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Generalized Bullen Type Inequalities and Their Applications

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Abstract

This paper presents a novel extension of Bullen-type inequalities for convex functions by leveraging recently established generalized identities. Through rigorous proofs, we derive new inequalities that exhibit strong connections to both the left- and right-hand sides of the Hermite-Hadamard inequalities for Riemann-integrable functions. Additionally, we apply these results to various special means of two positive numbers.

1. Introduction

Definition 1.1. *A function* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is said to be* convex *if the following inequality holds:*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ *and* $\lambda\in[0,1]$. *The function* f *is called* concave *if* $(-f)$ *is convex.*

The theory of convex functions is a fundamental area of mathematics with applications across a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. It is also highly relevant in other scientific disciplines such as economics, finance, engineering, and management sciences.

One of the most well-known results in this area is the Hermite–Hadamard integral inequality (see [1]), which serves as a fundamental tool for studying the behaviour of convex functions. This inequality has far-reaching implications and has been the subject of extensive research in recent years, giving rise to new and powerful mathematical techniques for addressing a broad spectrum of problems. The literature contains numerous extensions and refinements of this inequality (see [2]–[13]).

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on an interval I and $a,b\in I$ with $a<b$.
Suppose that $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$. Then the following chain of inequalities holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

The third inequality in (1.2) is commonly known as *Bullen's inequality*.

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These inequalities were first introduced independently by Charles Hermite and Jacques Hadamard in the late 19th century, and they have since found numerous applications in analysis, geometry, and probability theory. The Hermite–Hadamard inequalities state that if a function is convex on a closed interval, then the average value of the function over that interval lies between its value at the midpoint and the average of its values at the endpoints.

These inequalities serve as powerful tools for estimating integrals and are foundational results in the theory of convex functions. They have been applied in a variety of contexts, including integral calculus, probability theory, statistics, optimization, and number theory. Moreover, they are instrumental in solving physical and engineering problems where determining the average value of a function is required.

Hadamard's inequality, in particular, is widely applied and carries significant geometric interpretations. Bullen's inequality, on the other hand, can be interpreted as a convex combination of the midpoint and trapezoidal rules for numerical integration. This inequality has also been extensively investigated in the literature, leading to various generalizations and a rich body of related research (see [14]–[25]).

In this paper, we present a new extension of Bullen-type inequalities for convex functions by utilizing recently established generalized identities. Through rigorous proofs, we establish inequalities that are strongly connected to both sides of the Hermite-Hadamard inequalities for Riemann-integrable functions. Furthermore, we demonstrate the applicability of these inequalities to various special means of two positive numbers.

2. Main Results

To prove our main results, we require the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$, and suppose that $f' \in L[a,b]$. Then the following identity holds:*

$$\frac{1}{2}\int_a^b K(x,t) f''(t)\,dt = f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right] \tag{2.1}$$

where

$$K(x,t) = \begin{cases} \frac{1}{x-a}(x-t)(t-a) & \text{for } a \le t < x, \\ \\ \frac{1}{b-x}(t-x)(b-t) & \text{for } x \le t \le b. \end{cases}$$

Proof. By integration by parts, we have

$$\begin{aligned}
\int_a^b K(x,t) f''(t)\,dt &= \frac{1}{x-a}\int_a^x (x-t)(t-a) f''(t)\,dt + \frac{1}{b-x}\int_x^b (t-x)(b-t) f''(t)\,dt \\
&= \frac{1}{x-a}\left[(x-t)(t-a)f'(t)\right]_a^x - \frac{1}{x-a}\int_a^x (a+x-2t) f'(t)\,dt \\
&\quad + \frac{1}{b-x}\left[(t-x)(b-t)f'(t)\right]_x^b - \frac{1}{b-x}\int_x^b (b+x-2t) f'(t)\,dt \\
&= -\frac{1}{x-a}\left[(a+x-2t)f(t)\right]_a^x - \frac{2}{x-a}\int_a^x f(t)\,dt \\
&\quad - \frac{1}{b-x}\left[(b+x-2t)f(t)\right]_x^b - \frac{2}{b-x}\int_x^b f(t)\,dt \\
&= 2f(x) + f(a) + f(b) - \frac{2}{x-a}\int_a^x f(t)\,dt - \frac{2}{b-x}\int_x^b f(t)\,dt.
\end{aligned}$$

