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Prolongation of Generalized Metallic Structures Related to Weil Bundles

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ABSTRACT

Let (A, ℓ) be a Weil-Frobenius algebra, M a smooth manifold. In this paper, we study the prolongations of generalized metallic structures on manifold M to its Weil bundle $T^A M$ and we investigate some of their properties. In particular, we study the prolongation of calibrated generalized product structures and calibrated complex structures induced by metallic structures on M.

Keywords: *Weil-Frobenius algebra; Weil functor; A-jet, generalized metallic structures; natural transformations.*
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1. INTRODUCTION

Let M be a manifold of dimension $m > 0$. By $\pi_M : TM \to M$ we denote the tangent bundle and by $\pi^* : T^*M \to M$ the cotangent vector bundle. the concept of generalized geometry was originally introduced by N. Hitchin in [12] (as the differential geometry resulting from replacing the tangent bundle TM of a manifold M with the direct sum of the tangent and cotangent bundles $TM \oplus T^*M$) in order to unify complex and symplectic geometry. The particular case of this concept named generalized metallic structure has been defined and studied by A.M. Blaga and A. Nannicini in [2]. They defined the generalized metallic structure on M as an endomorphism $\Lambda_g : \mathbb{T}M = TM \oplus T^*M \to \mathbb{T}M = TM \oplus T^*M$ satisfying

$$(\Lambda_g)^2 = p\Lambda_g + qid$$

for some real numbers p and q.

On the other hand, for a given linear connection Γ^{TM} on vector bundle $(TM \to M)$ whose associated covariant derivative is ∇^{TM}, we consider the bracket $[\cdot,\cdot]_{\Gamma^{TM}}$ on the $C^\infty(M)$-module $\Gamma(\mathbb{T}M) \cong \Gamma(TM) \oplus \Gamma(T^*M)$ of sections of $(\mathbb{T}M \to M)$ defined for any $(X, \omega),\ (Y, \varpi) \in \Gamma(TM) \oplus \Gamma(T^*M)$ (see [2])by

$$[(X,\omega),\ (Y,\varpi)]_{\Gamma^{TM}} = \left([X,Y], \nabla^{T^*M}{}_X\varpi - \nabla^{T^*M}{}_Y\omega\right)$$

where ∇^{T^*M} is the covariant derivative induced by ∇^{TM} and defined by

$$\nabla^{T^*M} : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M),\ \ \left(\nabla^{T^*M}{}_X\varpi\right)(Y) = X\big(\omega(Y)\big) - \omega\big(\nabla^{TM}{}_XY\big).$$

The authors of [2] said that the generalized metallic structure Λ_g on M is Γ^{TM}-integrable if its Nijenhuis tensors field $N_{\Lambda_g}^{\Gamma^{TM}}$ with respect to Γ^{TM},

$$N_{\Lambda_g}^{\Gamma^{TM}}(\sigma,\widetilde{\sigma}) = (\Lambda_g)^2[\sigma,\widetilde{\sigma}]_{\Gamma^{TM}} + [\Lambda_g\sigma,\Lambda_g\widetilde{\sigma}]_{\Gamma^{TM}} - \Lambda_g[\Lambda_g\sigma,\widetilde{\sigma}]_{\Gamma^{TM}} - \Lambda_g[\sigma,\Lambda_g\widetilde{\sigma}]_{\Gamma^{TM}}$$

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vanishes for all $\sigma,\ \widetilde{\sigma} \in \Gamma(\mathbb{T}M)$.

It is known that the simplest example of the product preserving bundle functor on the category $\mathcal{M}f$ is a jet functor T_p^r. When $p = 1$, it is denoted by T^r and is called tangent functor of order r from the category $\mathcal{M}f$ of differential manifolds to the category $\mathcal{FM}$ of smooth fibered manifolds which preserves the products. In [16], the authors have proved that any bundle functor form $\mathcal{M}f$ to $\mathcal{FM}$ which preserves the products is a Weil functor. In other terms, the theory of Weil bundles represents a unified technique for studying a large class of geometric problems associated with product preserving functor. For all this reasons, we generalized the work of [25] by replacing the tangent functor of higher order by any Weil functor defined by the Weil-Frobenius algebra (A, ℓ) and we study some properties as in [25]. The research of this paper was motivated by the works of M. Doupovec, M. Kures and P.M. Kouotchop W.(see [6]-[17]).

So, the paper is organized as follows: In section 2, we recall briefly some results of [14]-[6] and [18] about the notion of Weil algebras, Weil-Frobenius algebras and Weil-Frobenuis functor. In section 3, we review some results about the prolongation of some tensor fields to Weil bundles. In section 4, the concept of metallic structures and generalized metallic structures is discuss and some properties are recall as in [2]. In the last section, some properties of the prolongation of generalized metallic structures related to Weil bundles are established witch generalized some similar results established in [25].

2. Preliminaries

2.1. Weil algebras

A Weil algebra A (initiated by A. Weil in 1953 to introduce the bundle T^AM of infinitely point of type A over a manifold M) is a finite dimensional real commutative, associative and unital algebra of the form $A = \mathbb{R} \cdot 1_A \oplus N_A$, where N_A is the ideal of nilpotent elements of A (see [14]-[15]-[16]). The simplest example of Weil algebras is

$$\mathbb{D}_k^r = \mathbb{R}[x_1, x_2, \cdots, x_k]/(x_1, x_2, \cdots, x_k)^{r+1} = J_0^r\left(\mathbb{R}^k, \mathbb{R}\right)$$

where $\mathbb{R}[x_1, x_2, \cdots, x_k]$ is the algebra of all polynomials on k undetermined. In particular, $\mathbb{D}_1^1 = \mathbb{D}$ is the classical algebra of dual (or Study) numbers. The ideal of nilpotent elements of $\mathbb{D}_k^r$ is the finite vector space $J_0^r\left(\mathbb{R}^k, \mathbb{R}\right)_0$.

Let $A = \mathbb{R} \cdot 1_A \oplus N_A$ be a Weil algebra, we will adopt the covariant approach of Weil functor described by I. Kolar in [14]. Let N_A^k be the ideal generated by the product of k elements of N_A, there is one and only one natural number r such that $N_A^r \neq 0$ and $N_A^{r+1} = 0$. In this case, the integer r is called the order $ord(A)$ of A and the dimension k of the vector space N_A/N_A^2 is said to be the width wA of A. Hence, a Weil algebra A of order r and width k will be called Weil (k, r)-algebra. We have the following results established in [14]

Proposition 2.1. *Every Weil (k, r)-algebra is a factor algebra of $\mathbb{D}_k^r$.*

Proof. See [14]. □

Proposition 2.2. *If $\rho, \widetilde{\rho} : J_0^r\left(\mathbb{R}^k, \mathbb{R}\right) \to A$ are two surjective algebra homomorphisms, then there is an algebra isomorphism $\nu : J_0^r\left(\mathbb{R}^k, \mathbb{R}\right) \to J_0^r\left(\mathbb{R}^k, \mathbb{R}\right)$ such that: $\widetilde{\rho} \circ \nu = \rho$.*

Proof. See [14]. □

We say that, two maps $\varphi, \widetilde{\varphi} : \mathbb{R}^k \to M$ determine the same A-velocity if for every smooth map $f : M \to \mathbb{R}$, one has

$$\rho\left(j_0^r\left(f \circ \varphi\right)\right) = \rho\left(j_0^r\left(f \circ \widetilde{\varphi}\right)\right)$$

This condition is independent of the choice of ρ. We also say that φ and $\widetilde{\varphi}$ determine the same A-jet. The equivalence class of the map $\varphi : \mathbb{R}^k \to M$ is denoted by $j^A\varphi$ and will called A-velocity at 0 (see [14]-[15]-[16] for more details). In [14], one has the following result

Proposition 2.3. *The space $\{j^A\varphi,\ \varphi : \mathbb{R}^k \to M\}$ of all A-velocities on M coincides with the smooth manifold T^AM of dimension $m \times \dim A$.*

Proof. See [14]. □

Hence, it is clair that $T^A\mathbb{R} \cong A$.

We denote by $\pi_M^A : T^AM \to M,\ j^A\varphi \mapsto \varphi(0)$ the natural projection so $\left(T^AM, M, \pi_M^A\right)$ is a well-defined fibered manifold. If $f : M_1 \to M_2$ is a smooth map, then it induces a smooth map

$$T^Af : T^AM_1 \to T^AM_2,\ \ j^A\varphi \mapsto j^A(f \circ \varphi)$$

In particular $\left(f, T^Af\right)$ is a fibered morphism from $\left(T^AM_1, M_1, \pi_{M_1}^A\right)$ to $\left(T^AM_2, M_2, \pi_{M_2}^A\right)$. This defines a bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$ on the category $\mathcal{M}f$ of all manifolds with values in the category $\mathcal{FM}$ of smooth fibered manifolds which is called the Weil functor induced by A. The bundle functor T^A preserves product in the sense, that for any smooth manifolds M_1 and M_2, the map

$$(T^Apr_1, T^Apr_2) : T^A(M_1 \times M_2) \to T^AM_1 \times T^AM_2$$

(where $pr_i : M_1 \times M_2 \to M_i$ i=1,2 is the projection) is an $\mathcal{FM}-$isomorphism. Hence we can identify $T^A(M_1 \times M_2)$ with $T^AM_1 \times T^AM_2$.

Let B be another (p, d) Weil algebra and $\mu : A \to B$ be an algebra homomorphism, $\widetilde{\rho} : J_0^d\left(\mathbb{R}^p, \mathbb{R}\right) \to B$ the surjective algebra homomorphism. Then there is an algebra homomorphism $\widetilde{\mu} : J_0^r\left(\mathbb{R}^k, \mathbb{R}\right) \to J_0^d\left(\mathbb{R}^p, \mathbb{R}\right)$ such $\widetilde{\rho} \circ \widetilde{\mu} = \mu \circ \rho$. In particular, there is map $f_\mu : \mathbb{R}^p \to \mathbb{R}^k$ such that, $\widetilde{\mu}\left(j_0^rg\right) = j_0^d\left(g \circ f_\mu\right)$, where $g \in C^\infty\left(\mathbb{R}^k\right)$. For any manifold M of dimension $m \geq 1$, there is smooth map $\mu_M : T^AM \to T^BM$ defined by:

$$\mu_M\left(j^A\varphi\right) = j^B\left(\varphi \circ f_\mu\right)$$

More precisely, $\mu_M : T^AM \to T^BM$ is the value of the natural transformation determined by μ on M [15].

Weil functors generalize through their covariant description the tangent functors, more precisely, when A is the space of all r-jets of $\mathbb{R}^k$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^k$ denoted by $J_0^r\left(\mathbb{R}^k, \mathbb{R}\right)$, the corresponding Weil functor is the functor of (k, r)-velocities and denoted by T_k^r. For $k = 1$, it is called tangent functor of order r and denoted by T^r, this functor plays an essential role in hamiltonian mechanic. The particular importance of Weil functors in differential geometry comes from the fact there is a bijective correspondence between them and the set of product preserving bundle functors on the category of smooth manifolds ([16]).

