Konuralp Journal of Mathematics, 12 (2) (2024) 112-119

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Refinements Jensen's Inequality and Some Their Applications

Mehmet Zeki Sarıkaya[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*

Abstract

This paper aims to present a new refinement of the Jensen inequality specifically for convex functions. Building on this refinement, the paper derives various related inequalities, with a particular focus on Bullen's inequality and Ostrowski's inequality. Furthermore, it explores practical applications of these derived inequalities in the context of mean inequalities, providing a deeper understanding and broader utility of these mathematical concepts.

Keywords: *Convex function, Bullen inequality, Jensen inequality, Ostrowski inequality and Special means.*
2010 Mathematics Subject Classification: *26A09, 26D10, 26D15.*

1. Introduction & Preliminaries

Definition 1.1. *Let $J \subseteq \mathbb{R}$ be an interval. The function $f : J \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$$

for all $a, b \in J$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

The above inequality can be rewritten as

$$f(\lambda_1 a + \lambda_2 b) \leq \lambda_1 f(a) + \lambda_2 f(b), \tag{1.1}$$

provided that $\lambda_1 + \lambda_2 = 1$, and $\lambda_1, \lambda_2 > 0$. A multivariable version of (1.1) is the celebrated Jensen inequality which asserts that if $f : J \subset \mathbb{R} \to \mathbb{R}$ is a convex function and $a_1, a_2, ..., a_n \in J$, then holds the inequality

$$f\left(\sum_{i=1}^{n} \lambda_i a_i\right) \leq \sum_{i=1}^{n} \lambda_i f(a_i)$$

where $\lambda_1, \lambda_2, ..., \lambda_n > 0$ are such that $\sum_{i=1}^{n} \lambda_i = 1$.

It is well-known that Jensen's inequality is a part of the classical mathematical analysis [19], and it has played an important role in other fields of mathematics, such as probability theory, optimization, control theory, etc. The problem of the generalization of Jensen's inequalities is a contemporary issue. The Jensen inequality, despite being a classical result, continues to captivate the interest of many researchers. For an in-depth exploration of convexity and the Jensen inequality, encompassing proofs, generalizations, and a wide array of applications, readers should consult the monographs [15] and [18] and the extensive references contained within them. Jensen's inequality, a cornerstone in the study of convex functions, has seen numerous refinements and extensions that enhance its applicability across various fields. These refinements have not only provided deeper insights into the nature of convexity but have also led to innovative applications in areas such as probability theory, information theory, and mathematical economics. This article delves into the latest advancements in Jensen's inequality, exploring the theoretical improvements and their practical implications, showcasing how these developments continue to be a focal point of research and interest in the mathematical community. In [1], the authors introduce refined versions of Jensen's inequality that offer more accurate bounds than the traditional formulation. These refinements are particularly useful in cases where the classical inequality may not be sufficiently tight. In [11], Horváth introduces refined bounds for the integral form of Jensen's inequality. These refinements provide tighter and more accurate estimates, improving the applicability of the inequality in theoretical and practical problems. In [13], the authors explores extensions of Jensen's and Jensen-Steffensen's inequalities using the generalized majorization theorem and they delve into theoretical advancements concerning these fundamental inequalities in mathematical analysis.

Email addresses: **sarikayamz@gmail.com (Mehmet Zeki Sarıkaya)**

The theory of convex functions is a crucial area of mathematics that has applications in a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. This theory is also highly relevant in other areas of science, such as economics, finance, engineering, and management sciences. In [14], C. Niculescu and L.E. Persson provides a comprehensive exploration of convex functions and their diverse applications. This book covers theoretical foundations, properties, and various practical uses of convex functions in mathematics, optimization, economics, and other fields. One of the most well-known inequalities in the literature is the Hermite-Hadamard integral inequality (see, [5]), which is a fundamental tool for studying the properties of convex functions.

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f: I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.
Suppose that $f:[a,b]\to R$ is a convex function on $[a,b]$. Then we have the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2} \end{aligned} \tag{1.3}$$

The third inequality in (1.3) is known in the literature as Bullen's inequality. Bullen's inequality corresponds to a convex combination of the midpoint and the trapezoidal rules. Bullen's inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [2],[3], [4], [6]-[10], [12], [17], [20]-[24]).

The main aim of the present paper is to establish a different refinement of the Jensen inequality for convex functions. Subsequently, several inequalities related to Bullen's inequality and Ostrowski's inequality are derived using the Jensen inequality. Additionally, some applications for mean inequalities are provided for these inequalities.