Multiplying both sides by $\frac{1}{2}$ yields the desired identity (2.1). □

Remark 2.2. *In Lemma 2.1, if we choose $x = \frac{a+b}{2}$, then identity (2.1) becomes:*

$$\frac{1}{2(b-a)}\int_a^b K(t) f''(t)\,dt = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt$$

where

$$K(t) = \begin{cases} \left(\frac{a+b}{2} - t\right)(t-a) & \text{for } a \le t < \frac{a+b}{2}, \\ \\ \left(t - \frac{a+b}{2}\right)(b-t) & \text{for } \frac{a+b}{2} \le t \le b. \end{cases}$$

Theorem 2.3. *Under the assumptions of Lemma 2.1, if $|f''|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right| \tag{2.2}$$

$$\leq \frac{(x-a)^2+(b-x)^2}{24}\left|f''(x)\right|+\frac{(x-a)^2\,|f''(a)|+(b-x)^2\,|f''(b)|}{24},$$

for all $x\in(a,b)$.

Proof. Since $|f''|$ is convex on $[a,b]$, we have

$$\left|f''(t)\right|\leq\frac{t-a}{x-a}\left|f''(x)\right|+\frac{x-t}{x-a}\left|f''(a)\right|,\quad \text{for } t\in[a,x],$$

and

$$\left|f''(t)\right|\leq\frac{b-t}{b-x}\left|f''(x)\right|+\frac{t-x}{b-x}\left|f''(b)\right|,\quad \text{for } t\in[x,b].$$

Taking the absolute value of both sides in equation (2.1), and applying the convexity estimates, we obtain:

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right|\\
&\leq\frac{1}{2(x-a)}\int_a^x(x-t)(t-a)\left|f''(t)\right|dt+\frac{1}{2(b-x)}\int_x^b(t-x)(b-t)\left|f''(t)\right|dt\\
&\leq\frac{|f''(x)|}{2(x-a)^2}\int_a^x(x-t)(t-a)^2dt+\frac{|f''(a)|}{2(x-a)^2}\int_a^x(x-t)^2(t-a)dt\\
&\quad+\frac{|f''(x)|}{2(b-x)^2}\int_x^b(t-x)(b-t)^2dt+\frac{|f''(b)|}{2(b-x)^2}\int_x^b(t-x)^2(b-t)dt.
\end{aligned}$$

Evaluating the integrals and simplifying yields:

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right|\leq\frac{(x-a)^2+(b-x)^2}{24}\left|f''(x)\right|+\frac{(x-a)^2\,|f''(a)|+(b-x)^2\,|f''(b)|}{24}.$$

□

Remark 2.4. *If we choose $x=\frac{a+b}{2}$ in Theorem 2.3, then inequality (2.2) becomes:*

$$\begin{aligned}
\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{(b-a)^2}{2}\left[\frac{1}{48}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{|f''(a)|+|f''(b)|}{96}\right]\\
&\leq \frac{(b-a)^2}{48}\left[\frac{|f''(a)|+|f''(b)|}{2}\right],
\end{aligned}$$

which coincides with the result previously obtained by Sarikaya and Aktan in [23].

Theorem 2.5. *Under the assumptions of Lemma 2.1, if $|f''|^q$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right| \tag{2.3}$$

$$\leq \frac{1}{2}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\cdot B^{\frac{1}{p}}(1+p,1+p)\cdot\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}},$$

for all $x\in(a,b)$.