2.2. *Weil-Frobenius algebras and Weil-Frobenius functors*

A Weil algebra $A = \mathbb{R} \cdot 1_A \oplus N_A$ is called a Weil-Frobenius algebra, if there is a symmetric nondegenerate bilinear form ξ on A such that

$$\xi(ab, c) = \xi(a, bc) \tag{2.1}$$

for all $a, b, c \in A$. This bilinear form is called the Frobenius form of the algebra A. Equivalently, A is a Weil-Frobenius algebra if there exists a linear map $\ell : A \to \mathbb{R}$ such that $\ker(\ell)$ contains nonzero ideal of A. More precisely, when ξ is given, ℓ is defined by $\ell(a) = \xi(a, 1_A)$ (such that $\ell(ab) = \xi(a, b)$, for all $a, b \in A$) and when ℓ is given, the map ξ defined by $\xi(a, b) = \ell(ab)$ is bilinear symmetric and verify the relation (2.1). The linear form $\ell : A \to \mathbb{R}$ is nondegenerate if the bilinear symmetry form $\xi : A \times A \to \mathbb{R},\ \ (a, b) \mapsto \ell(ab)$ is non degenerate. Hence, it follows that a Weil-Frobenius algebra is also a pair (A, ℓ) where A is a Weil algebra on which there exists a linear and nondegenerate form ℓ.

Example 2.1. The pair $\left(\mathbb{D}_1^r, \tau_r\right)$ is a Weil-Frobenius algebra where τ_r is the linear form on $\mathbb{D}_1^r$ defined

$$\begin{array}{rcl} \tau_r : \mathbb{D}_1^r & \to & \mathbb{R} \\ j_0^r\varphi & \mapsto & \dfrac{1}{r!}\dfrac{d^r}{dt^r}(\varphi(t))|_{t=0}. \end{array}$$

A Weil-Frobenius functor is a Weil functor T^A associated to a Weil-Frobenius algebra A. From the preserving of Weil property for tensor product, it follows that if T^A and T^B are Weil-Frobenius functors, then their iteration $T^A \circ T^B$ and fiber product $T^A \oplus T^B$ are Weil-Frobenius functor where $T^A \oplus T^B$ is defined for all $M \in \mathcal{M}f$ and $f \in C^\infty(M)$ (See [6])by

$$T^A \oplus T^B(M) = T^AM \times_M T^BM \quad \text{and} \quad T^A \oplus T^B(f) = T^Af \times T^Bf$$

2.3. Local coordinate system

Let M be and smooth manifold and $\left(A = \mathbb{R}\cdot 1_A \oplus N_A, \ell\right)$ a Weil-Frobenius algebra . For any $h \in C^\infty(\mathbb{R}^{wA}, M)$ and any local chart (U, u^i) of M in $h(0)$, one has

$$j^A(u^i \circ h) = u^i \circ h(0)\cdot 1_A + \sum_{1\le|\alpha|\le ord(A)} \frac{1}{\alpha!} D_\alpha(u^i\circ h)(0) j^A(x^\alpha).$$

It follows that the subset $\{d_\alpha = j^A(x^\alpha) : \ 1 \le |\alpha| \le ord(A)\}$ of is the ideal N_A which generates N_A. We denote by $\Re$ the subset of $\{\alpha \in \mathbb{N}^{wA} : 1 \le |\alpha| \le ord(A)\}$ such that $\{d_\alpha : \ \alpha \in \Re\}$ is a basis of N_A and $\Re^c = \mathbb{N}^{wA} \setminus \Re$. For $\beta \in \Re^c$, one has $d_\beta = \sum\limits_{\gamma\in\Re} \lambda_\beta^\gamma d_\gamma$. Hence

$$j^A(u^i \circ h) = u^i(h(0))\cdot 1_A + \sum_{\alpha\in\Re}\left(\frac{1}{\alpha!} D_\alpha(u^i\circ h)(0) + \sum_{\gamma\in\Re^c} \frac{\lambda_\gamma^\alpha}{\gamma!} D_\gamma(u^i\circ h)\right) d_\alpha.$$

Therefore, the coordinates system $(\overline{u}_0^i, \overline{u}_\alpha^i)$ of T^AM over T^AU is such that

$$\overline{u}_0^i = u^i \circ \pi_M^A \quad \text{and} \quad \overline{u}_\alpha^i = u_\alpha^i + \sum_{\gamma\in\Re^c} \lambda_\gamma^\alpha u_\gamma^i$$

where $u_\alpha^i(j^Af) = \frac{1}{\alpha!}D_\alpha(u_i \circ f)(0), \forall\, j^Af \in T^AU$. In the particular case where $A = \mathbb{D}$, the local coordinate system of TM induced by $\left(U, u^i\right)$ is denoted by $\left(u^i \circ \pi_M, \dot{u}^i\right)$.

2.4. Canonical flow natural equivalence and natural isomorphism $\varepsilon_{A,M}^\ell : T^AT^*M \to T^*T^AM$

Let A and B be two Weil algebras which widths are respectively wA and wB. The Weil algebra corresponding to the iteration $T^A \circ T^B$ of the two Weil functors T^A, $T^B : \mathcal{M}f \to \mathcal{FM}$ is in general $B \oplus A$. The exchange homomorphism $ex : B \oplus A \to A \oplus B$ induces the following natural transformation

$$ex_M : T^AT^BM \to T^BT^AM$$

Let $t \in \mathbb{R}^{wA}$ and $z \in \mathbb{R}^{wB}$, then every $\varsigma \in T^AT^BM$ is of the form

$$\varsigma = j^A\Big(t \mapsto j^B\big(z \mapsto \Psi(t, z)\big)\Big),$$

where $\Psi : \mathbb{R}^{wA} \times \mathbb{R}^{wB} \to M$. Hence,

$$ex_M(\varsigma) = j^B\Big(z \mapsto j^A\big(t \mapsto \Psi(t, z)\big)\Big). \tag{2.2}$$

In the particular case where $T^B = T$ i.e $B = \mathbb{D}$ (which means that $wB = 1$), write κ^A for $ex : \mathbb{D} \oplus A \to A \oplus \mathbb{D}$. Remarking that $(T^ATM \to T^AM)$ and $(TT^AM \to T^AM)$ are vector bundles, we deduce form (2.2) that

$$\kappa_M^A : T^ATM \to TT^AM$$

is a $\mathcal{VB}$-morphism over T^AM. For any smooth vector field X in M, one can define its flow prolongation

$$\mathcal{F}^AX : T^AM \to TT^AM,\ u \mapsto \frac{\partial}{\partial t} T^A(\exp(tX))(u),$$

where $\exp(tX)$ denoted the flow of X. On the other hand, we deduce from (2.2) that $\mathcal{F}^AX = \kappa_M^A \circ T^AX$ where $T^AX : T^AM \to T^ATM$ (see [15]). This show that κ_M^A is the flow natural exchange called the canonical flow natural equivalence related to Weil functor T^A.

For any vector bundle $(E \to M)$ and any linear map $\ell : A \to \mathbb{R}$, we consider the vector bundle morphism over id_{T^AM}

$$\zeta_{A,E}^\ell : T^AE^* \to (T^AE)^*$$

defined for $j^A\phi \in T^A E^*$ and $j^A\psi \in T^A E$ by

$$\zeta^{\ell}_{A,M}(j^A\phi)(j^A\psi) = \ell\big(j^A(\langle\psi,\phi\rangle_E)\big),$$

where $\langle\psi,\phi\rangle_E : \mathbb{R}^{wA} \to \mathbb{R},\ z \mapsto \langle\psi(z),\phi(z)\rangle_E$ and $\langle\cdot,\cdot\rangle_E$ the canonical pairing ([18]).

For any manifold M of dimension m, we consider the vector bundle morphism

$$\varepsilon^{\ell}_{A,M} = \left[(\kappa^A_M)^{-1}\right]^* \circ \zeta^{\ell}_{A,TM} : T^A T^* M \to T^* T^A M.$$

It is clear that the family of maps $\left(\varepsilon^{\ell}_{A,M}\right)$ defines a natural transformation between the functors $T^A \circ T^*$ and $T^* \circ T^A$ on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms, denoted by $\varepsilon^{\ell}_{A,*} : T^A \circ T^* \to T^* \circ T^A$. When (A,ℓ) is a Weil-Frobenius algebra (see [6]), the mapping $\varepsilon^{\ell}_{A,M}$ is an isomorphism of vector bundles over $id_{T^A M}$.

3. Prolongation of some tensor fields

3.1. Natural transformations $\chi^{(\alpha)} : T^A \to T^A$

Let $T^A : \mathcal{M}f \to \mathcal{FM}$ be a Weil functor associated to a Weil algebra A and $(q : E \to M)$ a vector bundle. Similarly to what is done in [19], let's denote $\mu_E : \mathbb{R} \times E \to E,\quad (x, e_u)(\in \mathbb{R} \times E_u) \mapsto x \cdot e_u \in E_u$ the fibered multiplication. This is a vector bundle morphism over the projection $\mathbb{R} \times M \to M$. Hence, for any $a \in A$, we have a natural transformation $\overline{Q}(a) : T^A \to T^A$ given by the partial maps

$$T^A \mu_E(a,\cdot) : T^A E \to T^A E.$$

When $d_\alpha = j^A(z^\alpha),\quad \alpha \in \mathbb{N}^{wA}$, the natural transformation $\overline{Q}(d_\alpha)$ is denoted $\chi^{(\alpha)} : T^A \to T^A$. Hence, for all $\varphi \in C^\infty(\mathbb{R}^{wA}, E)$, one has

$$\chi^{(\alpha)}_{A,E}(j^A\varphi) = j^A(z^\alpha\varphi)$$

where $z^\alpha\varphi$ is the smooth map defined for any $z \in \mathbb{R}^{wA}$ by $(z^\alpha\varphi)(z) = z^\alpha\varphi(z)$.
The maps $Q(a)_M = \kappa^A_M \circ \overline{Q}(a)_{TM} \circ (\kappa^A_M)^{-1}$ define the natural affinor $Q(a) : TT^A \to TT^A$ associated to $a \in A$ (see [7]).

For each multi-index $\alpha \in \mathbb{N}^{wA}$, we consider the map

$$\chi^{(\alpha)}_{A,E\oplus E^*} : T^A(E \oplus E^*) \to T^A(E \oplus E^*),\ \ j^A(\varphi_1,\varphi_2) \mapsto (j^A(z^\alpha\varphi_1), j^A(z^\alpha\varphi_2)),$$

where $(E^* \to M)$ is the dual bundle of $(E \to M)$. It is clair that

$$\chi^{(\alpha)}_{A,E\oplus E^*} = \chi^{(\alpha)}_{A,E} \oplus \chi^{(\alpha)}_{A,E^*}.$$

3.2. Lifts of functions

Let $\upsilon : A \to \mathbb{R}$ be a smooth function, for any smooth function $f : M \to \mathbb{R}$, we define the υ-lift of f to $T^A M$ by:

$$f^{(\upsilon)} = \upsilon \circ T^A(f)$$

$f^{(\upsilon)}$ is a smooth function on $T^A M$. One verifies easily that the map

$$\begin{array}{ccc} C^\infty(M) & \to & C^\infty\left(T^A M\right) \\ f & \mapsto & f^{(\upsilon)} \end{array}$$

is $\mathbb{R}$-linear.

