International Electronic Journal of Algebra
Published Online: October 1, 2024
DOI: 10.24330/ieja.1559027

STRONGLY $\mathcal{VW}$-GORENSTEIN N-COMPLEXES

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Received: 6 December 2023; Accepted: 4 August 2024
Communicated by Meltem Altun Özarslan

Abstract. Let $\mathcal{V}, \mathcal{W}$ be two classes of R-modules. The notion of strongly $\mathcal{VW}$-Gorenstein N-complexes is introduced, and under certain mild hypotheses on $\mathcal{V}$ and $\mathcal{W}$, it is shown that an N-complex $\boldsymbol{X}$ is strongly $\mathcal{VW}$-Gorenstein if and only if each term of $\boldsymbol{X}$ is a $\mathcal{VW}$-Gorenstein module and N-complexes $\mathrm{Hom}_R(V, \boldsymbol{X})$ and $\mathrm{Hom}_R(\boldsymbol{X}, W)$ are N-exact for any $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Furthermore, under the same conditions on $\mathcal{V}$ and $\mathcal{W}$, it is proved that an N-exact N-complex $\boldsymbol{X}$ is $\mathcal{VW}$-Gorenstein if and only if $\mathrm{Z}_n^t(\boldsymbol{X})$ is a $\mathcal{VW}$-Gorenstein module for each $n \in \mathbb{Z}$ and each $t = 1, 2, \ldots, N-1$. Consequently, we show that an N-complex $\boldsymbol{X}$ is strongly Gorenstein projective (resp., injective) if and only if $\boldsymbol{X}$ is N-exact and $\mathrm{Z}_n^t(\boldsymbol{X})$ is a Gorenstein projective (resp., injective) module for each $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N-1$.

Mathematics Subject Classification (2020): 18G25, 18G35, 18E10
Keywords: $\mathcal{VW}$-Gorenstein module, strongly $\mathcal{VW}$-Gorenstein N-complex, strongly Gorenstein projective N-complex, strongly Gorenstein injective N-complex

1. Introduction

Let $\mathcal{V}, \mathcal{W}$ be two classes of R-modules. Zhao and Sun [31] introduced and studied $\mathcal{VW}$-Gorenstein R-modules. Such class of R-modules is a common generalization of Gorenstein projective and Gorenstein injective R-modules [3,7], G_C-projective and G_C-injective R-modules (where C is a semidualizing R-module over commutative ring R) [8,24], $\mathcal{W}$-Gorenstein R-modules [5,23], and so on. In [32], Zhao and Ren extended the notion of $\mathcal{VW}$-Gorenstein R-modules to the category of R-complexes by introducing the notion of $\mathcal{VW}$-Gorenstein complexes. They showed that if $\mathcal{V}, \mathcal{W}$ are closed under extensions, isomorphisms and finite direct sums, $\mathcal{V} \perp \mathcal{W}, \mathcal{V} \perp \mathcal{V}, \mathcal{W} \perp \mathcal{W}$ and both modules in $\mathcal{V}, \mathcal{W}$ are $\mathcal{VW}$-Gorenstein, then $\mathcal{VW}$-Gorenstein complexes are just the complexes of $\mathcal{VW}$-Gorenstein modules, see [32, Theorem 3.8]. This result recovered the results on Gorenstein projective and injective complexes [26, Theorems 1, 2] and [30, Theorem 2.2, Proposition 2.8],

This work was partially supported by the National Natural Science Foundation of China (Grant nos. 11861055 and 12061061).

$\mathcal{W}$-Gorenstein complexes [12, Corollary 4.8] and [25, Theorem 3.12], G_C-projective and injective complexes [27, Theorems 4.6 and 4.7].

As a natural generalization of complexes, the N-complexes seem to have first introduced by Mayer [22] in his study of simplicial complexes. The study of homological theory of N-complexes was originated in the works of Kapranov[11] and Dubois-Violette[2]. From then, many results of complexes were extended to N-complexes, see for example [1,4,6,10,14,15,16,17,18,20,21,29,28] and the references therein. In particular, from [20, Theorem 3.5] or [15, Theorem 3.17] we know that an N-complex $\boldsymbol{X}$ is Gorenstein projective (resp., injective) if and only if each degree of $\boldsymbol{X}$ is a Gorenstein projective (resp., injective) module.

It is well known that an N-complex $\boldsymbol{X}$ is projective (resp., injective) if and only if $\boldsymbol{X}$ is N-exact (or simply exact) and $\mathrm{Z}_n^t(\boldsymbol{X})$ is projective (resp., injective) for each $n \in \mathbb{Z}$ and $1 \leqslant t \leqslant N-1$. The primary goal of this paper is to identify subcategories of N-complexes that will complete the following diagram:

$$\begin{array}{ccc}
\begin{array}{c}\boldsymbol{X} \text{ is a projective} \\ \text{(resp., injective) } N\text{-complex}\end{array} & \Longleftrightarrow & \begin{array}{c}\boldsymbol{X} \text{ is exact and } \mathrm{Z}_n^t(\boldsymbol{X}) \text{ is a} \\ \text{projective (resp., injective) module} \\ \text{for any } n \in \mathbb{Z} \text{ and } 1 \leqslant t \leqslant N-1\end{array} \\
\Downarrow & & \Downarrow \\
\boldsymbol{X} \text{ is a ? (resp., ?) } N\text{-complex} & \Longleftrightarrow & \begin{array}{c}\boldsymbol{X} \text{ is exact and } \mathrm{Z}_n^t(\boldsymbol{X}) \text{ is a} \\ \text{Gorenstein projective (resp., injective) module} \\ \text{for each } n \in \mathbb{Z} \text{ and } 1 \leqslant t \leqslant N-1\end{array} \\
\Downarrow & & \Downarrow \\
\begin{array}{c}\boldsymbol{X} \text{ is a Gorenstein projective} \\ \text{(resp., injective) } N\text{-complex}\end{array} & \Longleftrightarrow & \begin{array}{c}\text{each term of } \boldsymbol{X} \text{ is a Gorenstein projective} \\ \text{(resp., injective) module.}\end{array}
\end{array}$$

We achieve this goal as applications of the more general works that we develop for the so-called strongly $\mathcal{VW}$-Gorenstein N-complexes, where $\mathcal{V}$ and $\mathcal{W}$ are two classes of R-modules. Here is the outline: Section 2 contains preliminary notions, notation and lemmas for use throughout this paper. In Section 3, we first give definition of strongly $\mathcal{VW}$-Gorenstein N-complexes, see Definition 3.1. Then the main results Theorems 3.8 and 3.9 of this note characterise strongly $\mathcal{VW}$-Gorenstein N-complexes and exact strongly $\mathcal{VW}$-Gorenstein N-complexes, respectively. Finally, we apply these abstract results to deduce that an N-complex $\boldsymbol{X}$ is strongly Gorenstein projective (resp., injective) if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n^t(\boldsymbol{X})$ is a Gorenstein projective (resp., injective) module for each $n \in \mathbb{Z}$ and $1 \leqslant t \leqslant N-1$, see Corollaries 3.10 and 3.12. This arrives at our goal. Also, some other particular cases that fit to the main results are exhibited, see Corollaries 3.14-3.18.

2. Preliminaries

Throughout, R is a unitary ring and by an R-module we mean a left R-module, unless otherwise stated. We fix once and for all an integer $N \geqslant 2$. Next, we recollect some notation and terminology that will be needed in the rest of the paper.