Proof. Taking the absolute value of identity (2.1) and applying Hölder's integral inequality, together with the convexity of $|f''|^q$, we obtain:

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right|\\
&\leq\frac{1}{2}\left(\int_a^b|K(x,t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f''(t)\right|^q dt\right)^{\frac{1}{q}},\quad \text{where } \frac{1}{p}+\frac{1}{q}=1.\\
&\leq\frac{1}{2}\left(\frac{1}{(x-a)^p}\int_a^x(x-t)^p(t-a)^p\,dt+\frac{1}{(b-x)^p}\int_x^b(t-x)^p(b-t)^p\,dt\right)^{\frac{1}{p}}\times\left(\int_a^b\left[\frac{t-a}{b-a}\left|f''(b)\right|^q+\frac{b-t}{b-a}\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&=\frac{1}{2}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}B^{\frac{1}{p}}(1+p,1+p)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 2.6. *Under the assumptions of Theorem 2.5, if we take $x=\frac{a+b}{2}$ in inequality (2.3), then we obtain:*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^{\frac{1}{p}+1}}{8}B^{\frac{1}{p}}(1+p,1+p)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.4}$$

3. Applications

As in [1], we consider the means for arbitrary real numbers $a,b\in\mathbb{R}^+$ with $a\neq b$. The following classical means are defined as:

- $A(a,b)=\dfrac{a+b}{2}$ (Arithmetic Mean)
- $H(a,b)=\dfrac{2ab}{a+b}$ (Harmonic Mean)
- $K(a,b)=\sqrt{\dfrac{a^2+b^2}{2}}$ (Quadratic Mean)
- $G(a,b)=\sqrt{ab}$ (Geometric Mean)
- $L(a,b)=\dfrac{b-a}{\ln b-\ln a}$ (Logarithmic Mean)
- $I(a,b)=\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$ (Identric Mean)
- $L_n(a,b)=\left(\dfrac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)^{1/n}$, $n\in\mathbb{R}\setminus\{-1,0\}$ (Generalized Logarithmic Mean)

The following inequality among classical means is well known in the literature:

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b).$$

Proposition 3.1. *Assume that $n>3$ and $b>a>0$. Then the following inequality holds:*

$$\left|\frac{A^n(a,b)+A(a^n,b^n)}{2}-L_n^n(a,b)\right|\leq n(n-1)\frac{(b-a)^2}{48}\cdot\frac{A^{n-2}(a,b)+A(a^{n-2},b^{n-2})}{2}\leq n(n-1)\frac{(b-a)^2}{48}A(a^{n-2},b^{n-2}).$$

Proof. The result follows from Theorem 2.3 by choosing $x=\dfrac{a+b}{2}$ and $f(t)=t^n$ for $t>0$. Then we have:

$$f''(t)=n(n-1)t^{n-2}.$$

Since

$$\left(\left|f''(t)\right|\right)''=n(n-1)(n-2)(n-3)t^{n-4},$$

it follows that $|f''(t)|=n(n-1)t^{n-2}$ is convex on $[a,b]$ for $n>3$, and the inequality is obtained directly from Theorem 2.3. □

Proposition 3.2. *Let $b>a>0$. Then the following inequality holds:*

$$\left|\frac{\ln A(a,b)+A(\ln a,\ln b)}{2}-\ln I(a,b)\right|\leq\frac{(b-a)^2}{96}\left[A^{-2}(a,b)+H^{-1}(a^2,b^2)\right]\leq\frac{(b-a)^2}{48}\cdot H^{-1}(a^2,b^2).$$

Proof. This result also follows from Theorem 2.3 with $x=\dfrac{a+b}{2}$ and $f(t)=\ln t$ for $t>0$. Then,

$$f''(t)=-\frac{1}{t^2},\quad\text{and hence}\quad\left|f''(t)\right|=\frac{1}{t^2},$$

which is convex on $[a,b]$.
From Theorem 2.3, we obtain:

$$\left|\frac{1}{2}\left[\ln\left(\frac{a+b}{2}\right)+\frac{\ln a+\ln b}{2}\right]-\frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)+1\right|\leq\frac{(b-a)^2}{96}\left[\left(\frac{2}{a+b}\right)^{-2}+\frac{a^2+b^2}{2a^2b^2}\right]\leq\frac{(b-a)^2}{48}\cdot\frac{a^2+b^2}{2a^2b^2}.$$

