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q-Leonardo Hyper Dual Numbers

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Abstract

In the present paper, the hyper-Dual Leonardo numbers will be introduced with the use of q-integer. Some algebraic properties of these numbers such as recurrence relation, generating function, Honsberger identity, Catalan's identity, Cassini's identity,the Binet's formula and sum formulas will also be obtained.

Keywords: *Leonardo numbers, Hyper-Dual numbers, hyper Dual Leonardo numbers, q-integer.*
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1. Introduction

The algebra of dual numbers was introduced by W. Clifford in 1873 [37]. An extension of dual numbers called hyper-dual numbers have been introduced by Fike to accomplish second derivative problem in the complex-step derivative approximation [18]. For the calculations first and second derivatives, Fike and Alonso have used hyper-dual numbers in [19]. In [1,2], an expression for a hyper-dual numbers in terms of two dual numbers have been obtained by Cohen and Shoham, also, hyper-dual vectors have been examined in the sense of Study and Kotelnikov. It is important to study hyper dual numbers since its arithmetic allows us to calculate derivatives without truncation and subtractive cancellation errors and it is applicable to arbitrarily complex software [2,19].
There are several fascinating sequences of integers. The foremost wide studied sequence of numbers is the Fibonacci sequence. In the existing literature, one can find many papers on Fibonacci and Lucas numbers. (See [3,16,23]). Moreover, they have been examined on different number systems, for example, quaternions and hybrid numbers [9,11,21,30,38,41]. It is well-known that the Fibonacci and Lucas sequences are defined as follows: for $n \geq 0$,

$$F_{n+2} = F_{n+1} + F_n,$$

$$L_{n+2} = L_{n+1} + L_n,$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The Binet's formulas for F_n and L_n as

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi},$$

$$L_n = \phi^n + \psi^n, \tag{1.1}$$

where ϕ and ψ are the roots of the characteristic equation $x^2 - x - 1 = 0$.
During the paper, we consider the Leonardo sequence which has similar properties with the Fibonacci sequence and denoted the nth Leonardo number by L_{e_n}. Some properties of Leonardo numbers have been given by Catarino and Borges in [35] and it is noteworthy to recall that the Leonardo sequence is defined by the following recurrence relation: for $n \geq 2$,

$$L_{e_n} = L_{e_{n-1}} + L_{e_{n-2}} + 1, \tag{1.2}$$

with the initial conditions $L_{e_0} = L_{e_1} = 1$. One can find large number of sequences indexed in *The Online Encyclopedia of Integer Sequences*, being in this case $\{L_{e_n}\} : A001595$ in [33].

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Also, the following holds for Leonardo numbers for $n \geq 2$,

$$L_{e_{n+1}} = 2L_{e_n} - L_{e_{n-2}}. \tag{1.3}$$

The Binet's formula for Leonardo numbers is

$$L_{e_n} = \frac{2\phi^{n+1} - 2\psi^{n+1} - \phi + \psi}{\phi - \psi}. \tag{1.4}$$

Here ϕ and ψ are roots of characteristic equation $x^3 - 2x^2 + 1 = 0$.
From the Binet formula, the relationship between Leonardo and Fibonacci numbers is

$$L_{e_n} = 2F_{n+1} - 1, \tag{1.5}$$

where F_n is the $n-th$ the Fibonacci number.
In [35], Cassini's identity, Catalan's identity and d'Ocagne's identity have been obtained for Leonardo numbers by Catarino and et.al. Moreover they have presented the two-dimensional recurrences relations and matrix representation of Leonardo numbers. In [6], Shannon have defined generalized Leonardo numbers which are considered Asveld's extension and Horadam's generalized sequence.
Now, we are ready to recall some identities involving Fibonacci, Lucas and Leonardo numbers as follows. For more details related to them, please refer [23, 35, 38, 40].

$$F_n + L_n = 2F_{n+1}, \tag{1.6}$$

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j, \tag{1.7}$$

$$L_{e_{n+m}} + (-1)^m L_{e_{n-m}} = L_m(L_{e_n} + 1) - 1 - (-1)^m, \tag{1.8}$$

$$L_{e_{n+m}} - (-1)^m L_{e_{n-m}} = L_{n+1}(L_{e_{m-1}} + 1) - 1 + (-1)^m, \tag{1.9}$$

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r} F_r^2, \tag{1.10}$$

$$L_{r+s} - (-1)^s L_{r-s} = 5F_rF_s, \tag{1.11}$$

$$\sum_{k=1}^{n} (-1)^{k-1} F_{k+1} = (-1)^{n-1} F_n. \tag{1.12}$$

Recently, many researchers considered two dimensional number systems. One of them is hyper-dual numbers that are an extension of dual numbers. The set of hyper-dual hyper-dual numbers denoted by HD and defined as

$$HD = \left\{ \begin{array}{l} x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_1\varepsilon_2 : x_0, x_1, x_2, x_3 \in \mathbb{R}, \\ \varepsilon_1^2 = \varepsilon_2^2 = 0, \varepsilon_1 \neq 0, \varepsilon_2 \neq 0,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0 \end{array} \right\}.$$

