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ON THE EXTENSION PROBLEM FOR GYROGROUPS

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Abstract. A gyrogroup is an algebraic structure whose operation is in general non-associative and shares common properties with groups. In this paper, we introduce two disjoint families of gyrogroups. One family consists of gyrogroups whose operations are, in some sense, most far from being associative called contra-associative gyrogroups. The other family consists of gyrogroups that are, in some sense, most close to groups called g-extensive gyrogroups. We then describe their structural properties, which eventually lead to studying the extension problem for gyrogoups in detail using the notion of associators. In particular, we refine the hierarchy of gyrogroup structure by showing that generic gyrogroups are extensions of contra-associative gyrogroups or g-extensive gyrogroups.

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1. Introduction

A gyrogroup consists of one non-empty set, together with one binary operation, that satisfies some axioms similar to groups. In general, the operation of a generic gyrogroup is not associative. However, it satisfies a weaker form of associativity, the so-called gyroassociative law. Furthermore, the family of gyrogroups properly includes the family of groups, and so groups and gyrogroups have several common aspects; see, for instance, [1,3–8,11,16–19]. In [12], the author provides a constructive method to measure the deviation from associativity of gyrogroup operations by introducing the notion of associator normal subgyrogroups. It turns out that this notion can be used to examine the algebraic structure of a gyrogroup. This leads to a deep understanding of the gyrogroup structure, as shown in the present paper. In this work, we study two disjoint families of gyrogroups. One family contains gyrogroups whose operations are, in some sense, most far from being associative called contra-associative gyrogroups (see Section 3). The other family contains gyrogroups that are, in some sense, most close to groups called g-extensive gyrogroups (see Section 4). In particular, we prove some structural properties of gyrogroups in these families, which eventually lead to studying the extension problem for gyrogoups. We also show that a certain gyrogroup of order 8 can be constructed from the Klein 4-group and the cyclic group of order 2 and that a certain gyrogroup of order 15 can be constructed from the cyclic groups of order 3 and order 5 as an application of the obtaining results.

2. Preliminaries

The reader is referred to [9, 15] for the basic theory of gyrogroups. Terminology and basic definitions mentioned in this paper can be found in [12]. For the sake of completeness, the formal definition of a gyrogroup is presented here. Let G be a non-empty set, together with a binary operation $\oplus$ on G. The algebraic structure $(G, \oplus)$ is called a *gyrogroup* if the following properties are satisfied.

(1) G has a unique two-sided identity e: $a \oplus e = a$ and $e \oplus a = a$ for all $a \in G$.
(2) Each element of G has a unique two-sided inverse: if $a \in G$, then $\ominus a$ is the unique element of G such that $a \oplus (\ominus a) = e$ and $(\ominus a) \oplus a = e$.
(3) G satisfies the left and right *gyroassociative laws*: for all $a, b, c \in G$,
$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b](c)$$
and
$$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a](c)),$$
where $\mathrm{gyr}[a, b]$ denotes an automorphism of G called the *gyroautomorphism* generated by a and b.
(4) G has the *left loop property* and the *right loop property*: for all $a, b \in G$, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ and $\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$.

We summarize important results, proved in [12], that will be useful in studying gyrogroup structures in the sequel. Let G be a gyrogroup. For all $a, b, c \in G$, the *associator* of the triple (a, b, c) is defined by the formula
$$[a, b, c] = \ominus(a \oplus (b \oplus c)) \oplus ((a \oplus b) \oplus c). \tag{1}$$
Henceforward, we simply say the associator of a, b, and c. Let G^{a} be the normal closure of the set of all associators in G. In other words, G^{a} is the smallest normal subgyrogroup of G containing all the associators in G, called the *associator normal subgyrogroup* of G. Therefore, if N is a normal subgyrogroup of G containing all the associators in G, then $G^{\mathsf{a}} \subseteq N$ by the minimality of G^{a}. This fact will be used throughout the paper. The associator normal subgyrogroup, in some sense, measures the deviation from associativity of gyrogroup operations. Next, we quote

the following results, which will be used later on when we have examined structures of gyrogroups, for easy reference. Their proofs are presented in [12].

Proposition 2.1 (Proposition 3.2, [12]). *Let G be an arbitrary gyrogroup.*

(1) *Then, G/G^{a} is a group.*
(2) *If H is a subgyrogroup of G, then $H^{\mathsf{a}} \subseteq G^{\mathsf{a}}$.*

Proposition 2.2 (Proposition 3.4, [12]). *Let G be a gyrogroup. Then, G is a group under the same operation (or, alternatively, degenerate) if and only if $G^{\mathsf{a}} = \{e\}$.*

Proposition 2.3 (Proposition 3.5, [12]). *Let N be a normal subgyrogroup of a gyrogroup G. Then, G/N is a group if and only if $G^{\mathsf{a}} \subseteq N$.*

Proposition 2.4 (Proposition 3.6, [12]). *Let G be a gyrogroup. Then, G^{a} is the unique normal subgyrogroup of G such that G/G^{a} is a group and if φ is a homomorphism from G to a group, then $G^{\mathsf{a}} \subseteq \ker \varphi$.*

Proposition 2.5 (Proposition 3.7, [12]). *Let G and K be gyrogroups. If $\varphi : G \to K$ is a surjective homomorphism, then $\varphi(G^{\mathsf{a}}) = K^{\mathsf{a}}$.*

Let G be a gyrogroup. The *left nucleus* of G, denoted by $N_\ell(G)$, is defined as

$$N_\ell(G) = \{a \in G : a \oplus (b \oplus c) = (a \oplus b) \oplus c \text{ for all } b, c \in G\}. \tag{2}$$

It is proved in Theorem 3.8 of [10] that $N_\ell(G)$ forms a normal subgroup of G. Also, the *right nucleus* of G, denoted by $N_r(G)$, is defined as

$$N_r(G) = \{c \in G : a \oplus (b \oplus c) = (a \oplus b) \oplus c \text{ for all } a, b \in G\}. \tag{3}$$

The right nucleus of a certain gyrogroup is normal (see, for instance, Example 4.20). It is an open question whether the right nucleus of a gyrogroup is always normal in that gyrogroup. In certain circumstances, left and right nuclei can be used to analyze the algebraic structure of a gyrogroup, as we will see later.