2.1. ***N*-complexes.** The terminology is due to [6,10,28]. An N-complex $\boldsymbol{X}$ is a sequence of R-modules and R-homomorphisms

$$\cdots \xrightarrow{d^{\boldsymbol{X}}_{n+2}} X_{n+1} \xrightarrow{d^{\boldsymbol{X}}_{n+1}} X_n \xrightarrow{d^{\boldsymbol{X}}_{n}} X_{n-1} \xrightarrow{d^{\boldsymbol{X}}_{n-1}} \cdots$$

satisfying $d^{\boldsymbol{X}}_{n-(N-1)} \cdots d^{\boldsymbol{X}}_{n-1} d^{\boldsymbol{X}}_{n} = 0$ for any $n \in \mathbb{Z}$. So a 2-complex is a chain complex in the usual sense. For $0 \leqslant r \leqslant N$ and $n \in \mathbb{Z}$, we denote the composition $d^{\boldsymbol{X}}_{n-(r-1)} \cdots d^{\boldsymbol{X}}_{n-1} d^{\boldsymbol{X}}_{n}$ by $d^{\boldsymbol{X},\{r\}}_{n}$. Sometimes, we simply write $d^{\boldsymbol{X},\{r\}}$ without mentioning grades. In this notation, $d^{\boldsymbol{X},\{0\}}_{n} = \mathrm{Id}_{X_n}$, $d^{\boldsymbol{X},\{1\}}_{n} = d^{\boldsymbol{X}}_{n}$ and $d^{\boldsymbol{X},\{N\}}_{n} = 0$. A morphism $f : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ of N-complexes is collection of homomorphisms $f_n : X_n \longrightarrow Y_n$ that making all the rectangles commute. In this way, one gets a category of N-complexes of R-modules, denoted by $\mathcal{C}_N(R)$. This is an Abelian category having enough projectives and injectives. In what follows, N-complexes will always be the N-complexes of R-modules and the term complexes always means 2-complexes.

For an N-complexes $\boldsymbol{X}$, there are $N-1$ choices for homology. Indeed, one can define

$$\mathrm{Z}^r_n(\boldsymbol{X}) := \mathrm{Ker} d^{\boldsymbol{X},\{r\}}_{n}, \ \ \mathrm{B}^r_n(\boldsymbol{X}) := \mathrm{Im} d^{\boldsymbol{X},\{r\}}_{n+r} \ \text{ for } \ r = 1, 2, \ldots, N$$

and

$$\mathrm{H}^r_n(\boldsymbol{X}) := \mathrm{Z}^r_n(\boldsymbol{X})/\mathrm{B}^{N-r}_n(\boldsymbol{X}) \ \text{ for } \ r = 1, 2, \ldots, N-1.$$

An N-complex $\boldsymbol{X}$ is called N-exact, or just exact, if $\mathrm{H}^r_n(\boldsymbol{X}) = 0$ for all $n \in \mathbb{Z}$ and $r = 1, 2, \ldots, N-1$.

The following properties on exactness of N-complexes are useful.

Lemma 2.1. ([6, Proposition 2.2])

(1) *An N-complex $\boldsymbol{X}$ is exact if and only if for some $0 < r < N$ one has $\mathrm{H}^r_n(\boldsymbol{X}) = 0$ for each n.*

(2) *Suppose $0 \longrightarrow \boldsymbol{X} \longrightarrow \boldsymbol{Y} \longrightarrow \boldsymbol{Z} \longrightarrow 0$ is a short exact sequence of N-complexes. If any two out of the three are exact, then so is the third.*

A morphism $f : \boldsymbol{X} \to \boldsymbol{Y}$ of N-complexes is called *null-homotopic* if there exists a collection of homomorphisms $\{s_n | s_n \in \mathrm{Hom}_R(X_n, Y_{n+N-1}), n \in \mathbb{Z}\}$ such that

$$f_n = \sum_{i=1}^{N} d_{n+N-i}^{\boldsymbol{Y},\{N-i\}} s_{n+1-i} d_n^{\boldsymbol{X},\{i-1\}}$$

for each $n \in \mathbb{Z}$. Two morphisms $f, g : \boldsymbol{X} \to \boldsymbol{Y}$ of N-complexes are called *homotopic*, in symbols $f \sim g$, if $f - g$ is null-homotopic. We denote by $\mathcal{K}_N(R)$ the *homotopy category* of N-complexes, that is, the category consisting of N-complexes such that the morphism set between $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{K}_N(R)$ is given by $\mathrm{Hom}_{\mathcal{K}_N(R)}(\boldsymbol{X}, \boldsymbol{Y}) = \mathrm{Hom}_{\mathcal{C}_N(R)}(\boldsymbol{X}, \boldsymbol{Y})/\sim$. It is known that $\mathcal{K}_N(R)$ is a triangulated category, see [10, Theorem 2.3].

For any R-module M, any $n \in \mathbb{Z}$ and $1 \leqslant r \leqslant N$, we use $\mathrm{D}_n^r(M)$ to denote the N-complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{Id}_M} M \xrightarrow{\mathrm{Id}_M} \cdots \xrightarrow{\mathrm{Id}_M} M \xrightarrow{\mathrm{Id}_M} M \longrightarrow 0 \longrightarrow \cdots$$

with M in degrees $n, n-1, \ldots, n-(r-1)$. Let $\{M_n\}_{n\in\mathbb{Z}}$ be a collection of R-modules, it is obvious that $\bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(M_n) = \prod_{n\in\mathbb{Z}} \mathrm{D}_n^N(M_n)$.

Let $\boldsymbol{X} \in \mathcal{C}_N(R)$ be given. Then the identity map Id_{X_n} gives rise to two morphisms $\rho_n^{X_n} : \mathrm{D}_n^N(X_n) \longrightarrow \boldsymbol{X}$ and $\lambda_n^{X_n} : \boldsymbol{X} \longrightarrow \mathrm{D}_{n+N-1}^N(X_n)$ for any $n \in \mathbb{Z}$. Consequently, we have a degreewise split epimorphism $\rho^{\boldsymbol{X}} : \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(X_n) \longrightarrow \boldsymbol{X}$ and a degreewise split monomorphism $\lambda^{\boldsymbol{X}} : \boldsymbol{X} \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_{n+N-1}^N(X_n)$. Thus, there are degreewise split exact sequences of N-complexes

$$0 \longrightarrow \mathrm{Ker}\rho^{\boldsymbol{X}} \xrightarrow{\epsilon^{\boldsymbol{X}}} \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(X_n) \xrightarrow{\rho^{\boldsymbol{X}}} \boldsymbol{X} \longrightarrow 0$$

and

$$0 \longrightarrow \boldsymbol{X} \xrightarrow{\lambda^{\boldsymbol{X}}} \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_{n+N-1}^N(X_n) \xrightarrow{\eta^{\boldsymbol{X}}} \mathrm{Coker}\lambda^{\boldsymbol{X}} \longrightarrow 0,$$

where

$$(\mathrm{Ker}\rho^{\boldsymbol{X}})_n = \bigoplus_{i=1-N}^{-1} X_{n-i},$$

$$d^{\mathrm{Ker}\rho^{\boldsymbol{X}}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -d^{\{N-1\}} & -d^{\{N-2\}} & -d^{\{N-3\}} & \cdots & -d^{\{2\}} & -d \end{pmatrix},$$

$$\epsilon^{\boldsymbol{X}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -d^{\{N-1\}} & -d^{\{N-2\}} & -d^{\{N-3\}} & \cdots & -d^{\{2\}} & -d \end{pmatrix},$$

$$\rho^{\boldsymbol{X}} = \left(d^{\{N-1\}}, \dots, d, 1\right)$$

and

$$(\mathrm{Coker}\lambda^{\boldsymbol{X}})_n = \bigoplus_{i=1}^{N-1} X_{n-i}, \;\; d^{\mathrm{Coker}\lambda^{\boldsymbol{X}}} = \begin{pmatrix} -d & 1 & 0 & \cdots & 0 & 0 \\ -d^{\{2\}} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -d^{\{N-2\}} & 0 & 0 & \cdots & 0 & 1 \\ -d^{\{N-1\}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\lambda^{\boldsymbol{X}} = \begin{pmatrix} 1 \\ d \\ \vdots \\ d^{\{N-1\}} \end{pmatrix}, \qquad \eta^{\boldsymbol{X}} = \begin{pmatrix} -d & 1 & 0 & \cdots & 0 & 0 \\ -d^{\{2\}} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -d^{\{N-2\}} & 0 & 0 & \cdots & 1 & 0 \\ -d^{\{N-1\}} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Now, we define functors $\Sigma, \Sigma^{-1} : \mathcal{C}_N(R) \longrightarrow \mathcal{C}_N(R)$ by

$$\Sigma^{-1}\boldsymbol{X} = \mathrm{Ker}\rho^{\boldsymbol{X}} \;\text{ and }\; \Sigma\boldsymbol{X} = \mathrm{Coker}\lambda^{\boldsymbol{X}}$$

in the exact sequences above. Then Σ and Σ^{-1} induce the suspension functor and its quasi-inverse of the triangulated category $\mathcal{K}_N(R)$.