Remark 3.1. Let $(d_0 = 1_A, d_\beta)_{\beta\in\Re}$ a basis of A and $\left(d^*_0, d^*_\beta\right)_{\beta\in\Re}$ its dual basis . For $\upsilon = d^*_\alpha$, the smooth function $f^{(\upsilon)}$ is denoted by $f^{(\alpha)}$ and is defined for any $j^A\varphi \in T^A M$ by

$$f^{(\alpha)}\left(j^A\varphi\right) = \tfrac{1}{\alpha!} D_\alpha\left(f \circ \varphi\right)(z)\,|_{z=0} + \sum_{\beta\in\Re^c} \tfrac{\lambda^\alpha_\beta}{\beta!} D_\beta\left(f \circ \varphi\right)(z)\,|_{z=0}$$

with the convention $f^{(\gamma)} = 0$ for $\gamma \in \mathbb{Z}^{wA} \setminus (\Re \cup \{0\})$. $f^{(0)} = f \circ \pi^A_M$ is called the complete lift of f and is denoted f^c. In particular when $\left(U, u^i\right)$ is a local coordinate system in M, the adapted local coordinate system $\left\{u^i_0, \overline{u}^i_\alpha\right\}$ on $T^A M$ is such that, $u^i_0 = u^i \circ \pi^A_M$ and $\overline{u}^i_\alpha = \left(u^i\right)^{(\alpha)}$.

For any $\beta \in \Re$, $f \in C^{\infty}(M)$ and $j^A\varphi \in T^AM$, one has

$$d^*_\beta\big(j^A(z^\alpha \cdot (f\circ\varphi))\big) = \frac{1}{(\beta-\alpha)!}D_{\beta-\alpha}(\frac{1}{\alpha!}f\circ\varphi)\mid_{z=0} + \sum_{\gamma\in\Re^c}\frac{\lambda^\beta_{\gamma-\alpha}}{(\gamma-\alpha)!}D_{\gamma-\alpha}(\frac{1}{\alpha!}f\circ\varphi)\mid_{z=0}.$$

Hence, we define a function $\overline{f}^{(\beta-\alpha)}$ on T^AM by

$$\overline{f}^{(\beta-\alpha)}(j^A\varphi) = d^*_\beta\big(j^A(z^\alpha\cdot(f\circ\varphi))\big).$$

When $\alpha = 0$, one has $\overline{f}^{(\beta)} = f^{(\beta)}$.

3.3. *Prolongations of sections*

For a smooth section $\sigma : M \to E$ of a vector bundle $(E \to M)$, its α-prolongation ($\alpha \in \mathbb{N}^{wA}$ such that $1 \leq |\alpha| \leq ord(A)$) related to a Weil functor T^A is given by

$$\sigma^{(\alpha)} = \chi^{(\alpha)}_{A,E}\circ T^A\sigma$$

with the convention $\sigma^{(\gamma)} = 0,\ \forall\gamma \in \mathbb{Z}^{wA}\setminus(\Re\cup\{0\})$.

Proposition 3.1. *$\sigma^{(\alpha)}$ is a section of the vector bundle $(T^AE \to T^AM)$.*

In the particular case where $E = TM$ and $X \in \Gamma(TM)$, one has

$$X^{(\alpha)} := \kappa^A_M \circ \chi^{(\alpha)}_{A,TM}\circ T^A(X)$$

It is a vector field on $T^A(M)$ called α-prolongation of X to T^AM. In the particular case where $\alpha = 0$, the vector field $X^{(0)}$ is denoted by $X^{(c)}$ and it is called complete lift of X from M to T^AM. By convention we put $X^{(\gamma)} = 0,\ \forall\gamma\in\mathbb{Z}^{wA}\setminus(\Re\cup\{0\})$.

Remark 3.2. For any $|\alpha| \leq ord(A)$, the map

$$\mathfrak{X}(M) \to \mathfrak{X}\left(T^AM\right),\ \ X \mapsto X^{(\alpha)}$$

is $\mathbb{R}$-linear and for any smooth map $h : M \to N$ and any h-related vector fields $X \in \Gamma(TM)$, $Y \in \Gamma(TN)$, the vector fields $X^{(\alpha)} \in \Gamma\left(T^AM\right)$, $Y^{(\alpha)} \in \Gamma\left(T^AN\right)$ are $T^A(h)$ related. The set $\{X^{(\alpha)} :\ 1 \leq |\alpha| \leq ord(A)\}$ generates the $C^\infty(T^AM)$-module $\Gamma(T^AM)$ of vector fields on T^AM ([16]).

Proposition 3.2. *For any $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$ and $|\alpha,\beta| \leq ord(A)$, one has*

(i) $X^{(\alpha)}(\overline{f}^{\beta)}) = \overline{(X(f))}^{(\beta-\alpha)}$.

(ii) $\left[X^{(\alpha)}, Y^{(\beta)}\right] = [X,Y]^{(\alpha+\beta)}$.

Proof. See [7]. □

For additional properties of α-prolongation of X, see [7]-[17].

3.4. *Prolongation of* 1*-forms*

Let $\omega \in \Gamma(T^*M)$ be a 1-form on M and $\alpha \in \mathbb{N}^{wA}$ a multi-index such that $|\alpha| \leq ord(A)$. We set

$$\omega^{(\alpha)} = \varepsilon^\ell_{A,M}\circ\chi^{(\alpha)}_{A,T^*M}\circ T^A\omega. \tag{3.1}$$

$\omega^{(\alpha)}$ is clearly a 1-form on T^AM called the α-prolongation of ω to T^AM. [18].

The following remark is due to Wamba and Ntyam, [17]

Remark 3.3. For any $|\alpha| \leq ord(A)$, the map

$$\Gamma(T^*M) \to \Gamma\left(T^*T^AM\right),\ \ \omega \mapsto \omega^{(\alpha)}$$

is $\mathbb{R}$-linear. For any $\omega \in \Gamma(T^*M)$ and $X \in \Gamma(TM)$, one has

$$\omega^{(\alpha)}(X^{(\beta)}) = \sum_{\gamma\in\Re}\ell_\gamma\left(\overline{\omega(X)}\right)^{(\gamma-\alpha-\beta)},$$

and the set $\{\omega^{(\alpha)} :\ 1 \leq |\alpha| \leq ord(A)\}$ generates the $C^\infty(T^AM)$-module $\Gamma(T^*T^AM)$.

To learn more about the properties of α-prolongation of ω, see [7]-[17].

3.5. Lifts of tensor fields of type $(1, q)$.

Let S be a tensor field of type $(1, q)$, we interpret the tensor S as a q-linear mapping $S : TM \times_M \cdots \times_M TM \to TM$ of the bundle product over M of q copies of the tangent bundle TM. For all $0 \leq |\alpha| \leq ord(A)$, we put:

$$S^{(\alpha)} = \kappa_{A,M} \circ \chi^{(\alpha)}_{TM} \circ T^A(S) \circ \left((\kappa^A_M)^{-1} \times \cdots \times (\kappa^A_M)^{-1}\right) : T\left(T^A M\right) \times_{T^A M} \cdots \times_{T^A M} T\left(T^A M\right) \to T\left(T^A M\right)$$

It is tensor field of type $(1, q)$ on $T^A(M)$ called α-prolongation of the tensor field S from M to $T^A(M)$. In the particular case where $\alpha = 0$, it is denoted by $S^{(c)}$ or $\mathcal{T}^A S$ and is called complete lift of S from M to $T^A(M)$ ([3]).

The following remark is due to Gancarzewicz, Mikulski and Pogoda, [7].

Remark 3.4. The family of α-lift of vector fields is very important, because, if $\widehat{S}$ and $\widehat{S'}$ are two tensor fields of type $(1, p)$ or $(0, p)$ on $T^A(M)$ such that, for all $X_1, \cdots, X_p \in \mathfrak{X}(M)$, and multi-indices $\alpha_1, \cdots, \alpha_p$, the equality

$$\widehat{S}\left(X_1^{(\alpha_1)}, \cdots, X_p^{(\alpha_p)}\right) = \widehat{S'}\left(X_1^{(\alpha_1)}, \cdots, X_p^{(\alpha_p)}\right)$$

holds, then $\widehat{S} = \widehat{S'}$ (see [7]).

Proposition 3.3. *The tensor $S^{(\alpha)}$ is the only tensor field of type $(1, q)$ on $T^A(M)$ satisfying*

$$S^{(\alpha)}\left(X_1^{(\alpha_1)}, \cdots, X_q^{(\alpha_q)}\right) = (S(X_1, \cdots, X_q))^{\left(\alpha + \sum\limits_{i=1}^{p} \alpha_i\right)}$$

for all $X_1, \cdots, X_q \in \mathfrak{X}(M)$ and multi-index $\alpha_1, \cdots, \alpha_q$.

Proof. See [3]. □

Refer to [3] and [5] for a thorough examination of some of these lifts' characteristics.

3.6. Prolongation of pseudo-Riemannian metric

Let $(q : E \to M)$ be a vector bundle. A pseudo-Riemannian metric on E is a smooth section η of vectors bundle $E^* \otimes E^* \to M$, such that for each $x \in M$, the map

$$\eta_x \in (E^* \otimes E^*)_x \cong E^*_x \otimes E^*_x$$

is symmetric and nondegenerate. The pair (E, η) is called a pseudo-Riemannian vector bundle. If the bilinear form g_x are positive definite for every $x \in M$, then η is called a Riemannian metric and (E, η) is called a Riemannian vector bundle.
In particular, η is called pseudo-Riemannian metric on M when $E = TM$.

Proposition 3.4. *There exists a Riemannian metric on every vector bundle.*

Remark 3.5. Riemannian metric on vector bundles $E \to M$ and $F \to M$ induce canonical Riemannian metrics on E^*, $E \oplus F$, $E \otimes F$, $\bigwedge^r E$ and E/F(if F is a subbundle of E).

Consider a tensor field η of type $(0, k)$ on a smooth manifold M seen as k-linear mapping

$$\eta : TM \times_M \cdots \times_M TM \to \mathbb{R},$$

where the k-linearity means that the restriction of η to fibers are k-linear.
If $\ell : A \to \mathbb{R}$ is a linear function, we set

$$\eta^{(A,\ell)} := \ell \circ T^A \eta \circ \left((\kappa^A_M)^{-1} \times \cdots \times (\kappa^A_M)^{-1}\right) : TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M \to \mathbb{R}.$$

Hence, $\eta^{(A,\ell)}$ is a tensor of type $(0, k)$ on $T^A M$, which is called the (A, ℓ)-prolongation of η to $T^A M$.