3. Contra-associative gyrogroups

In this section, we study a family of gyrogroups that are defined using the notion of associator normal subgyrogroups. This family of gyrogroups, in some sense, contains building blocks of generic gyrogroups, as we will see shortly. First, let us introduce the formal definition of a contra-associative gyrogroup as follows.

Definition 3.1. A gyrogroup G is *contra-associative* if it is non-trivial and the associator normal subgyrogroup of G is G itself, that is, if $G \neq \{e\}$ and $G^{\mathsf{a}} = G$.

By definition, any contra-associative gyrogroup is never a group. More precisely, if a gyrogroup G is contra-associative, then $G^{\mathsf{a}} \neq \{e\}$, which implies that G is not a group by Proposition 2.2. In fact, we obtain the following proposition, which indicates that any contra-associative gyrogroup is most far from being a group.

Proposition 3.2. *Let G be a non-trivial gyrogroup. Then, the following statements are equivalent:*

(i) *G is contra-associative.*
(ii) *The associativization of G is trivial.*
(iii) *Every homomorphism from G to an arbitrary group is trivial.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows directly by definition. Suppose that G is contra-associative. Let Γ be a group, and let $\varphi : G \to \Gamma$ be a homomorphism. By Proposition 2.4, $G^{\mathsf{a}} \subseteq \ker\varphi$. By assumption, $G^{\mathsf{a}} = G$, which implies that $\ker\varphi = G$. Hence, φ is trivial. Suppose conversely that every homomorphism from G to a group is trivial. Hence, the canonical projection $\pi : G \to G/G^{\mathsf{a}}$ given by $\pi(a) = a \oplus G^{\mathsf{a}}$ is trivial. This implies that $a \oplus G^{\mathsf{a}} = e \oplus G^{\mathsf{a}}$ for all $a \in G$. It follows that $a \in G^{\mathsf{a}}$ for all $a \in G$. This shows that $G^{\mathsf{a}} = G$, which proves the equivalence (i) $\Leftrightarrow$ (iii). □

In general, subgyrogroups of a contra-associative gyrogroup need not be contra-associative. In fact, if G is a contra-associative gyrogroup, then any non-trivial subgroup of G is proper in G but is not contra-associative. Furthermore, every non-trivial gyrogroup contains at least one non-trivial subgroup, which is a cyclic subgyrogroup generated by a non-identity element (see Theorem 23 of [9]). Proposition 3.3 states that any homomorphic image of a contra-associative gyrogroup is contra-associative. Consequently, the property of being contra-associative becomes an invariant property of gyrogroups, as stated in Corollary 3.4. Moreover, Corollary 3.5 shows that the quotient of a contra-associative gyrogroup by any normal subgyrogroup is contra-associative and Corollary 3.6 shows that if an arbitrary direct product of gyrogroups is contra-associative, then each of its components must be contra-associative as well.

Proposition 3.3. *Let G and K be gyrogroups, and suppose that $\varphi : G \to K$ is a surjective homomorphism. If G is contra-associative, then so is K.*

Proof. Suppose that G is contra-associative. Then, by definition, $G^{\mathsf{a}} = G$. By Proposition 2.5, $K = \varphi(G) = \varphi(G^{\mathsf{a}}) = K^{\mathsf{a}}$. Hence, K is contra-associative. □

Corollary 3.4. *Let G and K be gyrogroups. If $G \cong K$, then G is contra-associative if and only if K is contra-associative.*

Corollary 3.5. *Let G be a gyrogroup, and let $N \trianglelefteq G$. If G is contra-associative, then so is G/N.*

Proof. Let $\pi_N : G \to G/N$ be the canonical projection. Then, π_N defines a surjective homomorphism from G to G/N, and so Proposition 3.3 applies. □

Corollary 3.6. *Let $\{G_i : i \in I\}$ be an indexed family of gyrogroups with $I \neq \emptyset$. If $\prod_{i\in I} G_i$ is contra-associative, then G_i is contra-associative for all $i \in I$.*

Proof. Set $G = \prod_{i\in I} G_i$. Suppose that G is contra-associative. Let $i \in I$. Define a map p_i by $p_i(f) = f(i)$ for all $f \in G$. Then, p_i is a surjective homomorphism from G to G_i. Hence, by Proposition 3.3, G_i is contra-associative. □

Recall that a (finite or infinite) gyrogroup G is said to be *simple* if G is non-trivial and the only normal subgyrogroups of G are $\{e\}$ and G itself (see Definition 14 of [9]). The following proposition states that the family of non-degenerate simple gyrogroups is included in the family of contra-associative gyrogroups.

Proposition 3.7. *If G is a non-degenerate simple gyrogroup, then G is contra-associative.*

Proof. Suppose that G is non-degenerate and simple. Hence, $G \neq \{e\}$ and $G^{\mathsf{a}} \neq \{e\}$. Since $G^{\mathsf{a}} \trianglelefteq G$, it follows that $G^{\mathsf{a}} = G$. Thus, G is contra-associative by definition. □

We complete this section by giving a concrete example of a contra-associative gyrogroup in the following example.

Example 3.8. Recall that the (complex) *Möbius gyrogroup* consists of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, together with *Möbius addition*, denoted by $\oplus_M$, defined by the formula $a \oplus_M b = \dfrac{a+b}{1+\overline{a}b}$ for all $a, b \in \mathbb{D}$. It is proved in [2] that every element of the Möbius gyrogroup can be expressed as an associator. This implies that the associator normal subgyrogroup of the Möbius gyrogroup is the Möbius gyrogroup itself, and so the Möbius gyrogroup is contra-associative.