On the other hand, we define the *shift functor* $\Theta : \mathcal{C}_N(R) \longrightarrow \mathcal{C}_N(R)$ by

$$\Theta(\boldsymbol{X})_n = X_{n-1}, \; d_n^{\Theta(\boldsymbol{X})} = d_{n-1}^{\boldsymbol{X}}$$

for $\boldsymbol{X} = (X_n, d_n^{\boldsymbol{X}}) \in \mathcal{C}_N(R)$. The N-complex $\Theta(\Theta X)$ is denoted $\Theta^2 X$ and inductively we define $\Theta^k X$ for all $k \in \mathbb{Z}$. This induces the shift functor $\Theta : \mathcal{K}_N(R) \longrightarrow \mathcal{K}_N(R)$ which is a triangle functor. Unlike classical case, Σ does not coincide with Θ. In fact, $\Sigma^2 \simeq \Theta^N$ on $\mathcal{K}_N(R)$, see [10, Theorem 2.4].

2.2. Hom N-complexes. Given two N-complexes $\boldsymbol{X}$ and $\boldsymbol{Y}$, the N-complex $\mathrm{Hom}_R(\boldsymbol{X}, \boldsymbol{Y})$ of Abelian groups is given by

$$\mathrm{Hom}_R(\boldsymbol{X}, \boldsymbol{Y})_n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}_R(X_t, Y_{n+t})$$

and

$$(d_n^{\mathrm{Hom}_R(\boldsymbol{X},\boldsymbol{Y})}(f))_m = d_{n+m}^{\boldsymbol{Y}} f_m - q^n f_{m-1} d_m^{\boldsymbol{X}}$$

for $f \in \mathrm{Hom}_R(\boldsymbol{X},\boldsymbol{Y})_n$, where q is the Nth root of unity, $q^N = 1$ and $q \neq 1$.

For $\boldsymbol{X},\boldsymbol{Y} \in \mathcal{C}_N(R)$, we denote the group of i-fold extensions by $\mathrm{Ext}^i_{\mathcal{C}_N(R)}(\boldsymbol{X},\boldsymbol{Y})$. Recall that $\mathrm{Ext}^0_{\mathcal{C}_N(R)}(\boldsymbol{X},\boldsymbol{Y})$ is naturally isomorphic to the group $\mathrm{Hom}_{\mathcal{C}_N(R)}(\boldsymbol{X},\boldsymbol{Y})$ of morphisms $\boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{X},\boldsymbol{Y})$ is the group of (equivalence classes) of short exact sequence $0 \longrightarrow \boldsymbol{Y} \longrightarrow \boldsymbol{Z} \longrightarrow \boldsymbol{X} \longrightarrow 0$ under the Baer sum. We let $\mathrm{Ext}^1_{dw_N}(\boldsymbol{X},\boldsymbol{Y})$ be the subgroup of $\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{X},\boldsymbol{Y})$ consisting of those short exact sequences which are split in each degree. The following lemma is a standard result relating $\mathrm{Ext}^1_{dw_N}(\boldsymbol{X},\boldsymbol{Y})$ to $\mathrm{Hom}_R(\boldsymbol{X},\boldsymbol{Y})$.

Lemma 2.2. ([15, Lemma 3.10]) *For any $\boldsymbol{X},\boldsymbol{Y} \in \mathcal{C}_N(R)$ and any $n \in \mathbb{Z}$, we have*

(1) $\mathrm{Ext}^1_{dw_N}(\Sigma\boldsymbol{X},\boldsymbol{Y}) \cong \mathrm{H}^1_n(\mathrm{Hom}_R(\boldsymbol{X},\Theta^n\boldsymbol{Y})) \cong \mathrm{Hom}_{\mathcal{K}_N(R)}(\boldsymbol{X},\boldsymbol{Y})$.
(2) $\mathrm{Ext}^1_{dw_N}(\boldsymbol{X},\boldsymbol{Y}) \cong \mathrm{H}^1_n(\mathrm{Hom}_R(\boldsymbol{X},\Theta^n\Sigma^{-1}\boldsymbol{Y})) \cong \mathrm{Hom}_{\mathcal{K}_N(R)}(\boldsymbol{X},\Sigma^{-1}\boldsymbol{Y})$.

2.3. Several classes of N-complexes. Let $\mathcal{X}$ be a class of R-modules. As the classical case, we have the following classes of N-complexes:

- $\widetilde{\mathcal{X}_N}$ is the class of all exact N-complex $\boldsymbol{X}$ with cycles $\mathrm{Z}^t_n(\boldsymbol{X}) \in \mathcal{X}$ for $n \in \mathbb{Z}$ and $t = 1,2,\ldots,N$;
- $\widetilde{\#\mathcal{X}_N}$ is the class of all N-complex $\boldsymbol{X}$ with terms $X_n \in \mathcal{X}$ for all $n \in \mathbb{Z}$;
- $\mathrm{CE}(\mathcal{X}_N)$ is the class of all N-complex $\boldsymbol{X}$ with X_n, $\mathrm{Z}^t_n(\boldsymbol{X})$, $\mathrm{B}^t_n(\boldsymbol{X})$, $\mathrm{H}^t_n(\boldsymbol{X}) \in \mathcal{X}$ for $n \in \mathbb{Z}$ and $t = 1,2,\ldots,N$.

2.4. Semidualizing modules and some related classes of modules.

Definition 2.3. ([24, 1.8]) Let R be a commutative ring. An R-module C is called *semidualizing* if

(1) C admits a degreewise finitely generated projective resolution,
(2) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_R(C,C)$ is an isomorphism,
(3) $\mathrm{Ext}^{\geqslant 1}_R(C,C) = 0$.

In the remainder of the paper, let C be an arbitrary but fixed semidualizing module over a commutative ring R.

Definition 2.4. ([9,24]) The *Auslander class* $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying:

(1) $\mathrm{Tor}^R_{\geqslant 1}(C,M) = 0 = \mathrm{Ext}^{\geqslant 1}_R(C, C\otimes_R M)$ and
(2) The natural evaluation homomorphism $\mu_M : M \longrightarrow \mathrm{Hom}_R(C, C\otimes_R M)$ is an isomorphism.

The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all R-modules M satisfying:

(1) $\mathrm{Ext}_R^{\geqslant 1}(C, M) = 0 = \mathrm{Tor}_{\geqslant 1}^R(C, \mathrm{Hom}_R(C, M))$ and
(2) The natural evaluation homomorphism $v_M : C \otimes_R \mathrm{Hom}_R(C, M) \longrightarrow M$ is an isomorphism.