2. Main Results

To prove our main results, we start the following theorem:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ with $a<b$. Then the following inequaties hold:*

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) &\leq \frac{x-a}{b-a}f\left(\frac{1}{x-a}\int_a^x g(t)\,dt\right)+\frac{b-x}{b-a}f\left(\frac{1}{b-x}\int_x^b g(t)\,dt\right) \\ &\leq \frac{1}{b-a}\int_a^b f(g(t))\,dt \\ &\leq f(x)-x\frac{f(b)-f(a)}{b-a}+\frac{f(x)-f(a)}{(x-a)(b-a)}\int_a^x g(t)\,dt+\frac{f(b)-f(x)}{(b-x)(b-a)}\int_x^b g(t)\,dt. \end{aligned} \tag{2.1}$$

Proof. Since f is a convex mappings, we have

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) &= f\left(\frac{x-a}{b-a}\frac{1}{x-a}\int_a^x g(t)\,dt+\frac{b-x}{b-a}\frac{1}{b-x}\int_x^b g(t)\,dt\right) \\ &\leq \frac{x-a}{b-a}f\left(\frac{1}{x-a}\int_a^x g(t)\,dt\right)+\frac{b-x}{b-a}f\left(\frac{1}{b-x}\int_x^b g(t)\,dt\right) \\ &\leq \frac{x-a}{b-a}\frac{1}{x-a}\int_a^x f(g(t))\,dt+\frac{b-x}{b-a}\frac{1}{b-x}\int_x^b f(g(t))\,dt \\ &= \frac{1}{b-a}\int_a^b f(g(t))\,dt. \end{aligned} \tag{2.2}$$

By using convexity of f, we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(g(t))\,dt &= \frac{1}{b-a}\int_a^x f\left(\frac{x-g(t)}{x-a}a+\frac{g(t)-a}{x-a}x\right)dt+\frac{1}{b-a}\int_x^b f\left(\frac{b-g(t)}{b-x}x+\frac{g(t)-x}{b-x}b\right)dt \\ &\le \frac{1}{b-a}\int_a^x \left[\frac{x-g(t)}{x-a}f(a)+\frac{g(t)-a}{x-a}f(x)\right]dt+\frac{1}{b-a}\int_x^b \left[\frac{b-g(t)}{b-x}f(x)+\frac{g(t)-x}{b-x}f(b)\right]dt \\ &= \frac{xf(a)-af(x)}{b-a}+\frac{f(x)-f(a)}{(x-a)(b-a)}\int_a^x g(t)\,dt+\frac{bf(x)-xf(b)}{b-a}+\frac{f(b)-f(x)}{(b-x)(b-a)}\int_x^b g(t)\,dt \\ &= f(x)-x\frac{f(b)-f(a)}{b-a}+\frac{f(x)-f(a)}{(x-a)(b-a)}\int_a^x g(t)\,dt+\frac{f(b)-f(x)}{(b-x)(b-a)}\int_x^b g(t)\,dt.\end{aligned} \tag{2.3}$$

By using (2.2) and (2.3), we obtain desired equality (2.1). □

Corollary 2.2. *With the assumptions in Theorem 2.1, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{x-a}{b-a}f\left(\frac{x+a}{2}\right)+\frac{b-x}{b-a}f\left(\frac{x+b}{2}\right) \\ &\le \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{2}f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}.\end{aligned} \tag{2.4}$$

Proof. If we take $g(t)=t$ for all $t\in[a,b]$ in the inequality (2.1), we get desired inequality (2.4). □

Remark 2.3. *If we take $x=\frac{a+b}{2}$ in (2.4), then we have the Bullen's inequalities which is given in (1.3).*

Proposition 2.4. *Assume that $n,p\ge 1$ and $b>a>0$. Then*

$$\begin{aligned} A^n(a,b) &\le \frac{1}{2}\left[A^n(a,\frac{a+b}{2})+A^n(\frac{a+b}{2},b)\right] \\ &\le L_n^n(a,b)\le \frac{A^n(a,b)+A(a^n,b^n)}{2}\end{aligned}$$

where $A(a,b)=\frac{a+b}{2}$ and $L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\right)^{1/p}$.