4. G-extensive gyrogroups and their extensions

In this section, we are going to study the extension problem of gyrogroups by exploring another family of gyrogroups that, in some sense, are most close to groups. We may generalize the definition of an extension of a group by a group to the case of gyrogroups in the same way as follows. A gyrogroup G is an *extension* of a gyrogroup X by a gyrogroup Y if there exists a normal subgyrogroup N of G such that $N \cong Y$ and $G/N \cong X$. The presence of associator normal subgyrogroups shows that every gyrogroup is an extension of a group by a gyrogroup; that is, the following is a short exact sequence:

$$\{e\} \longrightarrow G^{\mathsf{a}} \xrightarrow{\iota} G \xrightarrow{\pi} G/G^{\mathsf{a}} \longrightarrow \{e\}, \tag{4}$$

where ι is the inclusion map and π is the canonical projection. Moreover, the presence of left nuclei shows that every gyrogroup is an extension of a gyrogroup by a group; that is, the following is a short exact sequence:

$$\{e\} \longrightarrow N_\ell(G) \xrightarrow{\iota} G \xrightarrow{\pi} G/N_\ell(G) \longrightarrow \{e\}. \tag{5}$$

The following proposition indicates that any contra-associative extension arises from a contra-associative gyrogroup.

Proposition 4.1. *Let G be a gyrogroup that is an extension of a gyrogroup X by a gyrogroup Y. If G is contra-associative, then so is X.*

Proof. By definition, there is a normal subgyrogroup N of G such that $G/N \cong X$. Suppose that G is contra-associative. By Corollary 3.5, G/N is contra-associative. By Corollary 3.4, X is contra-associative. □

Motivated by the notion of normal series in group theory, we define a series of subgyrogroups related to associator normal subgyrogroups as follows. Let G be a gyrogroup. Define the following subgyrogroups inductively:

$$G^{\mathsf{a},0} = G, \quad G^{\mathsf{a},1} = G^{\mathsf{a}}, \text{ and } \quad G^{\mathsf{a},n} = \left(G^{\mathsf{a},n-1}\right)^{\mathsf{a}} \text{ for all integers } n \geq 2. \tag{6}$$

Then, the following proposition is obtained. Furthermore, part 3 of this proposition shows that series (6) is meaningless in the case of contra-associative gyrogroups.

Proposition 4.2. *Let G be a gyrogroup. Then, the following statements are true for all positive integers n:*

(1) $G^{\mathsf{a},n} \trianglelefteq G^{\mathsf{a},n-1}$.
(2) $G^{\mathsf{a},n-1}/G^{\mathsf{a},n}$ *is a group.*
(3) *If G is contra-associative, then $G^{\mathsf{a},n} = G$.*

Proof. Let $n \in \mathbb{N}$. By definition, $G^{\mathsf{a},n} = \left(G^{\mathsf{a},n-1}\right)^{\mathsf{a}} \trianglelefteq G^{\mathsf{a},n-1}$, which proves part 1. By part 1 of Proposition 2.1, $G^{\mathsf{a},n-1}/G^{\mathsf{a},n} = G^{\mathsf{a},n-1}/\left(G^{\mathsf{a},n-1}\right)^{\mathsf{a}}$ is a group. This proves part 2. Suppose that G is contra-associative. Then, $G^{\mathsf{a},1} = G^{\mathsf{a}} = G$. By induction, $G^{\mathsf{a},n} = G$ for all positive integers n. □

Proposition 4.2 motivates the following definition, which allows us to define a new family of gyrogroups that has strong connections with groups.

Definition 4.3. Let G be a gyrogroup. A finite sequence of subgyrogroups of G,

$$\{e\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = G$$

is called a *g-extensive series* if $N_{i-1} \trianglelefteq N_i$ and N_i/N_{i-1} is a group for $i = 1, 2, \ldots, k$. In this case, the quotient N_i/N_{i-1} is called a *g-extensive factor*. A gyrogroup is *g-extensive* if it has a g-extensive series.

The next theorem gives a characterization of g-extensive gyrogroups in terms of series (6) and leads to the formation of Definition 4.5.

Theorem 4.4. *Let G be an arbitrary gyrogroup. Then, G is g-extensive if and only if $G^{\mathsf{a},k} = \{e\}$ for some non-negative integer k.*

Proof. Suppose that G is g-extensive. Then, there is a g-extensive series, namely

$$\{e\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = G.$$

We first show that $G^{\mathsf{a},i} \subseteq N_{k-i}$ implies $G^{\mathsf{a},i+1} \subseteq N_{k-i-1}$. Suppose that $G^{\mathsf{a},i} \subseteq N_{k-i}$. By part 2 of Proposition 2.1, $G^{\mathsf{a},i+1} = (G^{\mathsf{a},i})^{\mathsf{a}} \subseteq N^{\mathsf{a}}_{k-i}$. By definition, $N_{k-i-1} \trianglelefteq N_{k-i}$ and N_{k-i}/N_{k-i-1} is a group. By Proposition 2.3, $N^{\mathsf{a}}_{k-i} \subseteq N_{k-i-1}$. This implies that $G^{\mathsf{a},i+1} \subseteq N_{k-i-1}$. Since $G^{\mathsf{a},0} = G \subseteq N_{k-0}$, it follows that $G^{\mathsf{a},1} \subseteq N_{k-1}$. Repeating the process, we obtain that $G^{\mathsf{a},k} \subseteq N_0$. Since $N_0 = \{e\}$, it follows that $G^{\mathsf{a},k} = \{e\}$.

To prove the converse, suppose that $G^{\mathsf{a},k} = \{e\}$ for some non-negative integer k. For each integer $i \in \{0, 1, \ldots, k\}$, define $N_i = G^{\mathsf{a},k-i}$. According to parts 1 and 2 of Proposition 4.2, we obtain that $N_{i-1} \trianglelefteq N_i$ and N_i/N_{i-1} is a group for all $i \in \{1, 2, \ldots, k\}$. It follows that $\{e\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = G$ defines a g-extensive series for G, and so G is g-extensive. This completes the proof. □

Definition 4.5. If G is a g-extensive gyrogroup, then the smallest non-negative integer k for which $G^{\mathsf{a},k} = \{e\}$ is called the *g-extensive length* of G.

The family of g-extensive gyrogroups properly includes the family of groups. In fact, we obtain the following proposition, which gives a characteristic property of g-extensive gyrogroups of length at most 2. This property resembles the property of nilpotent groups of class at most 2.