Let $x = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_1\varepsilon_2$ and $y = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2 + y_3\varepsilon_1\varepsilon_2$ be any two hyper-dual numbers. Then the main algebraic operations on hyper-dual numbers are defined as follows:
$x = y$ if and only if $x_0 = y_0, x_1 = y_1, x_2 = y_2, x_3 = y_3$.
$x + y = (x_0 + y_0) + (x_1 + y_1)\varepsilon_1 + (x_2 + y_2)\varepsilon_2 + (x_3 + y_3)\varepsilon_1\varepsilon_2$.
$kx = kx_0 + kx_1\varepsilon_1 + kx_2\varepsilon_2 + kx_3\varepsilon_1\varepsilon_2$ where $k \in \mathbb{R}$.
$x \cdot y = (x_0y_0) + (x_0y_1 + x_1y_0)\varepsilon_1 + (x_0y_2 + x_2y_0)\varepsilon_2 + (x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1)\varepsilon_1\varepsilon_2$.

Table 1: Multiplication scheme of hyper-dual units

$\cdot$	1	ε_1	ε_2	$\varepsilon_1\varepsilon_2$
1	1	ε_1	ε_2	$\varepsilon_1\varepsilon_2$
ε_1	ε_1	0	$\varepsilon_1\varepsilon_2$	0
ε_2	ε_2	$\varepsilon_1\varepsilon_2$	0	0
$\varepsilon_1\varepsilon_2$	$\varepsilon_1\varepsilon_2$	0	0	0

There are many generalizations of Fibonacci and Lucas numbers in the existing literature. One of these generalizations is the hyper-dual generalized Fibonacci and Lucas numbers and they have been defined by Omur and some properties have been presented in [28]. They defined hyper-dual generalized Fibonacci and Lucas numbers as follows:

$$\widetilde{U_{k_r}} = U_{k_r} + U_{k_{(r+1)}}\varepsilon_1 + U_{k_{(r+2)}}\varepsilon_2 + U_{k_{(r+3)}}\varepsilon_1\varepsilon_2, \tag{1.13}$$

and

$$\widetilde{V_{k_r}} = V_{k_r} + V_{k_{(r+1)}}\varepsilon_1 + V_{k_{(r+2)}}\varepsilon_2 + V_{k_{(r+3)}}\varepsilon_1\varepsilon_2, \tag{1.14}$$

where $\{U_{k_r}\}$ and $\{V_{k_r}\}$ are the generalized Fibonacci sequence and generalized Lucas sequences, respectively. In [24] the authors have investigated Leonardo Pisano polynomials and hybrinomials with the use of the Leonardo Pisano numbers and hybrid numbers. They also describe the basic algebraic properties and some identities of the Leonardo Pisano polynomials and hybrinomials.
With the motivation of there mentioned papers, here, we introduce a hyper-dual numbers with Leonardo number components. We also aim to obtain the generating function, the Binet's formula, recurrence relation, summation formula, Catalan's, Cassini's and other identities.
For more details about Leonardo numbers, see [4, 15, 24, 25, 34–36, 42]. There are many generalizations of Leonardo numbers in the existing literature. One of these generalizations is the hyper-dual Leonardo numbers and they have been defined and some properties have been presented in [32]. Hyper-Dual numbers have defined as follows:
There are many generalizations of Leonardo numbers in the existing literature. One of these generalizations is the hyper-dual Leonardo numbers and they have been defined and some properties have been presented in [32]. Hyper-Dual numbers have defined as follows:
For $n \geqslant 1$, the nth hyper-dual Leonardo numbers are defined by

$$HDL_{e_n} = L_{e_n} + L_{e_{n+1}}\varepsilon_1 + L_{e_{n+2}}\varepsilon_2 + L_{e_{n+3}}\varepsilon_1\varepsilon_2. \tag{1.15}$$

For $n \geq 2$,we get the following recurrence relation

$$HDL_{e_n} = HDL_{e_{n-1}} + HDL_{e_{n-2}} + A, \tag{1.16}$$

where here $A = 1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2$. This notation will be used in the whole paper. Also initial conditions are $HDL_{e_0} = 1+\varepsilon_1+3\varepsilon_2+5\varepsilon_1\varepsilon_2$ and $HDL_{e_1} = 1+3\varepsilon_1+5\varepsilon_2+9\varepsilon_1\varepsilon_2$.
Now, let us give another recurrence relation of hyper-dual Leonardo numbers. That is

$$HDL_{e_{n+1}} = 2HDL_{e_n} - HDL_{e_{n-2}}.$$

The generation function for the hyper-dual Leonardo numbers, denoted by $gHDL_{e_n}(t)$, is

$$gHDL_{e_n}(t) = \frac{HDL_{e_0} + t(-1+\varepsilon_1-\varepsilon_2-\varepsilon_1\varepsilon_2) + t^2(1-\varepsilon_1-\varepsilon_2-3\varepsilon_1\varepsilon_2)}{1-2t+t^3}.$$