Proof. The result is obtained from result in Corollary 2.2 by putting $f(t)=t^n$, $n\ge 1$, $t>0$ and $x=\frac{a+b}{2}$. □

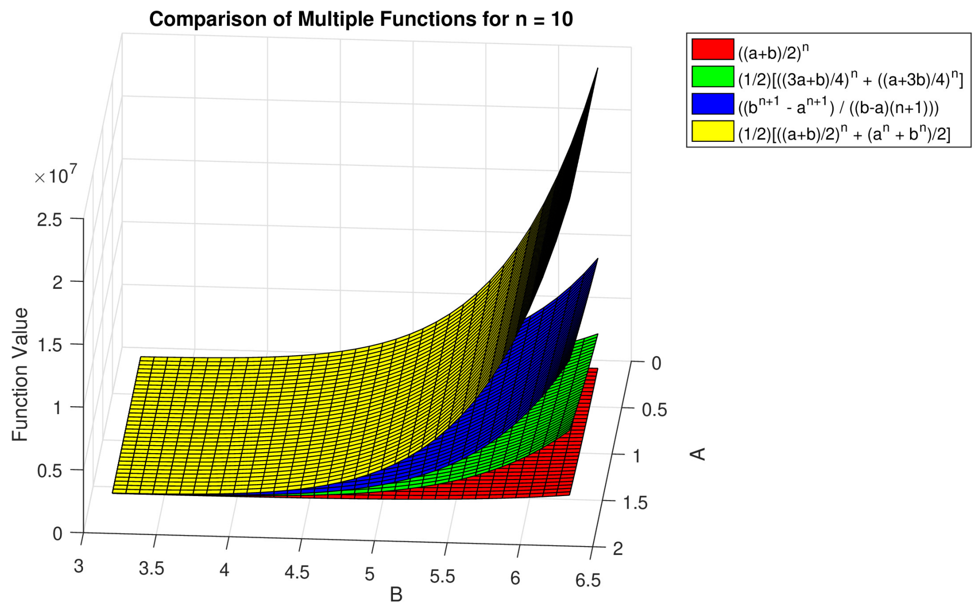

Figure 2.1: MATLAB has been used to compute and plot the graph of Proposition 2.4.

Proposition 2.5. *Assume that $b > a > 0$. Then*

$$\ln A^{-1}(a,b) \le \ln\left[A\left(a,\frac{a+b}{2}\right)A\left(\frac{a+b}{2},b\right)\right]^{-\frac{1}{2}} \quad \le \quad \ln I^{-1}(a,b) \le \ln\sqrt{\frac{A(a,b)}{G(a,b)}}.$$

where $G(a,b)=\sqrt{ab}$ and $I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$.

Proof. The result is obtained from result in Corollary 2.2 by putting $f(t)=-\ln t$, $t>0$ and $x=\frac{a+b}{2}$. □

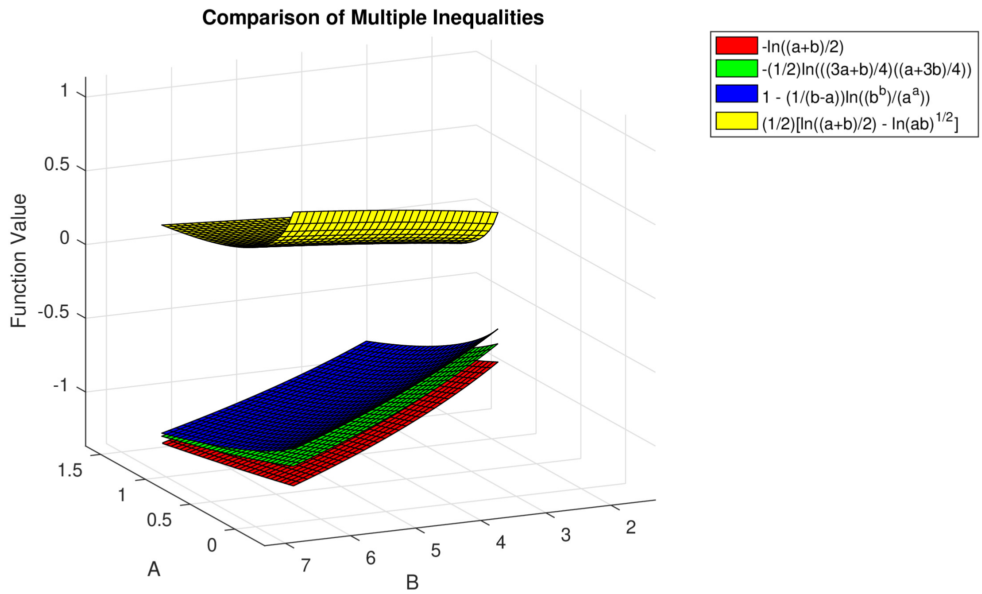

Figure 2.2: MATLAB has been used to compute and plot the graph of Proposition 2.5.