Proposition 4.6. *Let G be a gyrogroup.*

1. *Then, G is g-extensive of length* 0 *if and only if G is trivial.*
2. *Then, G is g-extensive of length* 1 *if and only if G is a non-trivial group.*
3. *There is a normal subgroup of G containing all the associators of G if and only if G is g-extensive of length at most* 2.

Proof. By definition, G is g-extensive of length 0 if and only if $G = G^{\mathsf{a},0} = \{e\}$. This proves part 1. Suppose that G is g-extensive of length 1. By definition, $G^{\mathsf{a}} = G^{\mathsf{a},1} = \{e\}$, and so G is a group by Proposition 2.2. Furthermore, G is not trivial since $G^{\mathsf{a},0} \neq \{e\}$. To prove the converse, suppose that G is a non-trivial group. Hence, $G^{\mathsf{a},0} \neq \{e\}$ and $G^{\mathsf{a},1} = \{e\}$. This shows that G is g-extensive and that the g-extensive length of G is 1, which proves part 2.

Suppose that Ξ is a normal subgroup of G such that $[a, b, c] \in \Xi$ for all $a, b, c \in G$. In the case when G is a group, G is g-extensive of length at most 1. Therefore, we may assume that G is not a group. Hence, $G^{\mathsf{a},0} \neq \{e\}$ and $G^{\mathsf{a},1} \neq \{e\}$. Since $\Xi \trianglelefteq G$, we obtain that $G^{\mathsf{a}} \subseteq \Xi$ by the minimality of G^{a}. Hence, $[a, b, c] = e$ for all $a, b, c \in G^{\mathsf{a}}$. This implies that $G^{\mathsf{a},2} = (G^{\mathsf{a}})^{\mathsf{a}} = \{e\}$, and so G is g-extensive of length 2. Suppose conversely that G is g-extensive of length at most 2. Hence, $G^{\mathsf{a},2} = \{e\}$. This implies that $(G^{\mathsf{a},1})^{\mathsf{a}} = \{e\}$, and so $G^{\mathsf{a}} = G^{\mathsf{a},1}$ is a group. Thus, G^{a} is a normal subgroup of G containing all the associators of G. This proves part 3. □

The main result of this section is exhibited in the following theorem, which shows some close relationships between contra-associative and g-extensive gyrogroups. In particular, part 1 of this theorem implies that if G is a g-extensive gyrogroup, then there is at least one non-trivial homomorphism from G to a non-trivial group, namely the canonical projection $\pi : G \to G/G^{\mathsf{a}}$. This fact gives us a better understanding of the structure of a g-extensive gyrogroup.

Theorem 4.7. *Let G be a gyrogroup.*

(1) *If G is contra-associative, then G is not g-extensive.*
(2) *If G is not g-extensive, then $G^{\mathsf{a},k}$ is contra-associative for some non-negative integer k or the sequence $G \supset G^{\mathsf{a},1} \supset G^{\mathsf{a},2} \supset \cdots \supset G^{\mathsf{a},k} \supset \cdots \supset \{e\}$ is an infinite ascending chain of subgyrogroups of G, where all containments are proper.*
(3) *If G is finite but is not g-extensive, then the sequence*

$$G = G^{\mathsf{a},0} \supset G^{\mathsf{a},1} \supset G^{\mathsf{a},2} \supset \cdots \supset G^{\mathsf{a},k} \supset \{e\}$$

is a finite ascending chain of subgyrogroups of G, where k is the smallest non-negative integer such that $G^{\mathsf{a},k}$ is contra-associative and all containments are proper.

(4) *If G is non-trivial, finite, and g-extensive, then the sequence*

$$G = G^{\mathsf{a},0} \supset G^{\mathsf{a},1} \supset \cdots \supset G^{\mathsf{a},k-1} \supset G^{\mathsf{a},k} = \{e\}$$

is a finite ascending chain of subgyrogroups of G, where k is the g-extensive length of G, $G^{\mathsf{a},k-1}$ is a group, and all containments are proper.

Proof. Suppose that G is contra-associative. By definition, $G \neq \{e\}$. By part 3 of Proposition 4.2, $G^{\mathsf{a},n} = G$ for all positive integers n. Hence, $G^{\mathsf{a},n} \neq \{e\}$ for all positive integers n. By Theorem 4.4, G is not g-extensive. This proves part 1.

To prove part 2, suppose that G is not g-extensive. Hence, $G \neq \{e\}$. Assume further that $G^{\mathsf{a},k}$ is not contra-associative for all integers $k \geq 0$. By Theorem 4.4, $G^{\mathsf{a},k} \neq \{e\}$ for all integers $k \geq 0$. Note that $G^{\mathsf{a}} \subsetneq G$ since $G = G^{\mathsf{a},0}$ is not contra-associative. Let $i \in \mathbb{N}$. Since $G^{\mathsf{a},i}$ is not contra-associative and $G^{\mathsf{a},i+1} = (G^{\mathsf{a},i})^{\mathsf{a}}$, it follows that $G^{\mathsf{a},i+1} \subsetneq G^{\mathsf{a},i}$. Thus, $G \supset G^{\mathsf{a},1} \supset G^{\mathsf{a},2} \supset \cdots \supset G^{\mathsf{a},k} \supset \cdots \supset \{e\}$ is an infinite ascending chain and all containments are proper. This proves part 2.

Suppose that G is finite but is not g-extensive. By part 2, $G^{\mathsf{a},i}$ is contra-associative for some non-negative integer i. Using the Well-Ordering Principle, we can let k be the smallest non-negative integer such that $G^{\mathsf{a},k}$ is contra-associative. If $k = 0$, we obtain the chain $G \supsetneq \{e\}$. Assume that $k \neq 0$. It can be proved by induction that $G^{\mathsf{a},k+i} = G^{\mathsf{a},k}$ for all $i \in \mathbb{N}$. Furthermore, $G^{\mathsf{a},k} \neq \{e\}$. Let j be an integer with $0 \leq j < k$. By the minimality of k, $G^{\mathsf{a},j}$ is not contra-associative. Hence, $G^{\mathsf{a},j+1} = (G^{\mathsf{a},j})^{\mathsf{a}} \subsetneq G^{\mathsf{a},j}$, which completes the proof of part 3.