Proposition 3.5. *Let η be a tensor field of type $(0, k)$ on a smooth manifold M, $\ell : A \to \mathbb{R}$ a linear form on a Weil algebra A. For a family of vector fields $\{X_i\}_{i=1}^k$ and a family of multi-index $\{\alpha_i \in \mathbb{N}^{wA} : 1 \leq |\alpha_i| \leq ord(A)\}_{i=1}^k$, one has*

$$\eta^{(A,\ell)}(X_1^{(\alpha_1)}, \cdots, X_k^{(\alpha_k)}) = \sum_{\gamma \in \Re} \ell_\gamma \overline{\eta(X_1, \cdots, X_k)}^{(\gamma - \sum\limits_{i=1}^{k} \alpha_i)}.$$

Proof. Let $\widetilde{x} = j^A\varphi \in T^AM$, one has

$$\begin{aligned}
\eta^{(A,\ell)}(X_1^{(\alpha_1)}, \cdots, X_k^{(\alpha_k)})(\widetilde{x}) &= \ell\Big(T^A\eta(\widetilde{x})\big(\chi^{\alpha_1}\circ T^AX_1(\widetilde{x}), \cdots, \chi^{\alpha_k}\circ T^AX_k(\widetilde{x})\big)\Big) \\
&= \ell\Big(T^A\eta(\widetilde{x})\big(j^A(z^{\alpha_1}X_1\circ\varphi), \cdots, j^A(z^{\alpha_k}X_k\circ\varphi)\big)\Big) \\
&= \ell\Big(j^A\big(z^{\alpha_1+\cdots\alpha_k}\eta(X_1, \cdots, X_k)\circ\varphi\big)\Big) \\
&= \sum_{\gamma\in\Re\cup\{0\}} \ell_\gamma d^\gamma\Big(j^A\big(z^{\alpha_1+\cdots\alpha_k}\eta(X_1, \cdots, X_k)\circ\varphi\big)\Big) \\
&= \sum_{\gamma\in\Re} \ell_\gamma\overline{\eta(X_1, \cdots, X_k)}^{(\gamma-\sum_{i=1}^{k}\alpha_i)}(\widetilde{x}).
\end{aligned}$$

□

We have following result due to Gancarzewicz, Mikulski and Pogoda

Proposition 3.6. *Let $\ell : A \to \mathbb{R}$ be a linear form on a Weil algebra A and η a pseudo-Riemannian metric on a smooth manifold M. The (A, ℓ)-prolongation $\eta^{(A,\ell)}$ of η is a pseudo-Riemannian metric on T^AM if and only if (A, ℓ) is a Weil-Frobenius algebra.*

Proof. See [7]. □

3.7. Prolongation of linear connections

Given an arbitrary vector bundle $(\nu : E \to M)$, a linear connection on E is a vector bundle morphism $\Gamma^E : TM \times_M E \to TE$ over id_{TM} and id_E such that $T\nu \circ \Gamma^E = \nu_1$, $\pi_E \circ \Gamma^E = \nu_2$, where ν_1, ν_2 are restrictions of canonical projection to $TM \times_M E$. In particular, Γ^E is called linear connection on M when $E = TM$.

Let Γ^E be a linear connection on $(E \to M)$. A covariant derivative associate to Γ^E is a map

$$\nabla^E : \Gamma(TM) \times \Gamma(E) \to \Gamma(E),\ (x, \sigma) \mapsto \nabla_X\sigma$$

which is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in σ and satisfies Leibnitz rule

$$\nabla^E_X(f \cdot \sigma) = X(f) \cdot \sigma + f \cdot \nabla^E_X\sigma.$$

A connection on E with a Riemannian metric g is called a metric connection if we have

$$X(\eta(\sigma_1, \sigma_2)) = \eta(\nabla^E_X\sigma_1, \sigma_2) + \eta(\sigma_1, \nabla^E_X\sigma_2),$$

for all $(X, \sigma_1, \sigma_2) \in \Gamma(TM) \times \Gamma(E) \times \Gamma(E)$.

Remark 3.6. (i) In case $E = TM$, the Riemannian connection (also called Levi-Civita connection) is the unique connection that is metric and torsion free.

(ii) If E and F are two vector bundles on the same base manifold M with linear connections Γ^E and Γ^F respectively, then these linear connections induce linear connections on the vector bundles E^*, $E \oplus F$ and $E \otimes F$ whose covariant derivatives are respectively defined by

$$\begin{aligned}
(\nabla^{E^*}_X\omega)(\sigma) &= X(\omega(\sigma)) - \omega(\nabla^E_X\sigma) \\
\nabla^{E\oplus F}_X\sigma_1 \oplus \sigma_2 &= \nabla^E_X\sigma_1 \oplus \nabla^F_X\sigma_2 \\
\nabla^{E\otimes F}_X\sigma_1 \otimes \sigma_2 &= (\nabla^E_X\sigma_1) \otimes \sigma_2 + \sigma_1 \otimes \nabla^F_X\sigma_2,
\end{aligned}$$

for all $(X, \omega) \in \Gamma(TM) \times \Gamma(E^*)$, $\sigma,\ \sigma_1,\ \sigma_2 \in \Gamma(E)$.
If Γ^E and Γ^F are metric connections, then the induced linear connections are metric connections with respect to the induced Riemannian metrics.

Given a Weil functor T^A and a linear connection Γ on a vector bundle $(E \to M)$, one defines the linear connection $\mathcal{T}^A\Gamma$ on $(T^A E \to T^A M)$ (see [24]) by

$$\mathcal{T}^A\Gamma^E := \kappa_E^A \circ T^A\Gamma^E \circ \left((\kappa_M^A)^{-1} \times_{T^A M} id_{T^A E}\right) : TT^A M \times_{T^A M} T^A E \to TT^A E.$$

In the particular case where Γ^E is a linear connection on M (i.e. $E = TM$), one define a linear connection on $(TT^A M \to T^A M)$ by

$$\mathcal{T}^{TT^A M}\Gamma^{TM} := T\kappa_M^A \circ \mathcal{T}^A\Gamma^{TM} \circ \left(id_{TT^A M} \times_{T^A M} (\kappa_M^A)^{-1}\right) : TT^A M \times_{T^A M} TT^A M \to TTT^A M,$$

which is called the canonical (or complete) lift of Γ^{TM} related to T^A and denoted by Γ^c.

We present the following result, which is due to Ntyam and Wouafo, [19]

Proposition 3.7. *Let Γ^{TM} be a linear connection on $(TM \to M)$ with associated covariant derivative ∇^{TM} and $\mathcal{T}^{TT^A M}\Gamma^{TM}$ its complete lift with associated covariant derivative $\mathcal{T}^{TT^A M}\nabla^{TM}$. Then, $\mathcal{T}^{TT^A M}\Gamma^{TM}$ is the only linear connection on $(TT^A M \to T^A M)$ satisfying the equality*

$$\mathcal{T}^{TT^A M}\nabla^{TM}{}_{X^{(\alpha)}}Y^{(\beta)} = \begin{cases} (\nabla^{TM}{}_X Y)^{(\alpha+\beta)}, & \forall \alpha, \beta \in \mathbb{N}^{wA} : \ 0 \le |\alpha + \beta| \le ord(A) \\ 0, & \forall \alpha, \beta \in \mathbb{N}^{wA} : \ |\alpha + \beta| > ord(A) \end{cases} \tag{3.2}$$

for all $X, \ Y \in \Gamma(TM)$.

Proof. See [19]. □

4. Generalized metallic pseudo-Riemannian manifolds

4.1. Metallic pseudo-Riemannian manifolds

Let M be a smooth manifold

The metallic means family (also called metallic proportions) was introduced by V.M. de Spinadel in [23] as a positive root $\tau_{(p,q)} := \dfrac{1}{2}(p + \sqrt{p^2 + 4q})$ of the algebraic equation

$$x^2 - px - q = 0$$

where p and q are two positive integers. In particular,

- ⋄ $\tau_{1,1}$ is golden ratio or divine ratio which is used in the field of architecture, medicine, financial market,...[26].
- ⋄ $\tau_{2,1}$ is silver ratio which is used in fractal geometry.
- ⋄ $\tau_{3,1}$ is bronze ratio which is in dynamical system.
- ⋄ $\tau_{1,3}$ is nickel ratio etc.

Inspired by the metallic proportions, Hreţcanu and Crâşmareanu, Ozkan and Yilmaz ([13]-[22]) introduced the notion of metallic structure on a smooth manifold M as an $C^\infty(M)$-endomorphism Λ of TM satisfying the following equality

$$\Lambda^2 - p\Lambda - qid = 0,$$

where $p, \ q$ are positive integers and id is the identity operator on the Lie algebra of vector fields on M. In this case ,the pair (M, Λ) is called a metallic manifold. Moreover, the triple (M, η, Λ) will be called a metallic pseudo-Riemannian manifold if the tensor fields η and Λ are compatible, that is

$$\eta(\Lambda X, Y) = \eta(X, \Lambda Y) \text{ (or } \eta(\Lambda X, \Lambda Y) = p\eta(\Lambda X, Y) + q\eta(X, Y)),$$

for all $X, Y \in \Gamma(TM)$.

Remark 4.1. In particular, an almost product structure (resp. an almost complex structure) is an $C^\infty(M)$-endomorphism Λ of TM which satisfies the algebraic equation $X^2 - id = 0$ (resp. $X^2 + id = 0$). When $\Lambda^2 = 0$ (resp. $\Lambda^2 - \Lambda - id = 0$), we have the notion of almost tangent structure (resp. Golden-structure) (See [4]).

Let (M, Λ) be a metallic manifold. We recall that Λ is integrable if its Nijenhuis tensor field

$$N_\Lambda(X, Y) = \Lambda^2[X, Y] + [\Lambda X, \Lambda Y] - \Lambda[\Lambda X, Y] - \Lambda[X, \Lambda Y]$$

vanishes for all vector fields X, Y in M.

4.2. Generalized metallic structures

Let $\mathbb{T}M = TM \oplus T^*M \to M$ be a generalized tangent bundle associated to a smooth manifold M and $\Gamma(\mathbb{T}M)$ its $C^\infty(M)$-module of smooth sections. The elements of $\Gamma(\mathbb{T}M)$ are of the form (X, ω) where $X \in \Gamma(TM)$ is a smooth vector field and $\omega \in \Gamma(T^*M)$ is a differential form of degree 1 on M. $\mathbb{T}M$ is equipped with a natural symplectic structure Φ and a natural indefinite metric $\widetilde{\Phi}$ defined for all $(X,\omega),\ (Y,\varpi) \in \Gamma(\mathbb{T}M)$ by

$$\Phi\big((X,\omega),(Y,\varpi)\big) = \frac{1}{2}\big(\varpi(X) - \omega(X)\big) \text{ and } \widetilde{\Phi}\big((X,\omega),(Y,\varpi)\big) = -\frac{1}{2}\big(\varpi(X) + \omega(X)\big)$$

Definition 4.1. ([2]) A generalized metallic structure on M is a $\mathcal{C}^\infty(M)$-endomorphism Λ_g of $\mathbb{T}M$ satisfying

$$(\Lambda_g)^2 = p\Lambda_g + qid,$$

for some real numbers p and q.