Corollary 2.6. *With the assumptions in Theorem 2.1, we have*

$$\begin{aligned} f\left(L_p^p(a,b)\right) &\le \frac{x-a}{b-a}f\left(L_p^p(a,x)\right)+\frac{b-x}{b-a}f\left(L_p^p(x,b)\right) \qquad (2.5)\\ &\le \frac{1}{b-a}\int_a^b f\left(t^p\right)dt \\ &\le f(x)\left(1+\frac{L_p^p(a,x)-L_p^p(x,b)}{b-a}\right)+\frac{1}{b-a}\left(x-L_p^p(a,x)\right)f(a)+\frac{1}{b-a}\left(L_p^p(x,b)-x\right)f(b) \end{aligned}$$

Proof. If we take $g(t)=t^p$, $p>1$ for all $t\in[a,b]$ in the inequality (2.1), we have

$$\begin{aligned} f\left(\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\right) &\le \frac{x-a}{b-a}f\left(\frac{x^{p+1}-a^{p+1}}{(x-a)(p+1)}\right)+\frac{b-x}{b-a}f\left(\frac{b^{p+1}-x^{p+1}}{(b-x)(p+1)}\right) \\ &\le \frac{1}{b-a}\int_a^b f\left(t^p\right)dt \\ &\le f(x)\left(1+\frac{1}{(p+1)(b-a)}\left[\frac{x^{p+1}-a^{p+1}}{(x-a)}-\frac{b^{p+1}-x^{p+1}}{(b-x)}\right]\right)+\left(\frac{x}{b-a}-\frac{\left(x^{p+1}-a^{p+1}\right)}{(p+1)(x-a)(b-a)}\right)f(a) \\ &\quad +\left(\frac{\left(b^{p+1}-x^{p+1}\right)}{(p+1)(b-x)(b-a)}-\frac{x}{b-a}\right)f(b) \end{aligned}$$

which implies desired inequality (2.5). □

Proposition 2.7. *Assume that $n,p\ge 1$ and $b>a>0$. Then*

$$\begin{aligned} L_p^{np}(a,b) \le \frac{1}{2}\left[L_p^{np}(a,\frac{a+b}{2}+L_p^{np}\left(\frac{a+b}{2},b\right)\right] \quad &\le \quad L_{np}^{np}(a,b) \le A^n(a,b)\left(1+\frac{4}{(p+1)(b-a)^2}\left[A^{p+1}(a,b)-A\left(a^{p+1},b^{p+1}\right)\right]\right) \\ &-nA(a,b)L_{n-1}^n(a,b)-\frac{4}{(p+1)(b-a)^2}\left[A^{p+1}(a,b)A\left(a^n,b^n\right)-A\left(a^{p+n+1},b^{p+n+1}\right)\right]. \end{aligned}$$

Proof. The result is obtained from result in Corollary 2.6 by putting $f(t)=t^n$, $n\geq 1$, $t>0$ and $x=\frac{a+b}{2}$. □

When we change the conditions of the functions f and g in Theorem 2.1, we get the following result:

Theorem 2.8. *Let $f:[a,b]\to\mathbb{R}$ be an increasing convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ be convex function on $[a,b]$ with $a<b$. Then the following inequaties hold:*

$$\begin{aligned}
f\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) &\leq \frac{x-a}{b-a}f\left(\frac{1}{x-a}\int_a^x g(t)\,dt\right)+\frac{b-x}{b-a}f\left(\frac{1}{b-x}\int_x^b g(t)\,dt\right)\\
&\leq \frac{1}{b-a}\int_a^b f(g(t))\,dt \qquad (2.6)\\
&\leq \frac{1}{b-a}\int_a^x f\left(\frac{x-t}{x-a}g(a)+\frac{t-a}{x-a}g(x)\right)dt+\frac{1}{b-a}\int_x^b f\left(\frac{b-t}{b-x}g(x)+\frac{t-x}{b-x}g(b)\right)dt\\
&\leq \frac{1}{2}f(g(x))+\frac{(x-a)f(g(a))+(b-x)f(g(b))}{2(b-a)}.
\end{aligned}$$