Suppose that G is non-trivial, finite, and g-extensive. Let k the g-extensive length of G. Since $G \neq \{e\}$, we have $k > 0$. By definition, $G^{\mathsf{a},k} = \{e\}$. This implies by Proposition 2.2 that $G^{\mathsf{a},k-1}$ is a group since $(G^{\mathsf{a},k-1})^{\mathsf{a}} = G^{\mathsf{a},k} = \{e\}$. Let j be an integer with $0 \leq j < k$. By the minimality of k, $G^{\mathsf{a},j} \neq \{e\}$. We claim that $G^{\mathsf{a},j+1} \subsetneq G^{\mathsf{a},j}$. Assume to the contrary that $G^{\mathsf{a},j+1} = G^{\mathsf{a},j}$. Then,

$$G^{\mathsf{a},j+2} = (G^{\mathsf{a},j+1})^{\mathsf{a}} = (G^{\mathsf{a},j})^{\mathsf{a}} = G^{\mathsf{a},j+1} = G^{\mathsf{a},j}.$$

Continuing in this fashion, we obtain that $G^{\mathsf{a},k} = G^{\mathsf{a},j}$, and so $G^{\mathsf{a},k} \neq \{e\}$, a contradiction. This proves part 4. □

Next, we prove that every subgyrogroup of a g-extensive gyrogroup is g-extensive (see Proposition 4.9) and that any homomorphic image of a g-extensive gyrogroup is g-extensive (see Proposition 4.11). In order to proceed, we need the following lemma.

Lemma 4.8. *Let G and K be gyrogroups, and let H be a subgyrogroup of G.*

(1) *Then, $H^{\mathsf{a},i} \subseteq G^{\mathsf{a},i}$ for all non-negative integers i.*
(2) *If $\varphi : G \to K$ is a surjective homomorphism, then $\varphi(G^{\mathsf{a},i}) = K^{\mathsf{a},i}$ for all non-negative integers i.*

Proof. We prove part 1 by induction on i. Clearly, $H^{\mathsf{a},0} = H \subseteq G = G^{\mathsf{a},0}$. Assume that $H^{\mathsf{a},i} \subseteq G^{\mathsf{a},i}$. By part 2 of Proposition 2.1, $H^{\mathsf{a},i+1} = (H^{\mathsf{a},i})^{\mathsf{a}} \subseteq (G^{\mathsf{a},i})^{\mathsf{a}} = G^{\mathsf{a},i+1}$, which completes the induction. To prove part 2, suppose that $\varphi : G \to K$ is a surjective homomorphism. We proceed by induction on i. Since φ is surjective, it follows that $\varphi(G^{\mathsf{a},0}) = \varphi(G) = K = K^{\mathsf{a},0}$. Assume that $\varphi(G^{\mathsf{a},i}) = K^{\mathsf{a},i}$. Then, the restriction of φ to $G^{\mathsf{a},i}$ is a surjective homomorphism from $G^{\mathsf{a},i}$ to $K^{\mathsf{a},i}$. By Proposition 2.5, $\varphi(G^{\mathsf{a},i+1}) = \varphi((G^{\mathsf{a},i})^{\mathsf{a}}) = (K^{\mathsf{a},i})^{\mathsf{a}} = K^{\mathsf{a},i+1}$, which completes the induction. □

Proposition 4.9. *Let G be a gyrogroup, and let H be a subgyrogroup of G. If G is g-extensive, then so is H.*

Proof. By Theorem 4.4, $G^{\mathsf{a},k} = \{e\}$ for some non-negative integer k. By part 1 of Lemma 4.8, $H^{\mathsf{a},k} \subseteq G^{\mathsf{a},k}$, and so $H^{\mathsf{a},k} = \{e\}$. Thus, by the same theorem, H is g-extensive. □

In light of the proof of Proposition 4.9, we immediately obtain the following corollary.

Corollary 4.10. *Let G be a gyrogroup, and let H be a subgyrogroup of G. If G is g-extensive of length n, then the g-extensive length of H does not exceed n.*

Proposition 4.11. *Let G and K be gyrogroups, and let $\varphi : G \to K$ be a surjective homomorphism. If G is g-extensive, then so is K.*

Proof. Suppose that G is g-extensive. By Theorem 4.4, $G^{\mathsf{a},k} = \{e\}$ for some integer $k \geq 0$. By part 2 of Lemma 4.8, $K^{\mathsf{a},k} = \varphi(G^{\mathsf{a},k}) = \varphi(\{e\}) = \{e\}$. It follows that K is g-extensive. □

As a consequence of Proposition 4.11, the property of being g-extensive becomes an invariant property of gyrogroups. This fact is proved in the following corollary.

Corollary 4.12. *Let G and K be gyrogroups. If $G \cong K$, then G is g-extensive if and only if K is g-extensive.*

The next proposition shows that the quotient of a g-extensive gyrogroup by any normal subgyrogroup is again g-extensive and vice versa.

Proposition 4.13. *Let G be a gyrogroup, and let $N \trianglelefteq G$. Then, G is g-extensive if and only if N and G/N are g-extensive.*

Proof. Suppose that G is g-extensive, and let $\pi_N : G \to G/N$ be the canonical projection. By Proposition 4.9, N is g-extensive. Note that $G/N = \pi_N(G)$. Hence, by Proposition 4.11, G/N is g-extensive. To prove the converse, suppose that N and G/N are g-extensive. Let m and n be the g-extensive lengths of N and G/N, respectively. By definition, $N^{\mathsf{a},m} = \{e\}$ and $(G/N)^{\mathsf{a},n} = \{e \oplus N\}$. By part 2 of

Lemma 4.8, $\pi_N(G^{\mathsf{a},n}) = (G/N)^{\mathsf{a},n} = \{e \oplus N\}$. Hence, $a \oplus N = e \oplus N$ for all $a \in G^{\mathsf{a},n}$. Then, $a \in N$ for all $a \in G^{\mathsf{a},n}$, which implies that $G^{\mathsf{a},n} \subseteq N$. By part 2 of Proposition 2.1, $G^{\mathsf{a},n+1} = (G^{\mathsf{a},n})^{\mathsf{a}} \subseteq N^{\mathsf{a}} = N^{\mathsf{a},1}$. Continuing in this fashion, we obtain that $G^{\mathsf{a},n+m} \subseteq N^{\mathsf{a},m}$. Since $N^{\mathsf{a},m} = \{e\}$, it follows that $G^{\mathsf{a},n+m} = \{e\}$. Thus, G is g-extensive, and the g-extensive length of G is less than or equal to $m + n$. □

In light of the proof of Proposition 4.13, we immediately obtain the following corollary.