We set,

$$\mathcal{P}_C(R) = \text{the subcategory of } R\text{-modules } C \otimes_R P \text{ where } P \text{ is } R\text{-projective,}$$
$$\mathcal{I}_C(R) = \text{the subcategory of } R\text{-modules } \mathrm{Hom}_R(C, I) \text{ where } I \text{ is } R\text{-injective.}$$

Modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-projective* and *C-injective*, respectively. When $C = R$, we omit the subscript and recover the classes of projective and injective R-modules.

2.5. Orthogonal subcategories. Let $\mathcal{A}$ be an Abelian category. For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, we say $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}_{\mathcal{A}}^{\geqslant 1}(X, Y) = 0$ for any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$. In particular, if $\mathcal{X} \perp \mathcal{X}$, then $\mathcal{X}$ is called *self-orthogonal.* According to [5, Theorem 3.1 and Corollary 3.2], $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$ are self-orthogonal and closed under finite direct sums and direct summands.

2.6. $\mathcal{VW}$-Gorenstein modules. Let $\mathcal{A}$ be an Abelian category and $\mathcal{X}, \mathcal{Y}$ two subcategories of $\mathcal{A}$. Recall that a sequence $\mathbb{S}$ in $\mathcal{A}$ is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$*-exact* (resp., $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{Y})$*-exact*) if the sequence $\mathrm{Hom}_{\mathcal{A}}(X, \mathbb{S})$ (resp., $\mathrm{Hom}_{\mathcal{A}}(\mathbb{S}, Y)$) is exact for any $X \in \mathcal{X}$ (resp., $Y \in \mathcal{Y}$).

Definition 2.5. ([31, Definition 3.1]) Let $\mathcal{V}, \mathcal{W}$ be two classes of R-modules. An R-module M is called *$\mathcal{VW}$-Gorenstein* if there exists a both $\mathrm{Hom}_R(\mathcal{V}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W})$-exact exact sequence

$$\cdots \longrightarrow V_1 \to V_0 \to W^0 \to W^1 \longrightarrow \cdots$$

with $V_i \in \mathcal{V}$ and $W^i \in \mathcal{W}$ for all $i \geq 0$ such that $M \cong \mathrm{Im}(V_0 \to W^0)$.

We denote the class of all $\mathcal{VW}$-Gorenstein modules by $\mathcal{G}(\mathcal{VW})$. The $\mathcal{VW}$-Gorenstein modules unifies the following notions: G_C-projective R-modules [8,24] (when $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{P}_C(R)$); G_C-injective R-modules [8,24] (when $\mathcal{V} = \mathcal{I}_C(R)$ and $\mathcal{W} = \mathcal{I}(R)$); modules in $\mathcal{A}_C(R)$ (when $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{I}_C(R)$, see [9, Lemma 6.1(1) and Theorem 2]); modules in $\mathcal{B}_C(R)$ (when $\mathcal{V} = \mathcal{P}_C(R)$ and $\mathcal{W} = \mathcal{I}(R)$, see [9, Lemma 6.1(2) and Theorem 6.1]); $\mathcal{W}$-Gorenstein modules [5,23] (when $\mathcal{V} = \mathcal{W}$), and of course Gorenstein projective R-modules (in the case $\mathcal{V} = \mathcal{W} = \mathcal{P}(R)$) and Gorenstein injective R-modules (in the case $\mathcal{V} = \mathcal{W} = \mathcal{I}(R)$), see [3,7].

3. Main results

In what follows, let $\mathcal{V}, \mathcal{W}$ be two classes of R-modules which are closed under isomorphisms, direct summands and finite direct sums.

Definition 3.1. An N-complex $\boldsymbol{X}$ is called *strongly* $\mathcal{VW}$*-Gorenstein* if there exists a both $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N), -)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-, \mathrm{CE}(\mathcal{W}_N))$-exact exact sequence

$$\cdots \longrightarrow \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{W}^0 \to \boldsymbol{W}^1 \longrightarrow \cdots,$$

where $\boldsymbol{V}_i \in \widetilde{\mathcal{V}_N}$ and $\boldsymbol{W}^i \in \widetilde{\mathcal{W}_N}$, such that $\boldsymbol{X} \cong \mathrm{Im}(\boldsymbol{V}_0 \to \boldsymbol{W}^0)$.

Remark 3.2. Here are some special cases of strongly $\mathcal{VW}$-Gorenstein N-complexes:

(1) If $\mathcal{V} = \mathcal{W}$, then we call strongly $\mathcal{VW}$-Gorenstein N-complexes strongly $\mathcal{W}$-Gorenstein N-complexes. In particular, if they are the class of projective (resp., injective) R-modules, then strongly $\mathcal{VW}$-Gorenstein N-complexes is particularly called strongly Gorenstein projective (respectively, injective) N-complexes. In the case of $N = 2$, strongly $\mathcal{W}$-Gorenstein N-complexes happen to be strongly $\mathcal{W}$-Gorenstein complexes in [13]. The strongly Gorenstein projective complexes were studied in [19].

(2) If $\mathcal{V} = \mathcal{P}(R)$, $\mathcal{W} = \mathcal{P}_C(R)$, then strongly $\mathcal{VW}$-Gorenstein N-complexes is particularly called strongly G_C-projective N-complexes; if $\mathcal{V} = \mathcal{I}_C(R)$, $\mathcal{W} = \mathcal{I}(R)$, then strongly $\mathcal{VW}$-Gorenstein N-complexes is particularly called strongly G_C-injective N-complexes.

To characterize strongly $\mathcal{VW}$-Gorenstein N-complexes, we need some preparations.

Lemma 3.3. ([15, Lemma 3.12]) *Let $\mathcal{X}, \mathcal{Y}$ be two classes of R-modules. If $\mathcal{X}$ is self-orthogonal, then the following statements hold:*

(1) *$\mathcal{X} \perp \mathcal{Y}$ if and only if $\widetilde{\mathcal{X}_N} \perp \widetilde{\#\mathcal{Y}_N}$.*

(2) *$\mathcal{Y} \perp \mathcal{X}$ if and only if $\widetilde{\#\mathcal{Y}_N} \perp \widetilde{\mathcal{X}_N}$.*

Corollary 3.4. *Let $\mathcal{X}, \mathcal{Y}$ be two classes of R-modules and $\mathcal{X} \perp \mathcal{Y}$.*

(1) *If $\mathcal{X}$ is self-orthogonal, then $\widetilde{\mathcal{X}_N} \perp \mathrm{CE}(\mathcal{Y}_N)$.*

(2) *If $\mathcal{Y}$ is self-orthogonal, then $\mathrm{CE}(\mathcal{X}_N) \perp \widetilde{\mathcal{Y}_N}$.*

Proof. It follows from $\mathrm{CE}(\mathcal{X}_N) \subseteq \widetilde{\#\mathcal{X}_N}$, $\mathrm{CE}(\mathcal{Y}_N) \subseteq \widetilde{\#\mathcal{Y}_N}$ and Lemma 3.3. □

Lemma 3.5. ([18, Theorem 1]) *Let $\boldsymbol{X}$ be an N-complex and $\mathcal{X}$ a class R-modules. If $\mathcal{X}$ is self-orthogonal, then $\boldsymbol{X} \in \mathrm{CE}(\mathcal{X}_N)$ if and only if $\boldsymbol{X} = \boldsymbol{X}' \bigoplus \boldsymbol{X}''$, where $\boldsymbol{X}' \in \widetilde{\mathcal{X}_N}$, $\boldsymbol{X}'' = \bigoplus_{n\in\mathbb{Z}} D_n^1(M_n)$ with $M_n \in \mathcal{X}$ for all $n \in \mathbb{Z}$.*

Lemma 3.6. *Let $\boldsymbol{X} \in \mathrm{CE}(\mathcal{X}_N)$. If $\mathcal{X}$ is closed under finite direct sums and self-orthogonal, then $\Sigma\boldsymbol{X}, \Sigma^{-1}\boldsymbol{X} \in \mathrm{CE}(\mathcal{X}_N)$.*