(Catalan's Identity) For positive integers n and r with $n \geq r$, we have

$$HDL_{e_n}^2 - HDL_{e_{n-r}}HDL_{e_{n+r}} = (HDL_{e_{n-r}} + HDL_{e_{n+r}} - 2HDL_{e_n})A + 4(-1)^{n-r+1}(2(\varepsilon_2+\varepsilon_1\varepsilon_2)+A)F_r^2,$$

where F_n is nth Fibonacci number.
(Cassini's Identity) For $n \geq 1$, we have

$$\begin{aligned} HDL_{e_n}^2 - HDL_{e_{n-1}}HDL_{e_{n+1}} &= (HDL_{e_{n-1}} - HDL_{e_n-2})A \\ &\quad +4(-1)^n(2(\varepsilon_2+\varepsilon_1\varepsilon_2)+A). \end{aligned}$$

q-integer $[n]_q$ is defined as follows:

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+...+q^{n-1}. \tag{1.17}$$

Let $HDF_n(\alpha;q)$ and $HDL_n(\alpha;q)$ be the q-Fibonnaci hyper-dual number is n-th q-Lucas hyper-dual number. For $n \geq 1$, the **Binet's formula** for these numbers respectively, is a follows:

$$HDF_n(\alpha;q) = \frac{\alpha^n\overline{\phi} - (\alpha q)^n\overline{\psi}}{\alpha - \alpha q}, \tag{1.18}$$

and

$$HDL_n(\alpha;q) = \alpha^n\overline{\phi} + (\alpha q)^n\overline{\psi}. \tag{1.19}$$

For $n,m \geq 0$, $HDF_n(\alpha;q)$ n-th q-Fibonnaci hyper Dual numbers, $HDF_m(\alpha;q)$ m-th q-Fibonnaci hyper-dual numbers, $HDL_n(\alpha;q)$ n-th q-Lucas hyper-dual numbers and $HDL_m(\alpha;q)$ m-th q-Lucas hyper Dual numbers, the **Honsberger identities** are as follows:

$$\begin{aligned} HDF_n(\alpha;q)HDF_m(\alpha;q) + HDF_{n+1}(\alpha;q)HDF_{m+1}(\alpha;q) &= \frac{\alpha^{n+m}}{(\alpha-\alpha q)^2}[(1+\alpha^2)\overline{\phi^2} \\ &- (1+\alpha(\alpha q))(q^n+q^m) \\ &\quad \overline{\phi}\,\overline{\psi} + (1+(\alpha q)^2)q^{n+m}\overline{\psi^2}], \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} HDL_n(\alpha;q)HDL_m(\alpha;q)+HDL_{n+1}(\alpha;q)HDL_{m+1}(\alpha;q) &= \alpha^{n+m}[(1+\alpha^2)\overline{\phi^2} \\ &+ (1+\alpha(\alpha q))(q^n+q^m) \\ & \overline{\phi}\overline{\psi}+(1+(\alpha q)^2)q^{n+m}\overline{\psi^2}]. \end{aligned} \tag{1.21}$$

For $n,m \geq 0$, $HDF_n(\alpha;q)$ is n-th q-Fibonnaci hyper Dual numbers and $HDL_n(\alpha;q)$ is n-th q-Lucas hyper Dual numbers, **d'Ocagne's identity** are as follows:

$$HDF_n(\alpha;q)HDF_{m+1}(\alpha;q)-HDF_{n+1}(\alpha;q)HDF_m(\alpha;q)=\frac{\alpha^{n+m-1}(q^m-q^n)\overline{\phi}\overline{\psi}}{(1-q)}, \tag{1.22}$$

$$HDL_n(\alpha;q)HDL_{m+1}(\alpha;q)-HDL_{n+1}(\alpha;q)HDL_m(\alpha;q)=\alpha^{n+m+1}[(1-q)(q^n-q^m)\overline{\phi}\overline{\psi}]. \tag{1.23}$$

For $n \geq r$, $HDF_n(\alpha;q)$ is n-th q-Fibonnaci hyper Dual numbers and $HDL_n(\alpha;q)$ is n-th q-Lucas hyper Dual numbers, **Catalan's identity** are as follows:

$$HDF_{n+r}(\alpha;q)HDF_{n-r}(\alpha;q)-HDF_n^2(\alpha;q)=\frac{\alpha^{2n-2}q^n(1-q^r)(1-q^{-r})\overline{\phi}\overline{\psi}}{(1-q)^2}, \tag{1.24}$$

$$HDL_{n+r}(\alpha;q)HDL_{n-r}(\alpha;q)-HDL_n^2(\alpha;q)=\alpha^{2n}q^{n-r}(1-q^r)^2\overline{\phi}\overline{\psi}. \tag{1.25}$$