Corollary 4.14. *Let G be a gyrogroup, and let $N \trianglelefteq G$. If N is g-extensive of length m and G/N is g-extensive of length n, then the g-extensive length of G does not exceed $m + n$.*

To prove that a direct product of any finite number of g-extensive gyrogroups is g-extensive, we need the following lemma.

Lemma 4.15. *Let $\{G_i : i \in I\}$ be an indexed family of gyrogroups with $I \neq \emptyset$. Then,*

$$\left(\prod_{i\in I} G_i\right)^{\mathsf{a},n} \subseteq \prod_{i\in I} G_i^{\mathsf{a},n}.$$

for all non-negative integers n.

Proof. The proof can be done by induction on n, using Proposition 3.9 of [12]. □

Proposition 4.16. *Let $G_1, G_2, \ldots, G_k$ be gyrogroups. Then, $G_1 \times G_2 \times \cdots \times G_k$ is g-extensive if and only if $G_1, G_2, \ldots, G_k$ are all g-extensive.*

Proof. Set $G = G_1 \times G_2 \times \cdots \times G_k$ and suppose that G is g-extensive. For each $i = 1, 2, \ldots, k$, let p_i be the projection from G to G_i defined by the formula

$$p_i(a_1, a_2, \ldots, a_k) = a_i.$$

Then, p_i is a surjective homomorphism from G to G_i. By Proposition 4.11, G_i is g-extensive. Suppose conversely that $G_1, G_2, \ldots, G_k$ are all g-extensive. Let m_i be the g-extensive length of G_i for all $i \in \{1, 2, \ldots, k\}$. Set $m = \max\{m_1, m_2, \ldots, m_k\}$. Then, ${G_i}^{\mathsf{a},m} = \{e\}$ for all $i \in \{1, 2, \ldots, k\}$. By Lemma 4.15,

$$G^{\mathsf{a},m} \subseteq G_1^{\mathsf{a},m} \times G_2^{\mathsf{a},m} \times \cdots \times G_k^{\mathsf{a},m} = \{(e, e, \ldots, e)\}.$$

Hence, $G^{\mathsf{a},m} = \{(e, e, \ldots, e)\}$. By Theorem 4.4, G is g-extensive. □

In light of the proof of Proposition 4.16, we immediately obtain the following corollary.

Corollary 4.17. *If $G_1, G_2, \ldots, G_k$ are g-extensive gyrogroups, then the g-extensive length of the direct product $G_1 \times G_2 \times \cdots \times G_k$ does not exceed the maximum of the g-extensive lengths of $G_1, G_2, \ldots, G_k$.*

The following theorem shows that any g-extensive gyrogroup arises from a group. For this reason, g-extensive gyrogroups and groups share some common properties, as we will see shortly.

Theorem 4.18. *If G is a gyrogroup that is an extension of a group by a group, then G is g-extensive. Conversely, if G is a g-extensive gyrogroup, then G can be obtained by a finite number of extensions of a group by a g-extensive gyrogroup.*

Proof. Let G be a gyrogroup. Suppose that G is an extension of X by Y, where X and Y are groups. Then, there exists a normal subgyrogroup N of G such that $N \cong Y$ and $G/N \cong X$. Since Y is a group, N is a group as well. Since $G/N \cong X$, it follows that G/N is a group, and so $G^{\mathsf{a}} \subseteq N$ by Proposition 2.3. This implies that G^{a} forms a normal subgroup of G, and so G is g-extensive by part 3 of Proposition 4.6.

Suppose conversely that G is g-extensive. By definition, G has a g-extensive series $\{e\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = G$, where $N_{i-1} \trianglelefteq N_i$ and N_i/N_{i-1} is a group for all $i \in \{1, 2, \ldots, k\}$. Since $N_1 \cong N_1/\{e\}$, we obtain that N_1 is a group. Since N_2/N_1 is a group, we obtain that N_2 is an extension of a group by a group. Hence, N_2 is g-extensive. Since $N_2 \trianglelefteq N_3$ and N_3/N_2 is a group, N_3 is an extension of a group by a g-extensive gyrogroup. By Proposition 4.13, N_3 is g-extensive. Continuing in this fashion, we see that N_{k-1} is g-extensive and that G is an extension of G/N_{k-1} by N_{k-1}. □

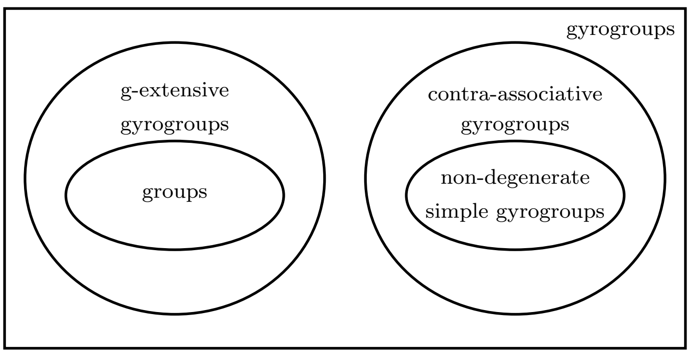

FIGURE 1. Relationships between some classes of gyrogroups.

In Figure 1, we summarize relationships between some classes of gyrogroups. In view of Theorems 4.7 and 4.18, we propose the following problems:

(1) (Existence Problem) Classify all contra-associative gyrogroups.
(2) (Extension Problem I) Find all ways of putting groups together to form other gyrogroups.
(3) (Extension Problem II) Find all ways of putting groups and contra-associative gyrogroups together to form other gyrogroups.

We end this section with a few concrete examples.