The desired result follows immediately from simplification of the terms on the right-hand side. □

Proposition 3.3. *Let $b > a > 0$. Then*

$$\left|\frac{A^{-1}(a,b)+H^{-1}(a,b)}{2}-L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{48}\left[A^{-3}(a,b)+H^{-1}(a^3,b^3)\right] \leq \frac{(b-a)^2}{24}H^{-1}(a^3,b^3).$$

Proof. The result is derived from Theorem 2.3 by choosing $x=\frac{a+b}{2}$ and setting $f(t)=\frac{1}{t}$ for $t>0$. Then,

$$f''(t)=\frac{2}{t^3}, \quad \text{so} \quad |f''(t)|=\frac{2}{t^3},$$

which is convex on $[a,b]$.
Thus,

$$\left|\frac{1}{2}\left[\frac{2}{a+b}+\frac{a+b}{2ab}\right]-\frac{\ln b-\ln a}{b-a}\right| \leq \frac{(b-a)^2}{48}\left[\left(\frac{a+b}{2}\right)^{-3}+\frac{a^3+b^3}{2a^3b^3}\right] \leq \frac{(b-a)^2}{24}\cdot\frac{a^3+b^3}{2a^3b^3}.$$

This completes the proof. □

Proposition 3.4. *Assume $n>2$, $q>1$, and $(n-2)q>1$ with $b>a>0$. Then*

$$\left|\frac{A^n(a,b)+A(a^n,b^n)}{2}-L_n^n(a,b)\right| \leq n(n-1)\cdot\frac{(b-a)^{\frac{1}{p}+1}}{8}\cdot B^{\frac{1}{p}}(1+p,1+p)\cdot A^{\frac{1}{q}}(a^{(n-2)q},b^{(n-2)q}).$$

Proof. This result follows from Corollary 2.6 by taking $f(t)=t^n$ for $t>0$, so that

$$f''(t)=n(n-1)t^{n-2}.$$

Then

$$\left(\left|f''(t)\right|^q\right)''=|n(n-1)|^q\cdot(n-2)q\cdot((n-2)q-1)\,t^{(n-2)q-2},$$

which shows that $|f''(t)|^q=|n(n-1)|^q\cdot t^{(n-2)q}$ is convex on $[a,b]$ under the assumption $(n-2)q>1$. The result then follows directly from Corollary 2.6. □

Proposition 3.5. *Let $q>1$ and $b>a>0$. Then*

$$\left|\frac{\ln A(a,b)+A(\ln a,\ln b)}{2}-\ln I(a,b)\right| \leq \frac{(b-a)^{\frac{1}{q}+1}}{8}\cdot B^{\frac{1}{q}}(1+q,1+q)\cdot H^{-\frac{1}{q}}(a^q,b^q).$$

Proof. The result follows from Corollary 2.6 by choosing $f(t)=\ln t$, so that

$$f''(t)=-\frac{1}{t^2}, \quad \text{and} \quad |f''^q=\frac{1}{t^{2q}}.$$

Since$|f''(t)|^q$ is convex on $[a,b]$ for $q>1$, the inequality

$$\left|\frac{1}{2}\left[\ln\left(\frac{a+b}{2}\right)+\frac{\ln a+\ln b}{2}\right]-\frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)+1\right| \leq \left(\frac{a^{2q}+b^{2q}}{2a^{2q}b^{2q}}\right)^{1/q}$$

holds. The desired result is then obtained using the harmonic mean representation:

$$\left(\frac{a^{2q}+b^{2q}}{2a^{2q}b^{2q}}\right)^{1/q}=H^{-\frac{1}{q}}(a^q,b^q).$$

□

4. Conclusion

In conclusion, this paper presents novel extensions of Bullen-type inequalities and establishes their applicability to functions whose absolute value of the first derivative is convex. Our contributions build upon existing research, offering refined insights and analytical techniques that can be utilized across a broad range of mathematical and scientific problems. Future work may focus on further exploring the implications and potential applications of these extensions, which hold promise for advancing theoretical knowledge and fostering innovation in various disciplines. for future directions.

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