For $n \geq 1$, $HDF_n(\alpha;q)$ is n-th q-Fibonnaci hyper Dual numbers and $HDL_n(\alpha;q)$ is n-th q-Lucas hyper Dual numbers, **Cassini identity** are as follows:

$$HDF_{n+1}(\alpha;q)HDF_{n-1}(\alpha;q)-HDF_n^2(\alpha;q)=\frac{\alpha^{2n-2}q^n(1-q^{-1})\overline{\phi}\overline{\psi}}{(1-q)}, \tag{1.26}$$

$$HDL_{n+1}(\alpha;q)HDL_{n-1}(\alpha;q)-HDL_n^2(\alpha;q)=\alpha^{2n}q^{n-1}(1-q^1)^2\overline{\phi}\overline{\psi}. \tag{1.27}$$

2. q-Leonardo Hyper-Dual Numbers

In this section, we define q-Fibonnaci hyper-dual numbers and q-Lucas hyper dual numbers by the basis $1,\varepsilon_1,\varepsilon_2,\varepsilon_1\varepsilon_2$. Some algebraic properties of q-Leonardo hyper Dual numbers such as recurrence relation, the generating function, Honsberger identity, Catalan's identity, Cassini's identity, Binet's formula and sum formulas will also be obtained in this part.
For q-Leonardo hyper-Dual numbers representation of is in following

$$\begin{aligned} HDL_{e_n}(\alpha;q) &= (2\alpha^n[n+1]_q-1)+(2\alpha^{n+1}[n+2]_q-1)\varepsilon_1+(2\alpha^{n+2}[n+3]_q-1)\varepsilon_2 \\ &+ (2\alpha^{n+3}[n+4]_q-1)\varepsilon_1\varepsilon_2. \end{aligned} \tag{2.1}$$

From the recurrence relation (1.16) and the definition of q-hyper-dual Leonardo numbers (2.1), for $n \geq 2$ we get

$$HDL_{e_n}(\alpha;q)+HDL_{e_{n+1}}(\alpha;q)=HDL_{e_{n+2}}(\alpha;q)-A, \tag{2.2}$$

where

$\alpha+1=\alpha^2, \quad \alpha^2+\alpha=\alpha^3, \quad \alpha^3+\alpha^2=\alpha^4, \quad \alpha^4+\alpha^3=\alpha^5,$

and

$(\alpha q)+1=(\alpha q)^2, \quad (\alpha q)^2+(\alpha q)=(\alpha q)^3,(\alpha q)^3+(\alpha q)^2=(\alpha q)^4,(\alpha q)^4+(\alpha q)^3=(\alpha q)^5.$

Let be $HDL^*_{e_n}$ set of all q-Leonardo hyper Dual numbers.

$$\begin{aligned} L_{HDL_{e_n}(\alpha;q)}: HDL^*_{e_n} &\rightarrow HDL^*_{e_n} \\ r &\rightarrow L_{HDL_{e_n}(\alpha;q)}(r)=rL_{HDL_{e_n}(\alpha;q)}. \end{aligned} \tag{2.3}$$

$L_{HB_n}(\alpha;q)$ should be isomorphic function from $HDL^*_{e_n}$ to $HDL^*_{e_n}$.
Firstly it can be seen that the equalities. For $Z_1=(2\alpha^n[n+1]_q-1)+(2\alpha^{n+1}[n+2]_q-1)\varepsilon_1+(2\alpha^{n+2}[n+3]_q-1)\varepsilon_2+(2\alpha^{n+3}[n+4]_q-1)\varepsilon_1\varepsilon_2$, $Z_2=(2\alpha^m[m+1]_q-1)+(2\alpha^{m+1}[m+2]_q-1)\varepsilon_1+(2\alpha^{m+2}[m+3]_q-1)\varepsilon_2+(2\alpha^{m+3}[m+4]_q-1)\varepsilon_1\varepsilon_2$ and $c \in \mathbb{R}$ equalities

$$L_{HDL_{e_n}(\alpha;q)}(Z_1+Z_2)=L_{HDL_{e_n}(\alpha;q)}(Z_1)+L_{HDL_{e_n}(\alpha;q)}(Z_2), \tag{2.4}$$