Example 4.19. Since the left nucleus of the gyrogroup G_8 (see Example 1 of [9]) is $\{0, 3\}$, which is the set of associators in G_8, it follows that $G_8^{\mathsf{a}} = \{0, 3\}$. Furthermore, $G_8^{\mathsf{a},2} = \{0\}$ since G_8^{a} is a group. Therefore, $G_8 \supset \{0, 3\} \supset \{0\}$ is a g-extensive series. A direct computation shows that the left cosets of $N_\ell(G_8)$ are precisely

$$0 \oplus N_\ell(G_8) = \{0, 3\},$$
$$1 \oplus N_\ell(G_8) = \{1, 2\},$$
$$4 \oplus N_\ell(G_8) = \{4, 6\},$$
$$5 \oplus N_\ell(G_8) = \{5, 7\}.$$

Note that G_8/G_8^{a} is a group whose operation is given by Table 1. Furthermore, the canonical projection $\pi : G_8 \to G_8/G_8^{\mathsf{a}}$ is a non-trivial homomorphism from a non-degenerate gyrogroup to a non-trivial group. It is easy to see that G_8/G_8^{a} is isomorphic to the Klein 4-group, and so G_8 is an extension of the Klein 4-group by the cyclic group of order two.

$\oplus$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{4}$	$\overline{5}$
$\overline{1}$	$\overline{1}$	$\overline{0}$	$\overline{5}$	$\overline{4}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$
$\overline{5}$	$\overline{5}$	$\overline{4}$	$\overline{1}$	$\overline{0}$

TABLE 1. Cayley table for G_8/G_8^{a}. Here, $\overline{a}$ denotes the left coset $a \oplus G_8^{\mathsf{a}}$.

Example 4.20. Recall that the right nucleus of the gyrogroup G_{15} (see Example 8 of [9]) is $N_r(G_{15}) = \{0, 4, 9, 12, 13\}$. A direct computation shows the left cosets of $N_r(G_{15})$ are precisely

$$0 \oplus N_r(G_{15}) = \{0, 4, 9, 12, 13\},$$
$$1 \oplus N_r(G_{15}) = \{1, 5, 6, 7, 10\},$$
$$2 \oplus N_r(G_{15}) = \{2, 3, 8, 11, 14\}.$$

Furthermore, the operation defined on $G_{15}/N_r(G_{15})$ by

$$(a \oplus N_r(G_{15})) \oplus (b \oplus N_r(G_{15})) = (a \oplus b) \oplus N_r(G_{15})$$

is well defined (see Table 2). This implies that $N_r(G_{15})$ is normal in G_{15}. Since the associators in G_{15} are precisely $0, 4, 9, 12$, and 13, $G_{15}^{\mathsf{a}} = \{0, 4, 9, 12, 13\}$. Moreover, $G_{15}^{\mathsf{a},2} = \{0\}$ since G_{15}^{a} forms a group. Therefore, the series

$$G_{15} \supset \{0, 4, 9, 12, 13\} \supset \{0\}$$

defines a g-extensive series for G_{15}. Note that $G_{15}/N_r(G_{15})$ is a group since $N_r(G_{15}) = G_{15}^{\mathsf{a}}$. Therefore, the canonical projection $\pi : G_{15} \to G_{15}/G_{15}^{\mathsf{a}}$ is a non-trivial homomorphism from a non-degenerate gyrogroup to a non-trivial group. Note that $G_{15}/N_r(G_{15})$ is isomorphic to the cyclic group of order three and that $N_r(G_{15})$ is isomorphic to the cyclic group of order five. Hence, G_{15} is an extension of the cyclic group of order three by the cyclic group of order five.

$\oplus$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

TABLE 2. Cayley table for $G_{15}/N_r(G_{15})$. Here, $\overline{a}$ denotes the left coset $a \oplus N_r(G_{15})$.

5. Gyrogroup properties

In this section, we formally define invariant properties of gyrogroups. By a *gyrogroup property* P (or, equivalently, an *invariant property* of gyrogroups) we mean a map from the collection of all gyrogroups to the two-element set, {true, false}, with the property that any two isomorphic gyrogroups get mapped to the same element. We say that a gyrogroup G has property P if and only if $\mathsf{P}(G)$ is true. A *finite-gyrogroup property* and *group property* are defined in the same way as a gyrogroup property. We can think of a gyrogroup property as an invariant property of gyrogroups: P is defined for all gyrogroups, and if $G \cong K$, then G has property

P if and only if K has property P. Now, we define several types of gyrogroup properties.

Definition 5.1. Let P be a gyrogroup property.

(1) P is *S-closed* if a gyrogroup G has property P implies every subgyrogroup of G has property P.
(2) P is *Q-closed* if a gyrogroup G has property P implies G/N has property P for all normal subgyrogroups N of G.
(3) P is *FD-closed* if $G_1, G_2, \dots, G_n$ are gyrogroups that have property P implies $G_1 \times G_2 \times \cdots \times G_n$ has property P.
(4) P is *AD-closed* if $\{G_i : i \in I\}$, $I \neq \emptyset$, is an indexed family of gyrogroups having property P implies $\prod_{i\in I} G_i$ has property P.

Parts 1–3 of Definition 5.1 are also defined for a finite-gyrogroup property with appropriate modifications. In view of Corollary 3.4, the property of being contra-associative is a gyrogroup property. In view of Corollary 4.12, the property of being g-extensive is a gyrogroup property. Note that the property of being contra-associative is Q-closed (see Corollary 3.5), but is not S-closed (see the remark after Proposition 3.2). Note also that the property of being g-extensive is S-closed (see Proposition 4.9), Q-closed (see Proposition 4.13) and FD-closed (see Proposition 4.16).

It frequently happens that we can understand the structure of a gyrogroup from an understanding of its normal subgyrogroups and its quotient gyrogroups. Thus, we make the following definition.

Definition 5.2. Let P be a gyrogroup property. We say that P is *NQ-inductive* if for any normal subgyrogroup N of a gyrogroup G, N and G/N have property P implies G has property P.