Proof. Since $\boldsymbol{X} \in \mathrm{CE}(\mathcal{X}_N)$, by Lemma 3.5 one has $\boldsymbol{X} = \boldsymbol{X}' \bigoplus \boldsymbol{X}''$, where $\boldsymbol{X}' \in \widetilde{\mathcal{X}_N}$ and $\boldsymbol{X}'' = \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^1(M_n)$ with all $M_n \in \mathcal{X}$. One then has $\Sigma\boldsymbol{X} = \Sigma\boldsymbol{X}' \bigoplus \Sigma\boldsymbol{X}''$. By assumption $\mathcal{X}$ is self-orthogonal, it follows that $\widetilde{\mathcal{X}_N} \subseteq \mathrm{CE}(\mathcal{X}_N)$, so one gets $\Sigma\boldsymbol{X}' \subseteq \mathrm{CE}(\mathcal{X}_N)$ from [15, Lemma 3.5]. To complete the proof, it is now sufficient to show that $\Sigma\boldsymbol{X}'' \in \mathrm{CE}(\mathcal{X}_N)$. Let $n \in \mathbb{Z}$, notice that

$$(\Sigma\boldsymbol{X}'')_n = M_{n-1} \oplus M_{n-2} \oplus \cdots \oplus M_{n-(N-1)}$$

and

$$d_n^{\Sigma\boldsymbol{X}''} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

then one has

$$\begin{aligned} \mathrm{Z}_n^1(\Sigma\boldsymbol{X}'') &\cong M_{n-1}, & \mathrm{B}_n^1(\Sigma\boldsymbol{X}'') &\cong M_{n-1} \oplus \cdots \oplus M_{n-(N-2)}, \\ \mathrm{Z}_n^2(\Sigma\boldsymbol{X}'') &\cong M_{n-1} \oplus M_{n-2}, & \mathrm{B}_n^2(\Sigma\boldsymbol{X}'') &\cong M_{n-1} \oplus \cdots \oplus M_{n-(N-3)}, \\ &\vdots & &\vdots \\ \mathrm{Z}_n^{N-1}(\Sigma\boldsymbol{X}'') &\cong M_{n-1} \oplus \cdots \oplus M_{n-(N-1)}, & \mathrm{B}_n^{N-1}(\Sigma\boldsymbol{X}'') &= 0. \end{aligned}$$

Since $\mathcal{X}$ is closed under finite direct sums, we have $(\Sigma\boldsymbol{X}'')_n, \mathrm{Z}_n^t(\Sigma\boldsymbol{X}''), \mathrm{B}_n^t(\Sigma\boldsymbol{X}'') \in \mathcal{X}$ and so $\mathrm{H}_n^t(\Sigma\boldsymbol{X}'') = M_{n-t} \in \mathcal{X}$ for $t = 1, 2, \ldots, N-1$. It now follows that $\Sigma\boldsymbol{X}'' \in \mathrm{CE}(\mathcal{X}_N)$, as desired. Similarly, one can show that $\Sigma^{-1}\boldsymbol{X} \in \mathrm{CE}(\mathcal{X}_N)$. $\square$

Lemma 3.7. *Let $\mathcal{V}, \mathcal{W}$ be two classes of R-modules and*

$$\cdots \longrightarrow \boldsymbol{X}_1 \to \boldsymbol{X}_0 \to \boldsymbol{X}_{-1} \longrightarrow \cdots$$

be a both $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N), -)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-, \mathrm{CE}(\mathcal{W}_N))$-exact exact sequence of N-complexes, then for any $n \in \mathbb{Z}$, the sequence

$$\cdots \longrightarrow (\boldsymbol{X}_1)_n \to (\boldsymbol{X}_0)_n \to (\boldsymbol{X}_{-1})_n \longrightarrow \cdots$$

is a $\mathrm{Hom}_R(\mathcal{V}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W})$-exact exact sequence of R-modules.

Proof. Let $V \in \mathcal{V}$, $W \in \mathcal{W}$ and $n \in \mathbb{Z}$. Then $\mathrm{D}_n^N(V) \in \mathrm{CE}(\mathcal{V}_N)$ and $\mathrm{D}_{n+N-1}^N(W) \in \mathrm{CE}(\mathcal{W}_N)$. Thus, we have the following exact sequences

$$\cdots \to \mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{D}_n^N(V), \boldsymbol{X}_1) \to \mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{D}_n^N(V), \boldsymbol{X}_0) \\ \to \mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{D}_n^N(V), \boldsymbol{X}_{-1}) \to \cdots,$$

$$\cdots \to \mathrm{Hom}_{\mathcal{C}_N(R)}(\boldsymbol{X}_{-1}, \mathrm{D}_{n+N-1}^N(W)) \to \mathrm{Hom}_{\mathcal{C}_N(R)}(\boldsymbol{X}_0, \mathrm{D}_{n+N-1}^N(W)) \\ \to \mathrm{Hom}_{\mathcal{C}_N(R)}(\boldsymbol{X}_1, \mathrm{D}_{n+N-1}^N(W)) \to \cdots.$$

It now follows from [15, Lemma 3.3] that the sequences

$$\cdots \to \mathrm{Hom}_R(V, (\boldsymbol{X}_1)_n) \to \mathrm{Hom}_R(V, (\boldsymbol{X}_0)_n) \to \mathrm{Hom}_R(V, (\boldsymbol{X}_{-1})_n) \to \cdots$$

and

$$\cdots \to \mathrm{Hom}_R((\boldsymbol{X}_{-1})_n, W) \to \mathrm{Hom}_R((\boldsymbol{X}_0)_n, W) \to \mathrm{Hom}_R((\boldsymbol{X}_1)_n, W) \to \cdots$$

are exact. □

With the above preparations, we are now in a position to prove our main results.

Theorem 3.8. *Let $\boldsymbol{X}$ be an N-complex. If $\mathcal{V}, \mathcal{W}$ are self-orthogonal, $\mathcal{V} \perp \mathcal{W}$ and $\mathcal{V}, \mathcal{W} \subseteq \mathcal{G}(\mathcal{V}\mathcal{W})$, then the following statements are equivalent:*

(1) *$\boldsymbol{X}$ is a strongly $\mathcal{V}\mathcal{W}$-Gorenstein N-complex.*
(2) *Each X_n is a $\mathcal{V}\mathcal{W}$-Gorenstein module, and both N-complexes $\mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{X})$ and $\mathrm{Hom}_R(\boldsymbol{X}, \boldsymbol{W})$ are exact for any $\boldsymbol{V} \in \mathrm{CE}(\mathcal{V}_N)$ and any $\boldsymbol{W} \in \mathrm{CE}(\mathcal{W}_N)$.*
(3) *Each X_n is a $\mathcal{V}\mathcal{W}$-Gorenstein module, and both N-complexes $\mathrm{Hom}_R(V, \boldsymbol{X})$ and $\mathrm{Hom}_R(\boldsymbol{X}, W)$ are exact for any $V \in \mathcal{V}$ and $W \in \mathcal{W}$.*

Proof. (1) ⇒ (3) Since $\boldsymbol{X}$ is a strongly $\mathcal{V}\mathcal{W}$-Gorenstein N-complex, there is a both $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N), -)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-, \mathrm{CE}(\mathcal{W}_N))$-exact exact sequence of N-complexes

$$\cdots \longrightarrow \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{W}^0 \to \boldsymbol{W}^1 \longrightarrow \cdots$$

such that $\boldsymbol{X} \cong \mathrm{Im}(\boldsymbol{V}_0 \to \boldsymbol{W}^0)$, where $\boldsymbol{V}_i \in \widetilde{\mathcal{V}_N}$ and $\boldsymbol{W}^i \in \widetilde{\mathcal{W}_N}$ for $i \geqslant 0$. Applying Lemma 3.7 one thus gets a $\mathrm{Hom}_R(\mathcal{V}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W})$-exact exact sequence of R-modules

$$\cdots \longrightarrow (\boldsymbol{V}_1)_n \to (\boldsymbol{V}_0)_n \to (\boldsymbol{W}^0)_n \to (\boldsymbol{W}^1)_n \longrightarrow \cdots$$

such that $X_n \cong \mathrm{Im}((\boldsymbol{V}_0)_n \to (\boldsymbol{W}^0)_n)$ for each $n \in \mathbb{Z}$. As $\mathcal{V}, \mathcal{W}$ are closed on finite direct sums and self-orthogonal, it follows from [20, Proposition 4.1] that $(\boldsymbol{V}_i)_n \in \mathcal{V}$ and $(\boldsymbol{W}^i)_n \in \mathcal{W}$ for any i and n. Therefore, each X_n is a $\mathcal{V}\mathcal{W}$-Gorenstein module.

