.1) and we obtain

$$\begin{aligned} &|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im| \leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}, \end{aligned}$$

on the other hand we have

$$\begin{aligned} \Big|w(x)-u(x)\Big| =&\Big|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im\Big|, \end{aligned}$$

$$\begin{aligned} \Big|w(x)-u(x)\Big| =&\Big|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im + \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im \\ &- \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im\Big|, \end{aligned}$$

$$\begin{aligned} &\leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} + \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} |f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} |f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im, \end{aligned}$$

$$\begin{aligned} |w(x)-u(x)| \leq& \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} + \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} k_1 |w(x)-u(x)| \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} |f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im, \end{aligned}$$

Let $\gamma_1 = \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1$ then, we obtain

$$|w(x)-u(x)| \leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)} + \frac{1}{1-\gamma_1}\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im,$$

$$|w(x)-u(x)| \leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)} e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|},$$

$$|w(x)-u(x)| \leq \varepsilon C_h.$$

Where $C_h = \frac{[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)} e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|}$. Hence, the solution of (1.1)-(1.2)) is Ulam-Hyers stable. □

4.2 Ulam-Hyers-Rassias stability

Theorem 4.2. *Suppose that the assumptions (**H2**)-(**H2**) are satisfied.*
(**H3**) *The function $\rho \in C(J,\mathbb{R})$ is increasing and there $\exists\ \Lambda_\rho > 0$, such that, for each $x \in J$ we have*

$$I^{\eth}\rho(t) < \Lambda_\rho \rho(x).$$

Then, the fractional differential equation (1.1) with the boundary condition (1.2) is Ulam-Hyers-Rassias stable with respect to ρ.

Proof. let $w \in C(J,\mathbb{R})$ be a solution of the inequality (2.3) i.e,

$$|^{C}D_{0^+}^{\eth,\mho(x)}w(x) - f(x,w(x))| \leq \varepsilon\rho(x), \quad x \in J. \tag{4.2}$$

If we defined $u \in C(J,\mathbb{R})$ the unique solution of the existence (1.1)-(1.2)

When u and w being continuous functions on J. From Lemma 2.14 we obtain

$$\begin{aligned} u(x) =& \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im \end{aligned}$$

we will take the integration of (4.2) and we obtain

$$\begin{aligned} &|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im| \leq \varepsilon\Lambda_\rho\rho(x), \end{aligned}$$

on the other hand we have

$$\begin{aligned} \Big|w(x)-u(x)\Big| = &\Big|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\Big|, \end{aligned}$$

$$\begin{aligned}\Big|w(x)-u(x)\Big| &= \Big|w(x)-\Omega_1+\Omega_3\left[\mho(x)-\mho(0)\right]-\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im\\ &+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im+\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im\\ &-\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im-\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\\ &+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\Big|,\end{aligned}$$

$$\begin{aligned}&\leq \varepsilon\Lambda_\rho\rho(x)+\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im\\ &+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im,\end{aligned}$$

$$|w(x)-u(x)|\leq \varepsilon\Lambda_\rho\rho(x)+\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1|w(x)-u(x)|+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im.$$

Let $\gamma_1=\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1$ then, we obtain

$$|w(x)-u(x)|\leq \frac{\varepsilon\Lambda_\rho\rho(x)}{(1-\gamma_1)}+\frac{1}{1-\gamma_1}\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im,$$

$$|w(x)-u(x)|\leq \frac{\varepsilon\Lambda_\rho\rho(x)}{1-\gamma_1}e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|},$$

$$|w(x)-u(x)|\leq \varepsilon C_h.$$

Here $C_h=\frac{\varepsilon\Lambda_\rho\rho(x)}{1-\gamma_1}e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|}$ hold, this show that the solution of the existence (1.1)-(1.2) is Ulam-Hyers-Rassias stable

□

5. Examples

Example 5.1. *Take the following existence*

$$^{C}D_{0^+}^{\eth,\sqrt{x}}u(x)=\frac{\sqrt{2-x}}{10+e^x}\frac{|u(x)|}{1+|u(x)|}, \tag{5.1}$$

with the boundary condition

$$\begin{cases}u(0)=3,\\ 2u(0)+3u(1)=2.\end{cases} \tag{5.2}$$

where $\eth=\frac{3}{2},\mho(x)=\sqrt{x}$ *and* $f(x,u(x))=\frac{\sqrt{2-x}}{10+e^x}\frac{|u(x)|}{1+|u(x)|}$
To prove Banach contraction mapping, let $x\in J$ *and* $u,v\in\mathbb{R}$
$|f(x,u_1)-f(x,u_2)|=\frac{\sqrt{2-x}}{10+e^x}|u_1-u_2|,$

We need to show that $|\frac{2[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|k_1| < 1$ *Then, the result become*

$$|(\Theta u)(x)-(\Theta v(t)| = |\frac{\sqrt{2-x}}{10+e^x}\frac{|u(x)|}{1+|u(x)|}-\frac{\sqrt{2-x}}{10+e^x}\frac{|v(x)|}{1+|v(x)|}|,$$

$$|(\Theta u)(x)-(\Theta v(t)| \leq |\frac{\sqrt{2-x}}{10+e^x}||\frac{|u(x)|}{1+|u(x)|}-\frac{|v(x)|}{1+|v(x)|}|,$$

$$\leq |\frac{\sqrt{2}}{11}||\frac{u(x)-v(x)}{(1+|u(x)|)(1+v(x))}|,$$

$$|(\Theta u)(x)-(\Theta v(t)| \leq |\frac{\sqrt{2}}{11}||u(x)-v(x)|.$$

*Thus, the assumption (**H2**) holds true with* $k_1 = \frac{\sqrt{2}}{11}$. *Moreover, we have*

$$|\frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|k_1| = |\frac{2(\sqrt{1}-\sqrt{0})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\frac{\sqrt{2}}{11}| = 0.19342 < 1.$$

Finally, all the conditions of Theorem 3.2 are satisfied, thus the B.V.P (5.1)-(5.2) has a unique solution on $[0,1]$.

6. Conclusion

In this paper, we examined the solutions for nonlinear FDEs with boundary conditions using the parameter of Ʊ-Caputo derivative. The Sadovskii fixed point theorem and Banach contraction principle ensure the existence and uniqueness of solutions to nonlinear problems. Additionally, the stability of Ulam-Hyers and Ulam-Hyers-Rassias solutions for the above issues is investigated. Finally, we provide an example to show the coherence of the theoretical conclusions. In the future, one can expand the provided fractional boundary value issue to more FDs, such as the Hilfer-Hadamard FDs and Caputo-Fabrizio FDs.

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