A given linear connection Γ^{TM} on $(TM \to M)$ with its associated covariant derivative ∇^{TM} defines a bracket $[\cdot,\cdot]_{\Gamma^{TM}}$ on $\Gamma(TM \oplus T^*M) \cong \Gamma(TM) \oplus \Gamma(T^*M)$ (see [2]-[20]) by

$$[(X,\omega),(Y,\varpi)]_{\Gamma^{TM}} := \big([X,Y], (\nabla^{T^*M}{}_X\varpi - \nabla^{T^*M}{}_Y\omega)\big),$$

for all $X, Y \in \Gamma(TM)$ and $\omega, \varpi \in \Gamma(T^*M)$, where ∇^{T^*M} is the extension of the covariant derivative ∇^{TM} to bundle of one forms. This bracket satisfies the following properties:

Proposition 4.1. *([20]) For all $\sigma, \widetilde{\sigma} \in \Gamma(TM \oplus T^*M)$ and all $f \in \mathcal{C}^\infty(M)$, one has*

(i) $[\sigma, \widetilde{\sigma}]_{\Gamma^{TM}} = -[\widetilde{\sigma}, \sigma]_{\Gamma^{TM}}$

(ii) $[\sigma, f\widetilde{\sigma}]_{\Gamma^{TM}} = f[\sigma, \widetilde{\sigma}]_{\Gamma^{TM}} + X(f)\widetilde{\sigma}$

(iii) $[\cdot,\cdot]_{\Gamma^{TM}}$ *verifies Jacobi's identity if and only if* Γ^{TM} *has zero curvature.*

Definition 4.2. ([20]) A generalized metallic structure Λ_g is ∇^{TM}-integrable if its Nijenhuis tensor field $N_{\Lambda_g}^{\nabla^{TM}}$ with respect to Γ^{TM}

$$N_{\Lambda_g}^{\Gamma^{TM}}(\sigma_1, \sigma_2) = \Lambda_g^2[\sigma, \widetilde{\sigma}]_{\nabla^{TM}} + [\Lambda_g\sigma, \Lambda_g\widetilde{\sigma}]_{\Gamma^{TM}} - \Lambda_g[\Lambda_g\sigma, \widetilde{\sigma}]_{\Gamma^{TM}} - \Lambda_g[\sigma, \Lambda_g\widetilde{\sigma}]_{\Gamma^{TM}}$$

vanishes for all $\sigma,\ \widetilde{\sigma} \in \Gamma(\mathbb{T}M)$.

5. Prolongation of generalized metallic structure to weil bundle and some properties

5.1. Prolongation of generalized metallic structure to weil bundle

Let M be a smooth pseudo-Riemannian manifold of dimension $m > 0$ and (A, ℓ) a Weil-Frobenius algebra. Let consider the following natural equivalences

$$\kappa_M^A : T^ATM \to TT^AM \quad \text{and} \quad \varepsilon_{A,M}^\ell : T^AT^*M \to T^*T^AM. \tag{5.1}$$

Hence, the bundle morphism

$$\kappa_M^A \oplus \varepsilon_{A,M}^\ell : T^ATM \oplus T^AT^*M \to TT^AM \oplus T^*T^AM \tag{5.2}$$

is an isomorphism of vector bundles over id_{T^AM}.

Let Λ_g be a generalized metallic structure on M that is an endomorphism of $\mathbb{T}M = TM \oplus T^*M$ defined by

$$(\Lambda_g)^2 = p\Lambda_g + qid,$$

where p and q are some real numbers.

We set:

$$\mathcal{T}^A\Lambda_g := \kappa_M^A \oplus \varepsilon_{A,M}^\ell \circ T^A\Lambda_g \circ (\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^\ell)^{-1}. \tag{5.3}$$

Theorem 5.1. *The endomorphism $\mathcal{T}^A\Lambda_g$ define a generalized metallic structure on T^AM.*

Proof. From the definition, it follows that $(\Lambda_g)^2 = p\Lambda_g + qid$. Therefore,

$$\begin{aligned}
(\mathcal{T}^A\Lambda_g)^2 &= \mathcal{T}^A\Lambda_g \circ \mathcal{T}^A\Lambda_g \\
&= \kappa^A_M \oplus \varepsilon^\ell_{A,M} \circ T^A(\Lambda_g \circ \Lambda_g) \circ (\kappa^A_M)^{-1} \oplus (\varepsilon^\ell_{A,M})^{-1} \\
&= \kappa^A_M \oplus \varepsilon^\ell_{A,M} \circ T^A(p\Lambda_g + qid) \circ (\kappa^A_M)^{-1} \oplus (\varepsilon^\ell_{A,M})^{-1} \\
&= p\mathcal{T}^A\Lambda_g + qid.
\end{aligned}$$

□

Definition 5.1. The endomorphism $\mathcal{T}^A\Lambda_g$ is called the ℓ-prolongation of generalized metallic structure Λ_g form M to T^AM.

Remark 5.1. In the particular case, where

$$\begin{aligned}
\ell = \tau_r : \mathbb{D}^r_1 &\to \mathbb{R} \\
j^r_0\varphi &\mapsto \frac{1}{r!}\frac{d^r}{dt^r}(\varphi(t))|_{t=0}.
\end{aligned}$$

We obtain the tangent generalized metallic structure of high order r of generalized metallic structure $\Lambda_g : TM \oplus T^*M \to TM \oplus T^*M$ from M to T^rM described in [25].

For each multi-index $\alpha \in \mathbb{N}^{wA}$, we have

$$\chi^{(\alpha)}_{A,TM\oplus T^*M} = \chi^{(\alpha)}_{A,TM} \oplus \chi^{(\alpha)}_{A,T^*M},$$

and let put

$$\sigma^{(\alpha)} = (X^{(\alpha)}, \omega^{(\alpha)}),$$

for any $\sigma = (X, \omega) \in \Gamma(TM) \oplus \Gamma(T^*M)$. we have the following proposition

Proposition 5.1. *Let Λ_g be a generalized metallic structure on M. For all $\sigma = (X, \omega) \in \Gamma(TM) \oplus \Gamma(T^*M)$ and $\alpha \in \mathbb{N}^{wA}$ such that $0 \leq |\alpha| \leq ord(A)$, one has*

(i) $T^A\Lambda_g \circ \chi^{(\alpha)}_{TM\oplus T^*M} \circ T^A(\sigma) = \chi^{(\alpha)}_{TM\oplus T^*M} \circ T^A(\Lambda_g(\sigma))$;

(ii) $\mathcal{T}^A\Lambda_g(\sigma^{(\alpha)}) = (\Lambda_g(\sigma))^{(\alpha)}$.

Proof. (i) For all $\sigma = (X, \omega) \in \Gamma(TM) \oplus \Gamma(T^*M)$ and $j^A\varphi \in T^AM$, one has

$$\begin{aligned}
T^A\Lambda_g \circ \chi^{(\alpha)}_{TM\oplus T^*M} \circ T^A(\sigma)(j^A\varphi) &= T^A\Lambda_g\Big(\chi^{(\alpha)}_{TM}(T^AX(j^A\varphi)), \chi^{(\alpha)}_{T^*M}(T^A\omega(j^A\varphi))\Big) \\
&= T^A\Lambda_g(j^Az^\alpha X \circ \varphi, j^At^\alpha\omega \circ \varphi) \\
&= T^A\Lambda_g(j^Az^\alpha(X \circ \varphi, \omega \circ \varphi)) \\
&= j^Az^\alpha\Lambda_g(X \circ \varphi, \omega \circ \varphi) \\
&= \chi^{(\alpha)}_{TM\oplus T^*M}(j^A\Lambda_g(X \circ \varphi, \omega \circ \varphi)) \\
&= \chi^{(\alpha)}_{TM\oplus T^*M} \circ T^A(\Lambda_g(\sigma))(j^A\varphi).
\end{aligned}$$

(ii) One has

$$\begin{aligned}
\mathcal{T}^A\Lambda_g(\sigma^{(\alpha)}) &= \kappa^A_M \oplus \varepsilon^\ell_{A,M} \circ T^A\Lambda_g \circ \eta^A_M \oplus \alpha^A_M(X^{(\alpha)}, \omega^{(\alpha)}) \\
&= \kappa^A_M \oplus \varepsilon^\ell_{A,M} \circ T^A\Lambda_g\Big(\chi^{(\alpha)}_{TM} \circ T^AX, \chi^{(\alpha)}_{T^*M} \circ T^A\omega\Big) \\
&= \kappa^A_M \oplus \varepsilon^\ell_{A,M} \circ T^A\Lambda_g \circ \chi^{(\alpha)}_{TM\oplus T^*M}(T^A(X, \omega)) \\
&= \kappa^A_M \oplus \varepsilon^\ell_{A,M} \circ \chi^{(\alpha)}_{TM\oplus T^*M} \circ T^A(\Lambda_g(X, \omega)) \\
&= (\Lambda_g\sigma)^{(\alpha)}
\end{aligned}$$

Hence, the proof is complete.

□

5.2. Prolongation of pseudo-Riemannian metrics and connections on vector bundle $(TM \oplus T^*M \to M)$

5.2.1. Prolongation of pseudo-Riemannian metric Let (A, ℓ) be a Weil-Frobenius algebra and G be a pseudo-Riemannian metric on a generalized tangent bundle $TM \oplus T^*M \to M$. We set

$$G^{(A,\ell)} \quad = \quad \ell \circ T^A G \circ \Big((\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^{\ell})^{-1} \times (\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^{\ell})^{-1} \Big).$$

Proposition 5.2. *The tensor $G^{(A,\ell)}$ is a pseudo-Riemannian metric on the vector bundle $TT^AM \oplus T^*T^AM \to T^AM$.*

Proof. See proposition 3.6. □

Definition 5.2. The tensor $G^{(A,\ell)}$ is called an (A, ℓ)-prolongation of the tensor G from $TM \oplus T^*M$ to $TT^AM \oplus T^*T^AM$.