Let $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Then $\mathrm{D}_n^1(V) \in \mathrm{CE}(\mathcal{V}_N), \mathrm{D}_n^1(W) \in \mathrm{CE}(\mathcal{W}_N)$. From Lemma 3.6 it follows that $\Sigma\mathrm{D}_n^1(V) \in \mathrm{CE}(\mathcal{V}_N), \Sigma\mathrm{D}_n^1(W) \in \mathrm{CE}(\mathcal{W}_N)$. Setting $\boldsymbol{K}_0 = \mathrm{Im}(\boldsymbol{V}_1 \to \boldsymbol{V}_0)$ and $\boldsymbol{K}^1 = \mathrm{Im}(\boldsymbol{W}^0 \to \boldsymbol{W}^1)$. Consider exact sequences

$$\mathrm{Hom}_{\mathcal{C}_N(R)}(\Sigma\mathrm{D}_n^1(V), \boldsymbol{K}^1) \to \mathrm{Ext}^1_{\mathcal{C}_N(R)}(\Sigma\mathrm{D}_n^1(V), \boldsymbol{X}) \to \mathrm{Ext}^1_{\mathcal{C}_N(R)}(\Sigma\mathrm{D}_n^1(V), \boldsymbol{W}^0)$$

and

$$\mathrm{Hom}_{\mathcal{C}_N(R)}(\boldsymbol{K}_0, \Sigma\mathrm{D}_n^1(W)) \to \mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{X}, \Sigma\mathrm{D}_n^1(W)) \to \mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{V}_0, \Sigma\mathrm{D}_n^1(W)).$$

By the assumptions on $\mathcal{V}$ and $\mathcal{W}$, Corollary 3.4 applies to yield that

$$\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\Sigma\mathrm{D}_n^1(V), \boldsymbol{W}^0) = 0 \text{ and } \mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{V}_0, \Sigma\mathrm{D}_n^1(W)) = 0.$$

The $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N), -)$-exactness of $0 \longrightarrow \boldsymbol{X} \longrightarrow \boldsymbol{W}^0 \longrightarrow \boldsymbol{K}^1 \longrightarrow 0$ and the $\mathrm{Hom}_{\mathcal{C}_N(R)}(-, \mathrm{CE}(\mathcal{W}_N))$-exactness of $0 \longrightarrow \boldsymbol{K}_0 \longrightarrow \boldsymbol{V}_0 \longrightarrow \boldsymbol{X} \longrightarrow 0$ now yield that $\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\Sigma\mathrm{D}_n^1(V), \boldsymbol{X}) = 0$ and $\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{X}, \Sigma\mathrm{D}_n^1(W)) = 0$. It then follows from Lemma 2.2 that $\mathrm{Hom}_R(V, \boldsymbol{X})$ and $\mathrm{Hom}_R(\boldsymbol{X}, W)$ are exact.

(3) $\Rightarrow$ (2) It follows by Lemma 3.5, [31, Proposition 3.5] and Lemmas 3.3, 2.1, 2.2.

(2) $\Rightarrow$ (1) For any $n \in \mathbb{Z}$, as X_n is a $\mathcal{VW}$-Gorenstein module, it follows that there is an exact sequence of R-modules

$$0 \longrightarrow G_n \longrightarrow V_n \xrightarrow{g_n} X_n \longrightarrow 0,$$

where $G_n \in \mathcal{G}(\mathcal{VW})$ and $V_n \in \mathcal{V}$ by [31, Corollary 4.6]. One thus gets an exact sequence of N-complexes

$$0 \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(G_n) \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(V_n) \xrightarrow{g} \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(X_n) \longrightarrow 0,$$

where $g = \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(g_n)$. Put $\boldsymbol{V}_0 = \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(V_n)$. By [20, Proposition 4.1] one has $\boldsymbol{V}_0 \in \widetilde{\mathcal{V}_N}$. On the other hand, there is always a degreewise split short exact sequence

$$0 \longrightarrow \Sigma^{-1}\boldsymbol{X} \xrightarrow{\epsilon^{\boldsymbol{X}}} \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(X_n) \xrightarrow{\rho^{\boldsymbol{X}}} \boldsymbol{X} \longrightarrow 0.$$

Let $\beta = \rho^{\boldsymbol{X}} g$. Then β is an epimorphism from $\boldsymbol{V}_0$ to $\boldsymbol{X}$. Setting $\boldsymbol{K}_0 = \mathrm{Ker}\beta$ yields an exact sequence of N-complexes

$$0 \longrightarrow \boldsymbol{K}_0 \longrightarrow \boldsymbol{V}_0 \longrightarrow \boldsymbol{X} \longrightarrow 0. \qquad (\dagger_0)$$

Now, we show that $\boldsymbol{K}_0$ has the same properties as $\boldsymbol{X}$, and that the exact sequence $(\dagger_0)$ is both $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N), -)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-, \mathrm{CE}(\mathcal{W}_N))$-exact. To this end, consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \boldsymbol{K}_0 \arrow[r] \arrow[d] & \Sigma^{-1}\boldsymbol{X} \arrow[d, "\epsilon^{\boldsymbol{X}}"] & \\
0 \arrow[r] & \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(G_n) \arrow[r] \arrow[d] & \boldsymbol{V}_0 \arrow[r, "g"] \arrow[d, "\beta"] & \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(X_n) \arrow[r] \arrow[d, "\rho^{\boldsymbol{X}}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \boldsymbol{X} \arrow[r, equal] \arrow[d] & \boldsymbol{X} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Apply the Snake Lemma to this diagram to get the exact sequence

$$0 \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(G_n) \longrightarrow \boldsymbol{K}_0 \longrightarrow \Sigma^{-1}\boldsymbol{X} \longrightarrow 0.$$

Notice that both $\bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(G_n)$ and $\Sigma^{-1}\boldsymbol{X}$ are N-complexes of $\mathcal{VW}$-Gorenstein modules, it follows from [31, Corollary 3.8] that each degree of $\boldsymbol{K}_0$ is $\mathcal{VW}$-Gorenstein. Let $\boldsymbol{V} \in \mathrm{CE}(\mathcal{V}_N)$, then $\mathrm{Ext}_R^1(V_k, (\boldsymbol{K}_0)_{n+k}) = 0$ for any $n, k \in \mathbb{Z}$ by [31, Proposition 3.5]. So we have the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(V_k, (\boldsymbol{K}_0)_{n+k}) \longrightarrow \mathrm{Hom}_R(V_k, (\boldsymbol{V}_0)_{n+k}) \longrightarrow \mathrm{Hom}_R(V_k, X_{n+k}) \longrightarrow 0.$$

One thus gets the following exact sequence of N-complexes

$$0 \longrightarrow \mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{K}_0) \longrightarrow \mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{V}_0) \longrightarrow \mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{X}) \longrightarrow 0.$$