$$(cL_{HDL_{e_n}(\alpha;q)}(Z_1)) = c(L_{HDL_{e_n},(\alpha;q)}(Z_1)) \tag{2.5}$$

are satisfied.
Secondly, the function should be injective for $Z_1 = (2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)\varepsilon_1 + (2\alpha^{n+2}[n+3]_q - 1)\varepsilon_2 + (2\alpha^{n+3}[n+4]_q - 1)\varepsilon_1\varepsilon_2$ and $Z_2 = (2\alpha^m[m+1]_q - 1) + (2\alpha^{m+1}[m+2]_q - 1)\varepsilon_1 + (2\alpha^{m+2}[m+3]_q - 1)\varepsilon_2 + (2\alpha^{m+3}[m+4]_q - 1)\varepsilon_1\varepsilon_2$ equality
If we take $L_{HDL_{e_n}(\alpha;q)}(Z_1) = L_{HDL_{e_n}(\alpha;q)}(Z_2)$, it can easily seen that

$$L_{HDL_{e_n}(\alpha;q)}(Z_1) = L_{HDL_{e_n}(\alpha;q)}(Z_2).$$

$$(2\alpha^n[n+1]_q - 1) = (2\alpha^m[m+1]_q - 1), \quad (2\alpha^{n+1}[n+2]_q - 1) = (2\alpha^{m+1}[m+2]_q - 1),$$
$$(2\alpha^{n+2}[n+3]_q - 1) = (2\alpha^{m+2}[m+3]_q - 1) \quad \text{and} \quad (2\alpha^{n+3}[n+4]_q - 1) = (2\alpha^{m+3}[m+4]_q - 1).$$

Finally, we can seen that her for $\forall A\ matrix \in HDL^*_{e_n}$ at least one $Z_1 \in HDL^*_{e_n} \ni L_{HDL_{e_n}(\alpha;q)}(Z_1) = A$
Thus $L_{HB_n}(\alpha;q)$ is isomorphism.
If we use units for q-Leonardo hyper Dual numbers, we obtain

$$\begin{aligned} L_{HDL_{e_n}(\alpha;q)}(1) &= 1[(2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)\varepsilon_1 \\ &\quad +(2\alpha^{n+2}[n+3]_q - 1)\varepsilon_2 + (2\alpha^{n+3}[n+4]_q - 1)\varepsilon_1\varepsilon_2] \\ &= (2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)\varepsilon_1 \\ &\quad +(2\alpha^{n+2}[n+3]_q - 1)\varepsilon_2 + (2\alpha^{n+3}[n+4]_q - 1)\varepsilon_1\varepsilon_2. \end{aligned}$$

$$\begin{aligned} L_{HDL_{e_n}(\alpha;q)}(\varepsilon_1) &= \varepsilon_1[(2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)\varepsilon_1 \\ &\quad +(2\alpha^{n+2}[n+3]_q - 1)\varepsilon_2 + (2\alpha^{n+3}[n+4]_q - 1)\varepsilon_1\varepsilon_2] \\ &= \varepsilon_1(2\alpha^n[n+1]_q - 1) + \varepsilon_1\varepsilon_2(2\alpha^{n+2}[n+3]_q - 1). \end{aligned}$$

$$\begin{aligned} L_{HDL_{e_n}(\alpha;q)}(\varepsilon_2) &= \varepsilon_2[(2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)\varepsilon_1 \\ &\quad +(2\alpha^{n+2}[n+3]_q - 1)\varepsilon_2 + (2\alpha^{n+3}[n+4]_q - 1)\varepsilon_1\varepsilon_2] \\ &= \varepsilon_2(2\alpha^n[n+1]_q - 1) + \varepsilon_1\varepsilon_2(2\alpha^{n+1}[n+2]_q - 1). \end{aligned}$$

$$\begin{aligned} L_{HDL_{e_n}(\alpha;q)}(\varepsilon_1\varepsilon_2) &= \varepsilon_1\varepsilon_2[(2\alpha^n[n+1]_q - 1) + (2\alpha^{n+1}[n+2]_q - 1)\varepsilon_1 \\ &\quad +(2\alpha^{n+2}[n+3]_q - 1)\varepsilon_2 + (2\alpha^{n+3}[n+4]_q - 1)\varepsilon_1\varepsilon_2] \\ &= \varepsilon_1\varepsilon_2(2\alpha^n[n+1]_q - 1). \end{aligned}$$

$$A = \begin{bmatrix} 2\alpha^n[n+1]_q - 1 & 0 & 0 & 0 \\ 2\alpha^{n+1}[n+2]_q - 1 & 2\alpha^n[n+1]_q - 1 & 0 & 0 \\ 2\alpha^{n+2}[n+3]_q - 1 & 0 & 2\alpha^n[n+1]_q - 1 & 0 \\ 2\alpha^{n+3}[n+4]_q - 1 & 2\alpha^{n+2}[n+3]_q - 1 & 2\alpha^{n+1}[n+2]_q - 1 & 2\alpha^n[n+1]_q - 1 \end{bmatrix}_{4\times 4} \tag{2.6}$$