In view of Proposition 4.13, the property of being g-extensive is NQ-inductive. Moreover, Definition 5.2 is defined for a finite-gyrogroup property with appropriate modifications. The next proposition shows that g-extensive gyrogroups and groups share NQ-inductive properties.

Proposition 5.3. *Let P be a gyrogroup (respectively, finite-gyrogroup) property. If P is NQ-inductive and if every (respectively, finite) group has property P, then every (respectively, finite) g-extensive gyrogroup has property P.*

Proof. We proceed by induction on the g-extensive length. In the case when G is g-extensive of length 0, G is the trivial gyrogroup $\{e\}$, which is a group, and so G

has property P by assumption. Assume inductively that if P is NQ-inductive and if every group has property P, then every g-extensive gyrogroup of length n has property P. Suppose that P is NQ-inductive and every group has property P. Let G be a g-extensive gyrogroup of length $n+1$. Then, we have the g-extensive series:

$$G = G^{\mathsf{a},0} \supset G^{\mathsf{a},1} \supset \cdots \supset G^{\mathsf{a},n} \supset G^{\mathsf{a},n+1} = \{e\}. \qquad (\star)$$

As in the proof of part 4 of Theorem 4.7, all containments in series $(\star)$ are proper. Note that $G^{\mathsf{a},1}$ is g-extensive of length n. By the inductive hypothesis, $G^{\mathsf{a},1}$ has property P. Since $G^{\mathsf{a},1} \trianglelefteq G$ and $G/G^{\mathsf{a},1} = G/G^{\mathsf{a}}$ is a group, $G/G^{\mathsf{a},1}$ has property P. Since P is NQ-inductive, G has property P, which completes the induction. The proof in the case of a finite-gyrogroup property can be done in a similar fashion. □

Next, we mention a few gyrogroup properties. Recall that a finite gyrogroup G is said to have the *Lagrange property* if H is a subgyrogroup of G implies the order of H divides the order of G (see Definition 5.1 of [13]). By Proposition 41 of [9], the Lagrange property is a finite-gyrogroup property. By Corollary 5.3 of [13], the Lagrange property is NQ-inductive. According to Lagrange's Theorem, every finite group has the Lagrange property. Thus, we obtain the following theorem immediately:

Theorem 5.4. *Every finite g-extensive gyrogroup has the Lagrange property.*

Proof. This is an application of Proposition 5.3. □

Recall that a finite gyrogroup G is said to have the *weak Cauchy property* if p is a prime dividing the order of G implies G has an element of order p (see Definition 6.3 of [13]). Furthermore, a finite gyrogroup G is said to have the *strong Cauchy property* if every subgyrogroup of G has the weak Cauchy property (see Definition 6.4 of [13]). By Corollary 6.6 of [13], the weak Cauchy property and the strong Cauchy property are finite-gyrogroup properties. By Corollary 6.8 of [13], the weak (respectively, strong) Cauchy property is NQ-inductive. According to Cauchy's Theorem in group theory, every finite group has the weak (respectively, strong) Cauchy property. Therefore, we obtain the following theorem immediately:

Theorem 5.5. *Every finite g-extensive gyrogroup has the weak (respectively, strong) Cauchy property.*

Proof. This is an application of Proposition 5.3. □

By Theorem 5.5, the weak Cauchy property and the strong Cauchy property are equivalent on the family of finite g-extensive gyrogroups. Furthermore, any finite gyrogroup not satisfying the weak Cauchy property cannot be g-extensive and finally leads to a contra-associative gyrogroup by part 3 of Theorem 4.7. We close this section with another gyrogroup property. Recall that a (finite or infinite) gyrogroup G is said to have the *left-coset decomposable property* if H is a subgyrogroup of G implies the set of left cosets of H, $G/H = \{a \oplus H : a \in G\}$, is a disjoint partition of G. As is well known, every group (not necessarily finite) has the left-coset decomposable property. Next, we show that the left-coset decomposable property is a gyrogroup property.

Proposition 5.6. *Let G and K be gyrogroups. If $G \cong K$, then G has the left-coset decomposable property if and only if K has the left-coset decomposable property.*

Proof. Let $\phi : K \to G$ be an isomorphism. Suppose that G possesses the left-coset decomposable property. Let H be a subgyrogroup of K. Clearly, $x \oplus H \neq \emptyset$ for all $x \in K$ and $\bigcup_{x\in K} x \oplus H = K$. Suppose that $(x \oplus H) \cap (y \oplus H) \neq \emptyset$. Let $z \in (x \oplus H) \cap (y \oplus H)$. Then, $z = x \oplus h_1 = y \oplus h_2$ for some elements $h_1, h_2 \in H$. As ϕ is a homomorphism, $\phi(H)$ is a subgyrogroup of G, and so $G/\phi(H)$ is a disjoint partition of G. Since $\phi(z) = \phi(x) \oplus \phi(h_1) = \phi(y) \oplus \phi(h_2)$, it follows that $(\phi(x)\oplus\phi(H))\cap(\phi(y)\oplus\phi(H)) \neq \emptyset$. Hence, $\phi(x)\oplus\phi(H) = \phi(y)\oplus\phi(H)$. This implies that $\phi(x \oplus H) = \phi(y \oplus H)$, which in turn implies that $x \oplus H = y \oplus H$ since ϕ is injective. This proves that K/H is a disjoint partition of K. Since H is arbitrary, K has the left-coset decomposable property. Since ϕ^{-1} is an isomorphism from G to K, the converse also holds. □

Unfortunately, the left-coset decomposable property is not, in general, NQ-inductive. In fact, the gyrogroup K_{16} (see page 41 of [14]), which has order 16, does not have the left-coset decomposable property (for instance, $4 \oplus \{0, 8\} = \{4, 15\}$ and $14 \oplus \{0, 8\} = \{4, 14\}$), whereas the left nucleus $N_\ell(K_{16}) = \{0, 1, 2, 3\}$ and the quotient $K_{16}/N_\ell(K_{16})$ do have the left-coset decomposable property because they form groups. Note also that K_{16} is g-extensive of length 2 by Proposition 4.6 for the associators of K_{16} are precisely 0 and 1. In particular, this situation indicates that g-extensive gyrogroups enjoy some (but not all) of the properties of groups.

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