Proposition 5.3. *If G is a tensor field of type $(0, 2)$ on $TM \oplus T^*M$, then for multi-indices $\alpha_1,\ \alpha_2 \in \mathbb{N}^{wA}$ and all $\sigma_1,\ \sigma_2 \in \Gamma(TM \oplus T^*M)$, we have*

$$G^{(A,\ell)}(\sigma_1^{\alpha_1}, \sigma_2^{\alpha_2}) \quad = \quad \sum_{\beta \in B_A} \ell_\beta \overline{G(\sigma_1, \sigma_2)}^{(\beta-(\alpha_1+\alpha_2))}.$$

Proof. For multi-indices $\alpha_1,\ \alpha_2 \in \mathbb{N}^{wA}$, $\sigma_1,\ \sigma_2 \in \Gamma(TM \oplus T^*M)$ and $\widetilde{x} = j^A\varphi \in T^AM$, one has

$$\begin{aligned} G^{(A,\ell)}(\sigma_1^{(\alpha_1)}, \sigma_2^{(\alpha_2)})(\widetilde{x}) &= \ell\Big[T^A G\Big((\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^{\ell})^{-1}(\sigma_1^{(\alpha_1)})(\widetilde{x})(\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^{\ell})^{-1}(\sigma_2^{(\alpha_2)})(\widetilde{x}) \Big)\Big] \\ &= \ell\Big[T^A G\Big(\chi_{A,TM\oplus T^*M}^{(\alpha_1)}(T^A\sigma_1^{(\alpha_1)}(\widetilde{x})), \chi_{A,TM\oplus T^*M}^{(\alpha_2)}(T^A\sigma_2^{(\alpha_2)}(\widetilde{x}))\Big)\Big] \\ &= \ell\Big(j^A(z^{\alpha_1+\alpha_2} G(\sigma_1 \circ \varphi, \sigma_2 \circ \varphi))\Big) \\ &= \sum_{\beta \in B_A} \ell_\beta \overline{G(\sigma_1, \sigma_2)}^{(\beta-(\alpha_1+\alpha_2))}(\widetilde{x}). \end{aligned}$$

□

Corollary 5.1. *Let (G, Λ_g) be a generalized Riemannian metallic structure on M. If Λ_g is G-symmetric, then $\mathcal{T}^A\Lambda_g$ is $G^{(A,\ell)}$-symmetric too.*

Proof. For all multi-indices $\alpha_1,\ \alpha_2 \in \mathbb{N}^{wA}$ and $\sigma_1,\ \sigma_2 \in \Gamma(TM \oplus T^*M)$, one has

$$\begin{aligned} G^{(A,\ell)}(\mathcal{T}\Lambda_g(\sigma_1^{(\alpha_1)}), \sigma_2^{(\alpha_2)}) &= G^{(A,\ell)}((\Lambda_g\sigma_1)^{(\alpha_1)}, \sigma_2^{(\alpha_2)}) \\ &= \sum_{\beta \in B_A} \ell_\beta \overline{G(\Lambda_g\sigma_1, \sigma_2)}^{(\beta-(\alpha_1+\alpha_2))} \\ &= \sum_{\beta \in B_A} \ell_\beta \overline{G(\sigma_1, \Lambda_g\sigma_2)}^{(\beta-(\alpha_1+\alpha_2))} \text{since } \Lambda_g \text{ is } G\text{-symmetric} \\ &= G^{(A,\ell)}(\sigma_1^{(\alpha_1)}, \mathcal{T}^A\Lambda_g(\sigma_2^{(\alpha_2)}). \end{aligned}$$

□

5.2.2. Prolongation of connections Let Γ^{TM} be a linear connection on vector bundle $(\pi_M : TM \to M)$, ∇^{TM} its covariant derivative and $_{\nabla^{T^*M}}$ the covariant derivative of the induced linear connection Γ^{T^*M} on $(T^*M \to M)$. We have

$$\nabla^{T^*M} : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M), \ \ (\nabla^{T^*M}{}_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla^{TM}{}_X Y).$$

for all $(X, Y, \omega) \in \Gamma(TM) \times \Gamma(TM) \times \Gamma((TM)^*)$. Let's denote by $\mathcal{T}^{T^*T^AM}\Gamma^{TM}$ the induced linear connection by $\mathcal{T}^{TT^AM}\Gamma^{TM}$ on vector bundle $(T^*T^AM \to T^AM)$ and $\mathcal{T}^{T^*T^AM}\nabla^{TM}$ its covariant derivative. Then, one has the following results:

Proposition 5.4. *For all $\sigma_1 = (X, \omega)$, $\sigma_2 = (Y, \varpi) \in \Gamma(TM \oplus T^*M)$, and all multi-indices $\alpha_1, \alpha_2 \in \mathbb{N}^{wA}$ such that $0 \leq |\alpha_i| \leq ord(A)$, $i = 1, 2$, one has*

(i) $\mathcal{T}^{T^*T^AM}\Gamma^{TM}$ is the only connection on $(T^*T^AM \to T^AM)$ verifying

$$\mathcal{T}^{T^*T^AM}\nabla^{TM}{}_{X^{(\alpha_1)}}\omega^{(\alpha_2)} = \left(\nabla^{T^*M}{}_X\omega\right)^{(\alpha_1+\alpha_2)},$$

(ii) $\left[\sigma_1^{(\alpha_1)}, \sigma_2^{(\alpha_2)}\right]_{\mathcal{T}^{TT^AM}\Gamma^{TM}} = \left([\sigma_1, \sigma_2]_{\Gamma^{TM}}\right)^{(\alpha_1+\alpha_2)}$

Proof. (i) One has

$$\begin{aligned}\left(\mathcal{T}^{T^*T^AM}\nabla^{TM}{}_{X^{(\alpha_1)}}\omega^{(\alpha_2)}\right)(Y^{(\gamma)}) &= X^{(\alpha_1)}\left(\omega^{(\alpha_2)}(Y^{(\gamma)})\right) - \omega^{(\alpha_2)}\left(\mathcal{T}^{TT^AM}\nabla^{TM}{}_{X^{(\alpha_1)}}Y^{(\gamma)}\right)\\ &= \sum_{\nu\in\Re}\ell_\nu X^{(\alpha_1)}\left(\overline{\omega(Y)}^{(\nu-\alpha_2-\gamma)}\right) - \omega^{(\alpha_2)}\left((\nabla^{TM}{}_XY)^{(\alpha_1+\gamma)}\right)\\ &= \sum_{\nu\in\Re}\ell_\nu\left(\overline{X(\omega(Y))}^{(\nu-(\alpha_1+\alpha_2+\gamma))}\right) - \sum_{\nu\in\Re}\ell_\nu\left(\overline{\omega(\nabla^{TM}{}_XY)}^{(\nu-(\alpha_1+\alpha_2+\gamma))}\right)\\ &= \left(\nabla^{T^*M}{}_X\omega\right)^{(\alpha_1+\alpha_2)}\left(Y^{(\gamma)}\right)\end{aligned}$$

(ii) One has

$$\begin{aligned}\left[\sigma_1^{(\alpha_1)}, \sigma_2^{(\alpha_2)}\right]_{\mathcal{T}^{TT^AM}\Gamma^{TM}} &= \left[(X^{(\alpha_1)}, \omega^{(\alpha_1)}), (Y^{(\alpha_2)}, \varpi^{(\alpha_2)})\right]_{\mathcal{T}^{TT^AM}\nabla^{TM}}\\ &= \left([X, Y]^{(\alpha_1+\alpha_2)}, (\nabla^{T^*M}{}_X\varpi)^{(\alpha_1+\alpha_2)} - (\nabla^{T^*M}{}_X\omega)^{(\alpha_1+\alpha_2)}\right)\\ &= \left([\sigma_1, \sigma_2]_{\Gamma^{TM}}\right)^{(\alpha_1+\alpha_2)}.\end{aligned}$$

□

Theorem 5.2. *For all $\sigma_1 = (X, \omega)$, $\sigma_2 = (Y, \varpi) \in \Gamma(TM \oplus T^*M)$, and all multi-index $\alpha, \beta \in \mathbb{N}^{wA}$ such that $0 \leq |\alpha, \beta| \leq ord(A)$, one has*

$$N^{\mathcal{T}^{TT^AM}\Gamma^{TM}}_{\mathcal{T}^A\Lambda_g}\left(\sigma_1^{(\alpha)},\ \sigma_2^{(\beta)}\right) = \left(N^{\Gamma^{TM}}_{\Lambda_g}(\sigma_1,\ \sigma_2)\right)^{(\alpha+\beta)}.$$

where $N^{\mathcal{T}^{TT^AM}\Gamma^{TM}}_{\mathcal{T}^A\Lambda_g}$ denote the Nijenhuis tensor field of $\mathcal{T}^A\Lambda_g$ with respect to $\mathcal{T}^{TT^AM}\Gamma^{TM}$.

Proof. The proof is based on the previous proposition. □

Corollary 5.2. *A generalized metallic structure $\mathcal{T}^A\Lambda_g$ on T^AM is $\mathcal{T}^{TT^AM}\Gamma^{TM}$-integrable if and only if Λ_g on M is Γ^{TM}-integrable too.*

Proposition 5.5. *If (Λ_g, G) is generalized metallic Riemannian structure on $TM \oplus T^*M$, then $(\mathcal{T}^A\Lambda_g, G^{(A,\ell)})$ is also a generalized metallic Riemannian structure on $TT^AM \oplus T^*T^AM$.*

5.3. Prolongation of generalized metallic structure induced by (Λ, η)

. Let (A, ℓ) be Weil-Frobenius algebra and M a smooth manifold. A pseudo-Riemannian manifold (M, η) gives rise to the musical vector bundle isomorphism $\flat_\eta : TM \to T^*M$ and its inverse $\sharp_\eta : T^*M \to TM$ naturally induced by the $C^\infty(M)$-module isomorphism

$$\begin{aligned}\flat_\eta : \Gamma(TM) &\to \Gamma(T^*M)\\ X &\mapsto i_X\eta\end{aligned}$$

and $\flat_\eta^{-1} = \sharp_\eta$. Remak that, the vector bundle morphism $\flat_\eta$ can be defined for any covariant tensor field η of type $(0,2)$ on M, but its inverse exists only if η is non-degenerate. Let Λ be a metallic structure on (M,η) such that $\Lambda^2 = p\Lambda + qid$, where p and q are some real numbers. Since Λ is η-symmetry, we have

$$\flat_\eta \circ \Lambda = \Lambda^* \circ \flat_\eta \ \text{ and } \ \sharp_\eta \circ \Lambda^* = \Lambda \circ \sharp_\eta \tag{5.4}$$

where $\Lambda^* : T^*M \to T^*M$ denote the dual map of Λ. This dual map is also a metallic structure such that $(\Lambda^*)^2 = p\Lambda^* + qid$.

Remark 5.2. For any $0 \le |\alpha| \le ord(A)$, one has

$$\begin{aligned} T^A\sharp \circ \chi^{(\alpha)}_{A,T^*M} &= \chi^{(\alpha)}_{A,TM} \circ T^A\sharp, \\ T^A\Lambda^* \circ \chi^{(\alpha)}_{A,T^*M} &= \chi^{(\alpha)}_{A,T^*M} \circ T^A\Lambda^* \end{aligned}$$

Proposition 5.6. *Let (M,η) is a pseudo-Riemannian manifold. The $C^\infty(M)$-module isomorphism*

$$\begin{aligned} \flat_{\eta^{(A,\ell)}} : \Gamma(TT^AM) &\to \Gamma(T^*T^AM) \\ \underline{X} &\mapsto i_{\underline{X}}\eta^{(A,\ell)} \end{aligned}$$

is defined by

$$\flat_{\eta^{(A,\ell)}} = \varepsilon^\ell_{A,M} \circ T^A\eta \circ \left(\kappa^A_M\right)^{-1}.$$

(where $\eta^{(A,\ell)} = \ell \circ T^A\eta \circ ((\kappa^A_M)^{-1} \times (\kappa^A_M)^{-1})$). In this case, its inverse is defined by

$$\sharp_{\eta^{(A,\ell)}} = (\kappa^A_M)^{-1} \circ T^A\eta \circ \left(\varepsilon^\ell_{A,M}\right)^{-1}.$$