$$detA = \begin{vmatrix} 2\alpha^n[n+1]_q - 1 & 0 & 0 & 0 \\ 2\alpha^{n+1}[n+2]_q - 1 & 2\alpha^n[n+1]_q - 1 & 0 & 0 \\ 2\alpha^{n+2}[n+3]_q - 1 & 0 & 2\alpha^n[n+1]_q - 1 & 0 \\ 2\alpha^{n+3}[n+4]_q - 1 & 2\alpha^{n+2}[n+3]_q - 1 & 2\alpha^{n+1}[n+2]_q - 1 & 2\alpha^n[n+1]_q - 1 \end{vmatrix}_{4\times 4} \tag{2.7}$$

$$= (2\alpha^n[n+1]_q - 1)^4.$$

Theorem 2.1. *For $n \in \mathbb{N}$, we have*

$$HDL_{e_n}(\alpha;q) = 2HDF_{n+1}(\alpha;q) - A. \tag{2.8}$$

Proof. By using definition of q-Leonardo hyper Dual number (2.1)

$$\begin{aligned} HDL_{e_n}(\alpha;q) &= (2\alpha^n[n+1]_q-1)+(2\alpha^{n+1}[n+2]_q-1)\varepsilon_1+(2\alpha^{n+2}[n+3]_q-1)\varepsilon_2 \\ &+ (2\alpha^{n+3}[n+4]_q-1)\varepsilon_1\varepsilon_2, \end{aligned}$$

by using $[n]_q=\frac{1-q^n}{1-q}$ and definition of the Binet's formula for $HDF_{n+1}(\alpha;q)$

$$\begin{aligned} &= 2(\frac{\alpha^{n+1}\overline{\phi}-(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q})-A, \\ &= 2HDF_{n+1}(\alpha;q)-A. \end{aligned}$$

where $A=1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2$, $\overline{\phi}=1+\alpha\varepsilon_1+\alpha^2\varepsilon_2+\alpha^3\varepsilon_1\varepsilon_2$ and $\overline{\psi}=1+(\alpha q)\varepsilon_1+(\alpha q)^2\varepsilon_2+(\alpha q)^3\varepsilon_1\varepsilon_2$. □

Theorem 2.2. *Let $HDL_{e_n}(\alpha;q)$ be the q-Leonardo hyper Dual number. For $n\geq 1$, the Binet's formula for this number, is a follow:*

$$HDL_{e_n}(\alpha;q)=\frac{2\alpha^{n+1}\overline{\phi}-2(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q}-A. \tag{2.9}$$

where $A=1+\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2$, $\overline{\phi}=1+\alpha\varepsilon_1+\alpha^2\varepsilon_2+\alpha^3\varepsilon_1\varepsilon_2$,$\overline{\psi}=1+(\alpha q)\varepsilon_1+(\alpha q)^2\varepsilon_2+(\alpha q)^3\varepsilon_1\varepsilon_2$, $\alpha=\frac{1+\sqrt{5}}{2}$ and $\frac{-1}{\alpha}$.

Proof. By using definition of q-Leonardo hyper Dual numbers (2.1) and definition of $[n]_q=\frac{1-q^n}{1-q}$

$$\begin{aligned} HDL_{e_n}(\alpha;q) &= (2\alpha^n[n+1]_q-1)+(2\alpha^{n+1}[n+2]_q-1)\varepsilon_1+(2\alpha^{n+2}[n+3]_q-1)\varepsilon_2, \\ &\quad +(2\alpha^{n+3}[n+4]_q-1)\varepsilon_1\varepsilon_2 \\ &= (2\alpha^n(\frac{1-q^{n+1}}{1-q})-1)+(2\alpha^{n+1}(\frac{1-q^{n+2}}{1-q})-1)\varepsilon_1, \\ &\quad +(2\alpha^{n+2}(\frac{1-q^{n+3}}{1-q})-1)\varepsilon_2+(2\alpha^{n+3}(\frac{1-q^{n+4}}{1-q})-1)\varepsilon_1\varepsilon_2, \\ &= \frac{2\alpha^{n+1}\overline{\phi}-2(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q}-A. \end{aligned}$$

□

Theorem 2.3. ***(Generating Function)*** *Let $HDL_{e_n}(\alpha;q)$ be the q-Leonardo hyper-dual number. For the exponential generating function for this number is as follows:*

$$gHDL_{e_n}(\alpha;q)(\frac{t^n}{n!})=\sum_{n=0}^{\infty}HDL_{e_n}(\alpha;q)\frac{t^n}{n!}=2(\frac{\overline{\phi}e^{\alpha t}-q\overline{\psi}e^{(\alpha q)t}}{1-q})-Ae^t. \tag{2.10}$$

Proof. Using Binet's formula for q Leonardo hyper Dual number (2.9)

$$\begin{aligned} \sum_{n=0}^{\infty}HDL_{e_n}(\alpha;q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}[2(\frac{\alpha^{n+1}\overline{\phi}-(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q})-A]\frac{t^n}{n!}, \\ &= 2(\frac{\overline{\phi}e^{\alpha t}-q\overline{\psi}e^{(\alpha q)t}}{1-q})-Ae^t. \end{aligned}$$