Proof. Since f is an increasing function and by using convexity of g and f, we get

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(g(t))\,dt &= \frac{1}{b-a}\int_a^x f\left(g\left(\frac{x-t}{x-a}a+\frac{t-a}{x-a}x\right)\right)dt+\frac{1}{b-a}\int_x^b f\left(g\left(\frac{b-t}{b-x}x+\frac{t-x}{b-x}b\right)\right)dt\\
&\leq \frac{1}{b-a}\int_a^x f\left(\frac{x-t}{x-a}g(a)+\frac{t-a}{x-a}g(x)\right)dt+\frac{1}{b-a}\int_x^b f\left(\frac{b-t}{b-x}g(x)+\frac{t-x}{b-x}g(b)\right)dt\\
&\leq \frac{1}{b-a}\int_a^x \frac{x-t}{x-a}f(g(a))\,dt+\frac{1}{b-a}\int_a^x \frac{t-a}{x-a}f(g(x))\,dt+\frac{1}{b-a}\int_x^b \frac{b-t}{b-x}f(g(x))\,dt+\frac{1}{b-a}\int_x^b \frac{t-x}{b-x}f(g(b))\,dt\\
&= \frac{1}{2}f(g(x))+\frac{(x-a)f(g(a))+(b-x)f(g(b))}{2(b-a)}.
\end{aligned}$$

By using (2.2) and (2.7), we obtain desired equality (2.6). □

Corollary 2.9. *With the assumptions in Theorem 2.8, we have*

$$\begin{aligned}
f\left(L_p^p(a,b)\right) &\leq \frac{x-a}{b-a}f\left(L_p^p(a,x)\right)+\frac{b-x}{b-a}f\left(L_p^p(x,b)\right)\\
&\leq \frac{1}{b-a}\int_a^b f(t^p)\,dt\\
&\leq \frac{x-a}{(b-a)(x^p-a^p)}\int_{a^p}^{x^p} f(t)\,dt+\frac{b-x}{(b-a)(b^p-x^p)}\int_{x^p}^{b^p} f(t)\,dt\\
&\leq \frac{1}{2}f(x^p)+\frac{(x-a)f(a^p)+(b-x)f(b^p)}{2(b-a)}.
\end{aligned}$$

Proof. If we take $g(t)=t^p$, $p>1$ for all $t\in[a,b]$ in the inequality (2.6), we have

$$\begin{aligned}
f\left(\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\right) &\leq \frac{x-a}{b-a}f\left(\frac{x^{p+1}-a^{p+1}}{(x-a)(p+1)}\right)+\frac{b-x}{b-a}f\left(\frac{b^{p+1}-x^{p+1}}{(b-x)(p+1)}\right)\\
&\leq \frac{1}{b-a}\int_a^b f(t^p)\,dt\\
&\leq \frac{1}{b-a}\frac{x-a}{x^p-a^p}\int_{a^p}^{x^p} f(t)\,dt+\frac{1}{b-a}\frac{b-x}{b^p-x^p}\int_{x^p}^{b^p} f(t)\,dt\\
&\leq \frac{1}{2}f(x^p)+\frac{(x-a)f(a^p)+(b-x)f(b^p)}{2(b-a)}
\end{aligned}$$

which implies desired inequality (2.7). □

Proposition 2.10. *Assume that $n,p\geq 1$ and $b>a>0$. Then*

$$\begin{aligned}
L_p^{np}(a,b)\leq\frac{1}{2}\left[L_p^{np}(a,\frac{a+b}{2})+L_p^{np}(\frac{a+b}{2},b)\right] &\leq L_{np}^{np}(a,b)\leq\frac{1}{2}\left[L_n^n\left(a^p,\left(\frac{a+b}{2}\right)^p\right)+L_n^n\left(\left(\frac{a+b}{2}\right)^p,b^p\right)\right]\\
&\leq \frac{A^{np}(a,b)+A(a^{np},b^{np})}{2}.
\end{aligned}$$

Proof. The result is obtained from result in Corollary 2.9 by putting $f(t) = t^n$, $n \geq 1$, $t > 0$ and $x = \frac{a+b}{2}$. $\square$

Let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $g \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|g'(t)| \leq M$, then the following inequality

$$\left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(t - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{2.7}$$

holds. This result is known in the literature as the Ostrowski inequality [16].
A function $f : J \to \mathbb{R}$ is said to satisfy a Lipschitz condition on the interval $J \subseteq \mathbb{R}$ if there exists a constant $L > 0$ such that the inequality

$$|f(b) - f(a)| \leq L|b-a|$$

holds for all $a,b \in J$. We also say that f is L-Lipschitzian (see, [18]). For example, since $b^2 - a^2 = |b+a||b-a| \leq 2max\{|a|,|b|\}|b-a|$, it follows that $f(t) = t^2$ is 2-Lipschitzian function on the unit interval.