Proof. Let $X,\ Y \in \Gamma(TM)$, $\widetilde{x} = j^A\varphi \in T^AM$ and $\alpha,\ \beta$ two multi-indices. One has,

$$\begin{aligned} \left(\varepsilon^\ell_{A,M} \circ T^A\eta \circ \left(\kappa^A_M\right)^{-1}\right)(X^{(\alpha)}(\widetilde{x}))(Y^{(\beta)}(\widetilde{x})) &= \varepsilon^\ell_{A,M}\left(\left(T^A\eta \circ \chi^\alpha_{A,TM} \circ T^AX\right)(\widetilde{x})\right)(Y^{(\beta)}(\widetilde{x})) \\ &= \varepsilon^\ell_{A,M}\left(\left(\chi^\alpha_{A,T^*M} \circ T^A\eta \circ T^AX\right)(\widetilde{x})\right)(Y^{(\beta)}(\widetilde{x})) \\ &= \varepsilon^\ell_{A,M}\left(j^A\left(z^\alpha\flat_\eta \circ X \circ \varphi\right)\right)(Y^{(\beta)}(\widetilde{x})) \\ &= \zeta^\ell_{A,TM}\left(j^A\left(z^\alpha\flat_\eta \circ X \circ \varphi\right)\right)\left(\kappa^A_M \circ Y^{(\beta)}(\widetilde{x})\right) \\ &= \zeta^\ell_{A,TM}\left(j^A\left(z^\alpha\flat_\eta \circ X \circ \varphi\right)\right)\left(j^A\left(z^\beta Y \circ \varphi\right)\right) \\ &= \ell\left(j^A z^{\alpha+\beta}\left(\langle Y \circ \varphi, \flat_\eta \circ X \circ \varphi\rangle_{TM}\right)\right) \\ &= \eta^{(A,\ell)}\left(X^{(\alpha)}, Y^{(\beta)}\right)(\widetilde{x}). \end{aligned}$$

□

Remark 5.3. For all $X \in \Gamma(TM)$ and $|\alpha| \le ord(A)$, one has $\flat_{\eta^{(A,\ell)}}\left(X^{(\alpha)}\right) = \left(\flat_\eta(X)\right)^{(\alpha)}$. Indeed, for all $X,\ Y \in \Gamma(TM)$ and $|\alpha,\ \beta| \le ord(A)$, one has

$$\begin{aligned} \flat_{\eta^{(A,\ell)}}\left(X^{(\alpha)}\right)\left(Y^{(\beta)}\right) &= \eta^{(A,\ell)}\left(X^{(\alpha)}, Y^{(\beta)}\right) \\ &= \sum_{\gamma \in \Re} \ell_\gamma \overline{\eta(X,Y)}^{(\gamma-(\alpha+\beta))} \\ &= \sum_{\gamma \in \Re} \ell_\gamma \overline{\eta(X,Y)}^{(\gamma-(\alpha+\beta))} \\ &= \left(\flat_\eta(X)\right)^{(\alpha)}\left(Y^{(\beta)}\right) \end{aligned}$$

On vector bundle $(\mathbb{T}M \to M)$, we consider the pseudo-Riemannian metric $\widehat{G}$ (induced by η) defined by

$$\widehat{G}\big((X,\omega),(Y,\varpi)\big) = \Big(\eta(X,Y), \eta(\sharp_g\omega, \sharp_g\varpi)\Big)$$

for all $(X,\omega),\ (Y,\varpi) \in \Gamma(\mathbb{T}M)$. A pair $(\Lambda_g, \widehat{G})$ of a generalized metallic structure Λ_g and a pseudo-Riemannian metric $\widehat{G}$ such that Λ_g is $\widehat{G}$-symmetric is called generalized metallic pseudo-Riemannian structure on M. Hence, a pair $\left(\widehat{\Lambda_g} := \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^* \end{pmatrix}, \widehat{G}\right)$ is a generalized metallic pseudo-Riemannian structure induced by the metallic pseudo-Riemannian structure (Λ, η) on M(see [2]).

We set

$$\widehat{G}^{(A,\ell)} := \eta^{(A,\ell)} \oplus (\eta^{(A,\ell)} \circ (\sharp_{\eta^{(A,\ell)}} \times \sharp_{\eta^{(A,\ell)}})) \text{ and } \mathcal{T}^A\widehat{\Lambda_g} := \begin{pmatrix} \mathcal{T}^A\Lambda & 0 \\ 0 & (\Lambda^*)^{(A,\ell)} \end{pmatrix}$$

where $\begin{cases} \mathcal{T}^A\Lambda = \kappa^A_M \circ T^A\Lambda \circ (\kappa^A_M)^{-1} \\ (\Lambda^*)^{(A,\ell)} = \varepsilon^{\ell}_{A,M} \circ T^A\Lambda^* \circ (\varepsilon^{\ell}_{A,M})^{-1}. \end{cases}$

Proposition 5.7. *The endomorphism $\mathcal{T}^A\widehat{\Lambda_g} : \mathbb{T}T^AM \to \mathbb{T}T^AM$ is a metallic structure such that*

$$(\mathcal{T}^A\widehat{\Lambda_g})^2 = p\mathcal{T}^A\widehat{\Lambda_g} + qid$$

Proposition 5.8. *For any $\omega \in \Gamma(T^*M)$ and any $0 \le |\alpha| \le ord(A)$, one has*

(i) $\sharp_{\eta^{(A,\ell)}}(\omega^{(\alpha)}) = (\sharp\omega)^{(\alpha)}$.

(ii) $(\Lambda^*)^{(A,\ell)}(\omega^{(\alpha)}) = (\Lambda^*(\omega))^{(\alpha)}$.

(iii) $\sharp_{\eta^{(A,\ell)}}((\Lambda^*)^{(A,\ell)}(\omega^{(\alpha)})) = (\Lambda(\sharp_\eta\omega))^{(\alpha)}$.

Proof. By using the remark 5.2 and relation (5.4). □

Proposition 5.9. *For any $\sigma,\ \widetilde{\sigma} \in \Gamma(\mathbb{T}M)$ and any multi-indices $\alpha,\ \beta \in \mathbb{N}^{wA}$ such that $0 \le |\alpha,\beta| \le ord(A)$, one has*

$$\widehat{G}^{(A,\ell)}\Big(\mathcal{T}^A\widehat{\Lambda_g}\sigma^{(\alpha)}, \widetilde{\sigma}^{(\beta)}\Big) = \widehat{G}^{(A,\ell)}\Big(\sigma^{(\alpha)}, \mathcal{T}^A\widehat{\Lambda_g}\widetilde{\sigma}^{(\beta)}\Big).$$

Proof. Let $\sigma = (X,\omega)$, $\widetilde{\sigma} = (Y,\varpi)$ be two elements of $\Gamma(\mathbb{T}M)$ and $\alpha,\ \beta \in \mathbb{N}^{wA}$ such that $0 \le |\alpha,\beta| \le ord(A)$, one has

$$\begin{aligned}
\widehat{G}^{(A,\ell)}\Big(\mathcal{T}^A\widehat{\Lambda_g}\sigma^{(\alpha)}, \widetilde{\sigma}^{(\beta)}\Big) &= \widehat{G}^{(A,\ell)}\Big(\mathcal{T}^A\widehat{\Lambda_g}(X^{(\alpha)},\omega^{(\alpha)}), (Y^{(\beta)},\varpi^{(\beta)})\Big) \\
&= \widehat{G}^{(A,\ell)}\Big((\mathcal{T}^A\Lambda(X^{(\alpha)}), (\Lambda^*)^{(A,\ell)}(\omega^{(\alpha)})), (Y^{(\beta)},\varpi^{(\beta)})\Big) \\
&= \Big(\eta^{(A,\ell)}\big(\mathcal{T}^A\Lambda(X^{(\alpha)}), Y^{(\beta)}\big), \eta^{(A,\ell)}\big(\mathcal{T}^A\sharp_\eta((\Lambda^*)^{(A,\ell)}(\omega^{(\alpha)})), \mathcal{T}^A\sharp_\eta\varpi^{(\beta)}\big)\Big) \\
&= \Big(\eta^{(A,\ell)}\big((\Lambda X)^{(\alpha)}, Y^{(\beta)}\big), \eta^{(A,\ell)}\big((\Lambda(\sharp_\eta\omega))^{(\alpha)}, (\sharp_\eta\varpi)^{(\beta)}\big)\Big) \\
&= \Big(\sum_{\gamma\in\Re} \ell_\gamma \overline{\eta(\Lambda X, Y)}^{(\gamma-\alpha-\beta)}, \sum_{\gamma\in\Re} \ell_\gamma \overline{\eta(\Lambda(\sharp_\eta\omega), \sharp_\eta\varpi)}^{(\gamma-\alpha-\beta)}\Big) \\
&= \Big(\sum_{\gamma\in\Re} \ell_\gamma \overline{\eta(X, \Lambda Y)}^{(\gamma-\alpha-\beta)}, \sum_{\gamma\in\Re} \ell_\gamma \overline{\eta(\sharp_\eta\omega, \Lambda(\sharp_\eta\varpi))}^{(\gamma-\alpha-\beta)}\Big) \\
&\quad \text{since } \eta \text{ is compatible with } \Lambda \\
&= \widehat{G}^{(A,\ell)}\Big(\sigma^{(\alpha)}, \mathcal{T}^A\widehat{\Lambda_g}\widetilde{\sigma}^{(\beta)}\Big).
\end{aligned}$$

□

Proposition 5.10. *The pair* $(\mathcal{T}^A\widehat{\Lambda_g}, \widehat{G}^{(A,\ell)})$ *is a generalized metallic pseudo-Riemannian structure on vector bundle* $(\mathbb{T}T^AM \to T^AM)$.

Proof. See remark 5.4 and proposition 5.9. □

Definition 5.3. The pair $(\mathcal{T}^A\widehat{\Lambda_g}, \widehat{G}^{(A,\ell)})$ is called a generalized metallic pseudo-Riemannian structure on T^AM induced by metallic pseudo-Riemannian structure (Λ, η) on M.

5.4. Prolongation of metallomorphisms

Definition 5.4. Let $f : (M_1, \Lambda_1) \to (M_2, \Lambda_2)$ be a smooth map between metallic pseudo-Riemannian structures. Then f is called a local metallomorphism if it satisfies

$$f_* \circ \Lambda_1 = \Lambda_2 \circ f_* \tag{5.5}$$

where $f_* : TM_1 \to TM_2$ is the tangent map of f. It will be called metallomorphism, when it is local metallomorphism and a diffeomorphism.

Remark 5.4. (a) If $f : (M_1, \Lambda_1) \to (M_2, \Lambda_2)$ is a local metallomorphism, then

$$(f_*)^* \circ \Lambda_2^* = \Lambda_1^* \circ (f_*)^*, \tag{5.6}$$

where $f^* : T^*M_2 \to T^*M_1$ is the dual map of f defined by $(f^*\omega)(X) := \omega(f_*X)$ for all $\omega \in \Gamma(T^*M_2)$ and $X \in \Gamma(TM)$.