□

Theorem 2.4. ***(Honsberger Identity)*** *For $n,m\geq 0$ the Honsberger identity for the q-Leonardo hyper Dual number is as follows:*

$$\begin{aligned} & HDL_{e_n}(\alpha;q)HDL_{e_m}(\alpha;q)+HDL_{e_{n+1}}(\alpha;q)HDL_{e_{m+1}}(\alpha;q) \\ =\ & \frac{4\alpha^{n+m+2}}{(\alpha-\alpha q)^2}[(1+\alpha^2)\overline{\phi}^2-(1+\alpha^2 q)(q^{n+1}+q^{m+1})\overline{\phi}\overline{\psi}+(1+(\alpha q)^2)q^{n+m+2}\overline{\psi}^2] \\ +\ & HDL_{e_{n+2}}(\alpha;q)+HDL_{e_{m+2}}(\alpha;q). \end{aligned} \tag{2.11}$$

Proof. By using the relation between q-Loenardo hyper Dual number and q-Fibonnaci hyper Dual number (2.8) and the Binet's formula for q-Fibonnaci hyper Dual number (1.18), we get
Take $HDL_{e_n}(\alpha;q)HDL_{e_m}(\alpha;q)+HDL_{e_{n+1}}(\alpha;q)HDL_{e_{m+1}}(\alpha;q)=$ **LHS**.

$$\begin{aligned} \textbf{LHS} &= (2HDF_{n+1}(\alpha;q)-A)(2HDF_{m+1}(\alpha;q)-A) \\ &\quad +(2HDF_{n+2}(\alpha;q)-A)(2HDF_{m+2}(\alpha;q)-A), \\ &= (\frac{2\alpha^{n+1}\overline{\phi}-2(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q}-A)(\frac{2\alpha^{m+1}\overline{\phi}-2(\alpha q)^{m+1}\overline{\psi}}{\alpha-\alpha q}-A) \\ &\quad +(\frac{2\alpha^{n+2}\overline{\phi}-2(\alpha q)^{n+2}\overline{\psi}}{\alpha-\alpha q}-A)(\frac{2\alpha^{m+2}\overline{\phi}-2(\alpha q)^{m+2}\overline{\psi}}{\alpha-\alpha q}-A). \end{aligned}$$

After this step we will use Binet's formula for q-Leonardo hyper Dual numbers (2.9).

$$\begin{aligned}
&= \frac{4\alpha^{n+m+2}}{(\alpha-\alpha q^2)}[(1+\alpha^2)\overline{\phi}^2-(1+\alpha^2 q)(q^{n+1}+q^{m+1})\overline{\phi}\,\overline{\psi}+(1+(\alpha q)^2)q^{n+m+2}\overline{\psi}^2] \\
&\quad -A[HDL_{e_n}(\alpha;q)+HDL_{e_{n+1}}(\alpha;q)]-A[HDL_{e_m}(\alpha;q)+HDL_{e_{m+1}}(\alpha;q)]-2A^2, \\
&= \frac{4\alpha^{n+m+2}}{(\alpha-\alpha q)^2}[(1+\alpha^2)\overline{\phi}^2-(1+\alpha^2 q)(q^{n+1}+q^{m+1})\overline{\phi}\,\overline{\psi}+(1+(\alpha q)^2)q^{n+m+2}\overline{\psi}^2] \\
&\quad +HDL_{e_{n+2}}(\alpha;q)+HDL_{e_{m+2}}(\alpha;q).
\end{aligned}$$

□

Theorem 2.5. ***(D'Ocagne's Identity)*** *For $n,m \geq 0$ the d'Ocagne's identity for q-Leonardo hyper Dual number, is as follows:*

$$\begin{aligned}
&HDL_{e_m}(\alpha;q)HDL_{e_{n+1}}(\alpha;q)-HDL_{e_{m+1}}(\alpha;q)HDL_{e_n}(\alpha;q) \\
&= \frac{4}{(1-q)}[\alpha^{n+m+7}q^2(q^n+q^{m+1}-q^{n+1}+q^m)]-HDL_{e_n}(\alpha;q)+HDL_{e_m}(\alpha;q).
\end{aligned} \tag{2.12}$$

Proof. By using the relation between q-Leonardo hyper Dual number, q-Fibonnaci hyper Dual number (2.8) and Binet's formula for q-Fibonnaci hyper Dual number (1.18), we get.
Let $HDL_{e_m}(\alpha;q)HDL_{e_{n+1}}(\alpha;q)-HDL_{e_{m+1}}(\alpha;q)HDL_{e_n}(\alpha;q)=\textbf{LHS}$.