As $\boldsymbol{V}_0 = \bigoplus_{n\in\mathbb{Z}} \mathrm{D}_n^N(V_n)$ is a contractible N-complex by [6, Theorem 3.3], it follows that $\boldsymbol{V}_0$ is a null object in $\mathcal{K}_N(R)$. Thus,

$$\mathrm{H}_n^1(\mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{V}_0)) \cong \mathrm{Hom}_{\mathcal{K}_N(R)}(\boldsymbol{V}, \Theta^{-n}\boldsymbol{V}_0) = 0$$

for each $n \in \mathbb{Z}$ by Lemma 2.2, and whence $\mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{V}_0)$ is exact by Lemma 2.1. The N-complex $\mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{K}_0)$ is now exact by Lemma 2.1, as $\mathrm{Hom}_R(\boldsymbol{V}, \boldsymbol{X})$ is exact by assumption. Similarly, one can show that $\mathrm{Hom}_R(\boldsymbol{K}_0, \boldsymbol{W})$ is exact for any $\boldsymbol{W} \in \mathrm{CE}(\mathcal{W}_N)$. Let $\boldsymbol{V} \in \mathrm{CE}(\mathcal{V}_N)$ and $\boldsymbol{W} \in \mathrm{CE}(\mathcal{W}_N)$. For any $n \in \mathbb{Z}$, by Lemma 3.6 one has $\Sigma\Theta^{-n}\boldsymbol{V} \in \mathrm{CE}(\mathcal{V}_N)$ and $\Theta^n\Sigma^{-1}\boldsymbol{W} \in \mathrm{CE}(\mathcal{W}_N)$, so $\mathrm{Hom}_R(\Sigma\Theta^{-n}\boldsymbol{V}, \boldsymbol{K}_0)$ is exact as above, and $\mathrm{Hom}_R(\boldsymbol{X}, \Theta^n\Sigma^{-1}\boldsymbol{W})$ is exact by assumption. Hence, it follows from [31, Proposition 3.5] and Lemma 2.2 that

$$\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{V}, \boldsymbol{K}_0) = \mathrm{Ext}^1_{dw_N}(\boldsymbol{V}, \boldsymbol{K}_0) \cong \mathrm{H}_n^1\left(\mathrm{Hom}_R(\Sigma\Theta^{-n}\boldsymbol{V}, \boldsymbol{K}_0)\right) = 0,$$

and

$$\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\boldsymbol{X},\boldsymbol{W}) = \mathrm{Ext}^1_{dw_N}(\boldsymbol{X},\boldsymbol{W}) \cong \mathrm{H}^1_n\left(\mathrm{Hom}_R(\boldsymbol{X},\Theta^n\Sigma^{-1}\boldsymbol{W})\right) = 0.$$

This implies that the sequence

$$0 \longrightarrow \boldsymbol{K}_0 \longrightarrow \boldsymbol{V}_0 \longrightarrow \boldsymbol{X} \longrightarrow 0$$

is both $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N),-)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-,\mathrm{CE}(\mathcal{W}_N))$-exact.

Since $\boldsymbol{K}_0$ has the same properties as $\boldsymbol{X}$, one may continue inductively to construct a both $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N),-)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-,\mathrm{CE}(\mathcal{W}_N))$-exact exact sequence of N-complexes

$$\cdots \longrightarrow \boldsymbol{V}_2 \longrightarrow \boldsymbol{V}_1 \longrightarrow \boldsymbol{V}_0 \longrightarrow \boldsymbol{X} \longrightarrow 0 \tag{†}$$

with all $\boldsymbol{V}_i \in \widetilde{\mathcal{V}}$.

Dually, one can get a $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N),-)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-,\mathrm{CE}(\mathcal{W}_N))$-exact exact sequence of N-complexes

$$0 \longrightarrow \boldsymbol{X} \longrightarrow \boldsymbol{W}^0 \longrightarrow \boldsymbol{W}^1 \longrightarrow \boldsymbol{W}^2 \longrightarrow \cdots \tag{‡}$$

with each $\boldsymbol{W}^i \in \widetilde{\mathcal{W}_N}$.

Finally, splicing together (†) and (‡) at $\boldsymbol{X}$, one gets a $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N),-)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-,\mathrm{CE}(\mathcal{W}_N))$-exact exact sequence N-complexes

$$\cdots \longrightarrow \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{W}^0 \to \boldsymbol{W}^1 \longrightarrow \cdots$$

with each $\boldsymbol{V}_i \in \widetilde{\mathcal{V}_N}$ and each $\boldsymbol{W}_i \in \widetilde{\mathcal{W}_N}$, such that $\boldsymbol{X} \cong \mathrm{Im}(\boldsymbol{V}_0 \to \boldsymbol{W}^0)$. Therefore, $\boldsymbol{X}$ is a strongly $\mathcal{VW}$-Gorenstein N-complex. $\square$

The next result gives a characterization of exact strongly $\mathcal{VW}$-Gorenstein N-complexes.

Theorem 3.9. *Let $\boldsymbol{X}$ be an exact N-complex. If $\mathcal{V},\mathcal{W}$ are self-orthogonal, $\mathcal{V}\perp\mathcal{W}$ and $\mathcal{V},\mathcal{W}\subseteq\mathcal{G}(\mathcal{VW})$, then $\boldsymbol{X}$ is strongly $\mathcal{VW}$-Gorenstein if and only if $\mathrm{Z}^t_n(\boldsymbol{X})$ is a $\mathcal{VW}$-Gorenstein module for any $n\in\mathbb{Z}$ and $t=1,2,\ldots,N-1$.*

Proof. ($\Rightarrow$) As $\boldsymbol{X}$ is strongly $\mathcal{VW}$-Gorenstein, there is a $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{CE}(\mathcal{V}_N),-)$-exact and $\mathrm{Hom}_{\mathcal{C}_N(R)}(-,\mathrm{CE}(\mathcal{W}_N))$-exact exact sequence of N-complexes

$$\mathbb{U} := \cdots \longrightarrow \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{W}^0 \to \boldsymbol{W}^1 \longrightarrow \cdots$$

with $\boldsymbol{V}_i \in \widetilde{\mathcal{V}_N}, \boldsymbol{W}^i \in \widetilde{\mathcal{W}_N}$ for all $i \geqslant 0$, such that $\boldsymbol{X} \cong \mathrm{Im}(\boldsymbol{V}_0 \to \boldsymbol{W}^0)$. We set $\boldsymbol{K}_i = \mathrm{Im}(\boldsymbol{V}_{i+1} \to \boldsymbol{V}_i)$ and $\boldsymbol{K}^i = \mathrm{Ker}(\boldsymbol{W}^i \to \boldsymbol{W}^{i+1})$ for $i \geqslant 0$. Since $\boldsymbol{X} = \boldsymbol{K}^0$ and all $\boldsymbol{V}_i, \boldsymbol{W}^i$ are exact, Lemma 2.1 implies $\boldsymbol{K}_i$ and $\boldsymbol{K}^i$ are exact N-complexes for $i = 0, 1, 2, \ldots$. It now follows from [14, Lemma 3.4] that there exists an exact sequence of R-modules

$$\mathrm{Z}_n^t(\mathbb{U}) := \cdots \longrightarrow \mathrm{Z}_n^t(\boldsymbol{V}_1) \to \mathrm{Z}_n^t(\boldsymbol{V}_0) \to \mathrm{Z}_n^t(\boldsymbol{W}^0) \to \mathrm{Z}_n^t(\boldsymbol{W}^1) \longrightarrow \cdots$$

such that $\mathrm{Z}_n^t(\boldsymbol{X}) \cong \mathrm{Im}(\mathrm{Z}_n^t(\boldsymbol{V}_0) \to \mathrm{Z}_n^t(\boldsymbol{W}^0))$ for all $n \in \mathbb{Z}$ and all $t = 1, 2, \ldots, N-1$. Given an $n \in \mathbb{Z}$ and a $t = 1, 2, \ldots, N-1$, to show $\mathrm{Z}_n^t(\boldsymbol{X})$ is a $\mathcal{VW}$-Gorenstein module, it remains to show that $\mathrm{Z}_n^t(\mathbb{U})$ is both $\mathrm{Hom}_R(\mathcal{V}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{W})$-exact.