(b) If $f : (M_1, \Lambda_1) \to (M_2, \Lambda_2)$ is a local metallomorphism, then it induced an isomorphism of vector bundles between their generalized tangent bundles defined by

$$\psi_f : \mathbb{T}M_1 \to \mathbb{T}M_2,\ (X, \omega) \mapsto (f_*X, ((f_*)^*)^{-1}\omega).$$

(i.e $\psi :== \begin{pmatrix} f_* & 0 \\ 0 & ((f_*)^*)^{-1} \end{pmatrix}$)which preserves the induced generalized metallic structures $\widehat{\Lambda_{ig}} = \begin{pmatrix} \Lambda_i & 0 \\ 0 & \Lambda_i^* \end{pmatrix}$ where $i = 1, 2$.
In particular case where $f : (M, \Lambda) \to (M, \Lambda)$ is a metallomorphism, ψ_f can be defined by

$$\psi_f : \mathbb{T}M_1 \to \mathbb{T}M_2,\ (X, \omega) \mapsto (f_*X, ((f_*)^*)\omega).$$

which coincides with the generalized metallic structure Λ_g when $\Lambda = f^*$. In this case, Λ is invertible and $\Lambda^{-1} = \frac{1}{q}\Lambda - \frac{p}{q}id$, for $q \neq 0$.

let $\theta : \mathbb{T}M_1 \to \mathbb{T}M_2$ be a smooth map between two generalized tangent bundles and (A, ℓ) a Weil-Frobenius algebra. We set

$$\theta^{(A,\ell)} := \kappa^A_{M_2} \oplus \varepsilon^\ell_{A,M_2} \circ T^A\theta \circ (\kappa^A_{M_1})^{-1} \oplus (\varepsilon^\ell_{A,M_1})^{-1}.$$

We have the following results

Proposition 5.11. *If* $\theta : (\mathbb{T}M_1, \Lambda_1) \to (\mathbb{T}M_2, \Lambda_2)$ *is the smooth map between two generalized metallic manifolds such that* θ *preserves metallic structures* Λ_1 *and* Λ_2 *(that is* $\theta \circ \Lambda_1 = \Lambda_2 \circ \theta$*), then* $\theta^{(A,\ell)}$ *preserves also metallic structures* $\mathcal{T}^A\Lambda_{1g}$ *and* $\mathcal{T}^A\Lambda_{2g}$.

Proof. The proof is based on straightforward calculations. □

Proposition 5.12. *If* $f(M_1, \Lambda_1) \to (M_2, \Lambda_2)$ *is a metallomorphism, then the map*

$$(\psi_f)^{(A,\ell)} : \left(\mathbb{T}T^AM_1, \mathcal{T}^A\widehat{\Lambda_{1g}}\right) \to \left(\mathbb{T}T^AM_2, \mathcal{T}^A\widehat{\Lambda_{2g}}\right)$$

between the induced generalized metallic manifolds preserves the metallic structures $\mathcal{T}^A\widehat{\Lambda_{1g}}$ *and* $\mathcal{T}^A\widehat{\Lambda_{1g}}$.

Proof. Let $(X,\omega)\in\Gamma(\mathbb{TM})$, $\widetilde{x}=j^A\varphi\in T^AM$ and $|\alpha|\leq ord(A)$. One has

$$\begin{aligned}
\mathcal{T}^A\widehat{\Lambda_{2g}}\circ(\psi_f)^{(A,\ell)} \quad & \big(X^{(\alpha)},\omega^{(\alpha)}\big)(\widetilde{x}) = \mathcal{T}^A\widehat{\Lambda_{2g}}\Big(\kappa^A_{M_2}\oplus\varepsilon^\ell_{A,M_2}\circ T^A\psi_f\big(j^A(z^\alpha X\circ\varphi),j^A(z^\alpha\omega\circ\varphi)\big)\Big)\\
&= \Big(\mathcal{T}^A\Lambda_{2g}\circ\circ\kappa^A_{M_2}\big(j^A(z^\alpha f_*\circ X\circ\varphi)\big),(\Lambda^*_{2g})^{(A,\ell)}\circ\varepsilon^\ell_{A,M_2}\big(j^A(z^\alpha[(f_*)^*]^{-1}\circ\omega\circ\varphi)\big)\Big)\\
&= \Big(\kappa^A_{M_2}T^A\Lambda_2\big(j^A(z^\alpha f_*\circ X\circ\varphi)\big),\varepsilon^\ell_{A,M_2}\circ T^A\Lambda^*_2\big(j^A(z^\alpha[(f_*)^*]^{-1}\circ\omega\circ\varphi)\big)\Big)\\
&= \Big(\kappa^A_{M_2}\big(j^A(z^\alpha\Lambda_2\circ f_*\circ X\circ\varphi)\big),\varepsilon^\ell_{A,M_2}\big(j^A(z^\alpha\Lambda^*_2[(f_*)^*]^{-1}\circ\omega\circ\varphi)\big)\Big)\\
&= \Big(\kappa^A_{M_2}\big(j^A(z^\alpha f_*\circ\Lambda_1\circ X\circ\varphi)\big),\varepsilon^\ell_{A,M_2}\big(j^A(z^\alpha[(f_*)^*]^{-1}\circ\Lambda^*_1\circ\omega\circ\varphi)\big)\Big)\ \text{see (5.6)}\\
&= \kappa^A_{M_2}\oplus\varepsilon^\ell_{A,M_2}\circ T^A\psi_f\Big(\chi^\alpha_{A,TM}\circ T^A(\Lambda_1\circ X),\chi^\alpha_{A,T^*M}\circ T^A(\Lambda^*_1\circ\omega)\Big)(\widetilde{x})\\
&= (\psi_f)^{(A,\ell)}\circ\mathcal{T}^A\widehat{\Lambda_{1g}}\big(X^{(\alpha)},\omega^{(\alpha)}\big)(\widetilde{x})
\end{aligned}$$

□

6. Prolongation of generalized product (resp. complex) structure induced by a metallic structure

Let (Λ,η) be a metallic Riemannian structure on M such that $\Lambda^2=p\Lambda+qid$, for some real numbers p and q. Then

$$\widehat{\Lambda_{g_p}}:=\begin{pmatrix}\Lambda & (id-\Lambda^2)\circ\sharp_\eta\\ \flat_\eta & -\Lambda^*\end{pmatrix}\ \left(\text{resp. }\widehat{\Lambda_{g_c}}:=\begin{pmatrix}\Lambda & -(id+\Lambda^2)\circ\sharp_\eta\\ \flat_\eta & -\Lambda^*\end{pmatrix}\right)$$

is a generalized product (resp. generalized complex) structure on M (i.e. $\widehat{\Lambda_{g_p}}^2=id$ (resp. $\widehat{\Lambda_{g_p}}^2=-id$)) induced by (Λ,η) [2].

We set

$$\mathcal{T}^A\widehat{\Lambda_{g_p}}:=\begin{pmatrix}\mathcal{T}^A\Lambda & \big(id-(\mathcal{T}^A\Lambda)^2\big)\circ\sharp_{\eta^{(A,\ell)}}\\ \flat_{\eta^{(A,\ell)}} & -(\Lambda^*)^{(A,\ell)}\end{pmatrix}\ \left(\text{resp. }\mathcal{T}^A\widehat{\Lambda_{g_c}}:=\begin{pmatrix}\mathcal{T}^A\Lambda & -\big(id+(\mathcal{T}^A\Lambda)^2\big)\circ\sharp_{\eta^{(\ell)}}\\ \flat_{\eta^{(A,\ell)}} & -(\Lambda^*)^{(A,\ell)}\end{pmatrix}\right)$$

Proposition 6.1. *$\mathcal{T}^A\widehat{\Lambda_{g_p}}$ (resp. $\mathcal{T}^A\widehat{\Lambda_{g_c}}$)is a generalized product (resp. complex) structure on T^AM.*

Proposition 6.2. *For all $\sigma\in\Gamma(\mathbb{T}M)$ and $|\alpha|\leq ord(A)$, one has*

$$\mathcal{T}^A\widehat{\Lambda_{g_p}}(\sigma^{(\alpha)})=\Big(\widehat{\Lambda_{g_p}}(\sigma)\Big)^{(\alpha)}\ \text{and}\ \mathcal{T}^A\widehat{\Lambda_{g_c}}(\sigma^{(\alpha)})=\Big(\widehat{\Lambda_{g_c}}(\sigma)\Big)^{(\alpha)}.$$

Proof. It comes from proposition 5.8 and remark 5.3 □

Hence, we have the following result,

Proposition 6.3. *Let Γ^{TM} be a linear connection on M. If $\widehat{\Lambda_{g_p}}$ is Γ^{TM}-integrable, then $\mathcal{T}^A\widehat{\Lambda_{g_p}}$ (resp. $\mathcal{T}^A\widehat{\Lambda_{g_c}}$) is $\mathcal{T}^{TT^AM}\Gamma^{TM}$-integrable too.*

Definition 6.1. ([2]) A generalized product (resp. generalized complex) structure Λ_g is called pseudo-calibrated (resp. anti-calibrated) with respect to Φ if it is Φ-invariant (resp. Φ-anti-invariant) and the bilinear form $\Phi\circ(id\times\Lambda_g)$ on TM is non-degenerate (resp. non-degenerate and positive definite).

Remark 6.1. The generalized product structure $\widehat{\Lambda_{g_p}}$ (resp. generalized complex $\widehat{\Lambda_{g_c}}$) structure induced by a metallic structure (η,Λ) on M is pseudo-calibrated (resp. anti-calibrated) with respect to Φ.

Proposition 6.4. *(i) If a generalized product structure Λ_g on M is pseudo-calibrated with respect to Φ, then $\mathcal{T}^A\Lambda_g$ is also pseudo-calibrated with respect to $\Phi^{(A,\ell)}$.*

(ii) If a generalized complex structure Λ_g on M is anti-calibrated with respect to Φ, then $\mathcal{T}^A\Lambda_g$ is also anti-pseudo-calibrated with respect to $\Phi^{(A,\ell)}$.

where $\Phi^{(A,\ell)} = \ell \circ T^A\Phi \circ \left((\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^{\ell})^{-1} \times (\kappa_M^A)^{-1} \oplus (\varepsilon_{A,M}^{\ell})^{-1} \right)$

Proof. The proof is based on straightforward calculations. □

Corollary 6.1. *The tensor field $\mathcal{T}^A\widehat{\Lambda_{g_p}}$ (resp. $\mathcal{T}^A\widehat{\Lambda_{g_c}}$) is pseudo-calibrated (resp. anti-calibrated) with respect to $\Phi^{(A,\ell)}$.*

Proposition 6.5. *If The pair $\left(\Lambda_{g_c}, \Lambda_{g_p}\right)$ of generalized complex structure and generalized product structure is a generalized complex-product structure (i.e. $\Lambda_{g_c} \circ \Lambda_{g_p} = -\Lambda_{g_p} \circ \Lambda_{g_c}$) then, the pair $\left(\mathcal{T}^A\Lambda_{g_c}, \mathcal{T}^A\Lambda_{g_p}\right)$ is also a generalized complex-product structure.*

Proof. The proof stems from direct and simple calculations. □

Corollary 6.2. *The pair $\left(\mathcal{T}^A\widehat{\Lambda_{g_c}}, \mathcal{T}^A\widehat{\Lambda_{g_p}}\right)$ is a generalized complex-product structure.*

Proof. The proof is based on the fact that The pair $\left(\widehat{\Lambda_{g_c}}, \widehat{\Lambda_{g_p}}\right)$ is a generalized complex-product structure and the use of previous proposition. □

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