Theorem 2.11. *Let $f : [0,\infty) \to \mathbb{R}$ be a L-Lipschitzian mapping and let $g : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $g' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|g'(t)| \leq M$, then the following inequality holds*

$$\left| \frac{1}{b-a}\int_a^b f(g(t))\,dt - f\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right) \right| \leq \frac{(b-a)}{3}ML. \tag{2.8}$$

Proof. Since f is a Lipshitzian function and using $|g'(t)| \leq M$ on $[a,b]$, it follows that

$$\begin{aligned}
\left| \frac{1}{b-a}\int_a^b f(g(t))\,dt - f\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right) \right| &\leq \frac{1}{b-a}\int_a^b \left| f(g(t)) - f\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right) \right| dt \\
&\leq \frac{L}{b-a}\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt \\
&\leq \frac{L}{b-a}\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt
\end{aligned}$$

Since $|g'(t)| \leq M$, by using (2.7), we have

$$\left| \frac{1}{b-a}\int_a^b f(g(t))\,dt - f\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right) \right| \leq \frac{(b-a)}{3}ML$$

which implies desired inequality (2.8). $\square$

Corollary 2.12. *With the assumptions in Theorem 2.11, we have*

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{3}L. \tag{2.9}$$

Proof. If we take $g(t) = t$ for all $t \in [a,b]$ in the inequality (2.8), we get desired inequality (2.9). $\square$

Proposition 2.13. *Let $b > a > 0$. Then*

$$\left| \frac{\sin b - \sin a}{b-a} - \cos\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{3}.$$

Proof. Since $|cosy - cosx| \leq |y-x|$, for all $x,y \in R$, by the Lagrange mean value theorem, it follows that the cosine function is 1-Lipschitzian on R. Hence, the inequality (2.9) get

$$\left| \frac{\sin b - \sin a}{b-a} - \cos\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{3}.$$

The desired result can now be obtained. $\square$

3. Conclusion

In conclusion, this paper successfully introduces a new refinement of the Jensen inequality for convex functions. Through this refinement, several significant inequalities related to Bullen's inequality and Ostrowski's inequality are derived. The practical applications of these inequalities in the context of mean inequalities are also explored, demonstrating their broader utility and relevance. The findings enhance our understanding of these mathematical concepts, offering new perspectives and potential for further research in this field.

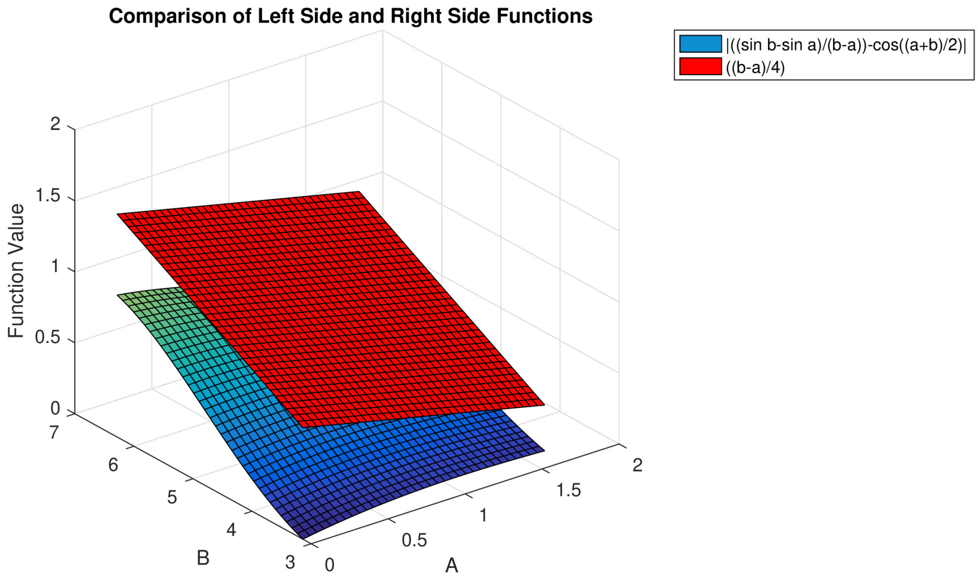

Figure 2.3: MATLAB has been used to compute and plot the graph of Proposition 2.13.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