$$\begin{aligned}
\textbf{LHS} &= (2HDF_{m+1}(\alpha;q)-A)(2HDF_{n+2}(\alpha;q)-A) \\
&\quad -(2HDF_{m+2}(\alpha;q)-A)(2HDF_{n+1}(\alpha;q)-A), \\
&= (\frac{2\alpha^{m+2}\overline{\phi}-2(\alpha q)^{m+2}\overline{\psi}}{\alpha-\alpha q}-A)(\frac{2\alpha^{n+3}\overline{\phi}-2(\alpha q)^{n+3}\overline{\psi}}{\alpha-\alpha q}-A), \\
&\quad -(\frac{2\alpha^{m+3}\overline{\phi}-2(\alpha q)^{m+3}\overline{\psi}}{\alpha-\alpha q}-A)(\frac{2\alpha^{n+2}\overline{\phi}-2(\alpha q)^{n+2}\overline{\psi}}{\alpha-\alpha q}-A).
\end{aligned}$$

Finally, if we use the recurrence relation for q-Leonardo hyper Dual number in equation (2.2), we obtain

$$=\frac{4}{(1-q)}[\alpha^{n+m+7}q^2(q^n+q^{m+1}-q^{n+1}+q^m)]-HDL_{e_n}(\alpha;q)+HDL_{e_m}(\alpha;q).$$

□

Theorem 2.6. ***(Catalan Identity)*** *For $n \geq r$ the Catalan's identity for the q-Leonardo hyper Dual number is follows:*

$$\begin{aligned}
HDL_{e_{n+r}}(\alpha;q)HDL_{e_{n-r}}(\alpha;q)-HDL^2_{e_n}(\alpha;q) &= \frac{4}{(1-q)^2}\overline{\phi}\,\overline{\psi}[\alpha^{2n}q^{n+1}(1-q^{-r})(1-q^r)] \\
&\quad +A[2HDL_{e_n}(\alpha;q)-HDL_{e_{n-r}}(\alpha;q) \\
&\quad -HDL_{e_{n+r}}(\alpha;q)].
\end{aligned} \tag{2.13}$$

Proof. By using relation between q-Loenardo hyper Dual number and q-Fibonnaci hyper Dual number (2.8) and Binet's formula for q-Fibonnaci hyper Dual number (1.18), we get
Let be $HDL_{e_{n+r}}(\alpha;q)HDL_{e_{n-r}}(\alpha;q)-HDL^2_{e_n}(\alpha;q)=\textbf{LHS}$

$$\begin{aligned}
\textbf{LHS} &= (2HDF_{n+r+1}(\alpha;q)-A)(2HDF_{n-r+1}(\alpha;q)-A) \\
&\quad -[((2HDF_{n+1}(\alpha;q)-A))(2HDF_{n+1}(\alpha;q)-A)], \\
&= (\frac{2\alpha^{n+r+1}\overline{\phi}-2(\alpha q)^{n+r+1}\overline{\psi}}{\alpha-\alpha q}-A)(\frac{2\alpha^{n-r+1}\overline{\phi}-2(\alpha q)^{n-r+1}\overline{\psi}}{\alpha-\alpha q}-A), \\
&\quad -[(\frac{2\alpha^{n+1}\overline{\phi}-2(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q}-A)(\frac{2\alpha^{n+1}\overline{\phi}-2(\alpha q)^{n+1}\overline{\psi}}{\alpha-\alpha q}-A)] \\
&= \frac{4}{(1-q)^2}\overline{\phi}\,\overline{\psi}[\alpha^{2n}q^{n+1}(1-q^{-r})(1-q^r)] \\
&\quad +A[2HDL_{e_n}(\alpha;q)-HDL_{e_{n-r}}(\alpha;q)-HDL_{e_{n+r}}(\alpha;q)].
\end{aligned}$$

Where $\overline{\phi}\ \overline{\psi}=\overline{\psi}\ \overline{\phi}$. □

If we take take $r=1$ on Catalan identity, we will obtain Cassini's identity as follows.

Theorem 2.7. ***(Cassini's Identity)*** *For $n \geq 1$, the Cassini's identity for the q-Leonardo hyper Dual numbers is*

$$\begin{aligned}
HDL_{e_{n+1}}(\alpha;q)HDL_{e_{n-1}}(\alpha;q)-HDL^2_{e_n}(\alpha;q) &= \frac{4}{1-q}\overline{\phi}\,\overline{\psi}[\alpha^{2n}q^{n+1}(1-q^{-1})] \\
&\quad +A[2HDL_{e_n}(\alpha;q)-HDL_{e_{n-1}}(\alpha;q) \\
&\quad -HDL_{e_{n+1}}(\alpha;q)].
\end{aligned} \tag{2.14}$$

3. Conclusion

In the present paper, the hyper-Dual Leonardo numbers are introduced with the use of q-integer. First of all, the recurrence relation and generating function for these numbers have been obtained. Then, summation formulas for these numbers have been provided. Furthermore, Catalan's and Cassini's identities and some interesting properties have been given.

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