References

[1] S. Abramovich, G. Jameson and G. Sinnamon, Refining Jensen's inequality. Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, 2004, 3-14.
[2] M. Çakmak, The differentiable h-convex functions involving the Bullen inequality, Acta Univ. Apulensis 2021, 65, 29–36.
[3] M. Çakmak, On some Bullen-type inequalities via conformable fractional integrals, J. Sci. Perspect. 3(4), 285–298, 2019.
[4] S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value, Tbilisi Mathematical Journal 7(1) (2014), pp. 1–17.
[5] S. S. Dragomir and C. E. M. Pearce, Selected topics on Hermite–Hadamard inequalities and applications, RGMIA Monographs, Victoria University, 2000.
[6] T. Du, C. Luo and Z. Cao, On the Bullen-type inequalities via generalized fractional integrals and their applications, Fractals, 29(07), 2150188, 2021.
[7] S. Erden and M. Z. Sarikaya, Generalized Bullen type inequalities for local fractional integrals and its applications, Palestine Journal of Mathematics, 9(2), 2020.
[8] A. Fahad, S. I. Butt, B. Bayraktar, M. Anwar and Y. Wang, Some new Bullen-type inequalities obtained via fractional integral operators. Axioms, 12(7), 691, 2023.
[9] F. Hezenci, H. Budak and H. Kara, A study on conformable fractional version of Bullen-type inequalities. Turkish Journal of Mathematics, 47(4), 1306-1317, 2023.
[10] S. Hussain and S. Mehboob, On some generalized fractional integral Bullen type inequalities with applications, J. Fract. Calc. Nonlinear Syst. 2021, 2, 93–112.
[11] L Horváth, A refinement of the integral form of Jensen's inequality. Journal of Inequalities and Applications, 2012, 1-19, 2012.
[12] I. Iscan, T. Toplu and F. Yetgin, Some new inequalities on generalization of Hermite–Hadamard and Bullen type inequalities, applications to trapezoidal and midpoint formula, Kragujev. J. Math. 45(4), 647–657, 2021.
[13] M.A Khan, J. Khan and J. Pečarić, Generalization of Jensen's and Jensen-Steffensen's inequalities by generalized majorization theorem. J. Math. Inequal, 11(4), 2017, 1049-1074.
[14] C. Niculescu and L.E. Persson, Convex functions and their applications (Vol. 23). New York: Springer, 2006.
[15] D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink, Classical and New Inequalities in Analysis, Mathematics and its Applications (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993.
[16] A. Ostrowski, Über die Absolutabweichung einer differenzierbaren Funktion von ihrem Integralmittelwert, Comment. Math. Helv., 10(1) (1937), 226227.
[17] H. Öğülmüş and M. Z. Sarikaya, Some Hermite–Hadamard type inequalities for h-convex functions and their applications. Iranian Journal of Science and Technology, Transactions A: Science, 44, 813-819, (2020).

[18] J. E. Pecaric, F. Proschan, and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
[19] H. L. Royden, Real analysis, Macmillan, New York, 1988.
[20] M. Z. Sarikaya, On the some generalization of inequalities associated with Bullen, Simpson, midpoint and trapezoid type, Acta Universitatis Apulensis: Mathematics-Informatics, 73, (2023).
[21] M. Z. Sarikaya and N. Aktan, On the generalization of some integral inequalities and their applications, Mathematical and computer Modelling, 54(9-10), 2175-2182, (2011).
[22] B.N. Yasar, N. Aktan and G.Ç. Kizilkan, Generalization of Bullen type, trapezoid type, midpoint type and Simpson type inequalities for s-convex in the fourth sense, Turk. J. Inequal. 2022, 6, 40–51.
[23] J. Wang, C. Zhu and Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, Journal of Inequalities and Applications, 2013(1), 1-15, (2013).
[24] B. Y. Xi and F. Qi, Some Hermite-Hadamard type inequalities for differentiable convex functions and applications, Hacettepe Journal of Mathematics and Statistics, 42(3), 243-257, 2013.