Claim 1. $\mathrm{Z}_n^t(\mathbb{U})$ is $\mathrm{Hom}_R(\mathcal{V}, -)$-exact.

Let $V \in \mathcal{V}$. Then $\mathrm{D}_n^t(V) \in \mathrm{CE}(\mathcal{V}_N)$. Thus, $\mathrm{Hom}_{\mathcal{C}_N(R)}(\mathrm{D}_n^t(V), \mathbb{U})$ is exact. It now follows from [29, Lemma 2.2] that $\mathrm{Hom}_R(V, \mathrm{Z}_n^t(\mathbb{U}))$ is exact. This yields the claim 1.

Claim 2. $\mathrm{Z}_n^t(\mathbb{U})$ is $\mathrm{Hom}_R(-, \mathcal{W})$-exact.

It is sufficient to show that

$$0 \longrightarrow \mathrm{Z}_n^t(\boldsymbol{K}_i) \xrightarrow{\varphi} \mathrm{Z}_n^t(\boldsymbol{V}_i) \to \mathrm{Z}_n^t(\boldsymbol{K}_{i-1}) \longrightarrow 0 \qquad (*_i)$$

and

$$0 \longrightarrow \mathrm{Z}_n^t(\boldsymbol{K}^i) \longrightarrow \mathrm{Z}_n^t(\boldsymbol{W}^i) \longrightarrow \mathrm{Z}_n^t(\boldsymbol{K}^{i+1}) \longrightarrow 0 \qquad (*^i)$$

are $\mathrm{Hom}_R(-, \mathcal{W})$-exact for all $i \geqslant 0$, where $\boldsymbol{K}_{-1} = \boldsymbol{X}$. We will prove $(*_i)$ is $\mathrm{Hom}_R(-, \mathcal{W})$-exact, the proof of the $\mathrm{Hom}_R(-, \mathcal{W})$-exactness of $(*^i)$ is similar.

Let $W \in \mathcal{W}$. As $\mathcal{V} \perp \mathcal{V}, \mathcal{W} \perp \mathcal{W}$, it follows that $\widetilde{\mathcal{V}_N} \subseteq \widetilde{\#\mathcal{V}_N}, \widetilde{\mathcal{W}_N} \subseteq \widetilde{\#\mathcal{W}_N}$, so $\boldsymbol{K}_{i-1}$ consists of $\mathcal{VW}$-Gorenstein modules by Lemma 3.7 and [31, Corollary 4.6]. Hence, [31, Proposition 3.5] implies the sequence

$$0 \longrightarrow \mathrm{Hom}_R(\boldsymbol{K}_{i-1}, W) \longrightarrow \mathrm{Hom}_R(\boldsymbol{V}_i, W) \longrightarrow \mathrm{Hom}_R(\boldsymbol{K}_i, W) \longrightarrow 0$$

is exact. Because $\boldsymbol{V}_i \in \widetilde{\mathcal{V}_N}$ and $\mathcal{V} \perp \mathcal{W}$, the N-complex $\mathrm{Hom}_R(\boldsymbol{V}_i, W)$ is exact, and so $\mathrm{Hom}_R(\boldsymbol{K}_i, W)$ is exact by an induction argument since $\mathrm{Hom}_R(\boldsymbol{X}, W)$ is exact. To show that

$$0 \longrightarrow \mathrm{Hom}_R(\mathrm{Z}_n^t(\boldsymbol{K}_{i-1}), W) \to \mathrm{Hom}_R(\mathrm{Z}_n^t(\boldsymbol{V}_i), W) \xrightarrow{\varphi^*} \mathrm{Hom}_R(\mathrm{Z}_n^t(\boldsymbol{K}_i), W) \longrightarrow 0$$

is exact, let $\alpha \in \mathrm{Hom}_R(\mathrm{Z}_n^t(\boldsymbol{K}_i), W)$. As $\mathrm{Hom}_R(\boldsymbol{K}_i, W)$ is exact, applying $\mathrm{Hom}_R(-, W)$ to the exact sequence

$$0 \longrightarrow \mathrm{Z}_n^t(\boldsymbol{K}_i) \xrightarrow{\varepsilon} (\boldsymbol{K}_i)_n \longrightarrow \mathrm{Z}_{n-t}^{N-t}(\boldsymbol{K}_i) \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\mathrm{Z}_{n-t}^{N-t}(\boldsymbol{K}_i), W) \longrightarrow \mathrm{Hom}_R((\boldsymbol{K}_i)_n, W) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(\mathrm{Z}_n^t(\boldsymbol{K}_i), W) \longrightarrow 0.$$

Thus, there is a $\beta \in \operatorname{Hom}_R((\boldsymbol{K}_i)_n, W)$ such that $\alpha = \beta\varepsilon$. Notice that $\mathrm{D}^N_{n+N-1}(W) \in \mathrm{CE}(\mathcal{W}_N)$, $\operatorname{Hom}_{\mathcal{C}_N(R)}(-, D^N_{n+N-1}(W))$ leaves the sequence

$$0 \longrightarrow \boldsymbol{K}_i \to \boldsymbol{V}_i \to \boldsymbol{K}_{i-1} \longrightarrow 0$$

exact. So by [29, Lemma 2.2], the sequence

$$0 \longrightarrow \operatorname{Hom}_R((\boldsymbol{K}_{i-1})_n, W) \longrightarrow \operatorname{Hom}_R((\boldsymbol{V}_i)_n, W) \xrightarrow{\delta^*} \operatorname{Hom}_R((\boldsymbol{K}_i)_n, W) \longrightarrow 0$$

is exact, where $\delta \in \operatorname{Hom}_R((\boldsymbol{K}_i)_n, (\boldsymbol{V}_i)_n)$. Then we obtain a $\gamma \in \operatorname{Hom}_R((\boldsymbol{V}_i)_n, W)$ such that $\beta = \gamma\delta$. It now follows from the commutative diagram

$$\begin{tikzcd} \mathrm{Z}^t_n(\boldsymbol{K}_i) \arrow[r, "\varepsilon"] \arrow[d, "\varphi"] & (\boldsymbol{K}_i)_n \arrow[d, "\delta"] \\ \mathrm{Z}^t_n(\boldsymbol{V}_i) \arrow[r, "e"] & (\boldsymbol{V}_i)_n \end{tikzcd}$$

that $\gamma e \in \operatorname{Hom}_R(\mathrm{Z}^t_n(\boldsymbol{V}_i), W)$ and $\alpha = \beta\varepsilon = \gamma\delta\varepsilon = \gamma e\varphi = \varphi^*(\gamma e)$. This finishes the proof of Claim 2.

Now, the proof of the necessity is complete.

(2) $\Rightarrow$ (1) Let $n \in \mathbb{Z}$. Take a $1 \leqslant t \leqslant N-1$, the exactness of $\boldsymbol{X}$ provides an exact sequence

$$0 \longrightarrow \mathrm{Z}^t_n(\boldsymbol{X}) \longrightarrow X_n \longrightarrow \mathrm{Z}^{N-t}_{n-t}(\boldsymbol{X}) \longrightarrow 0.$$

Since $\mathcal{G}(\mathcal{VW})$ is closed under extensions by [31, Corollary 3.8], the displayed sequence implies that $X_n \in \mathcal{G}(\mathcal{VW})$. To prove that $\boldsymbol{X}$ is strongly $\mathcal{VW}$-Gorenstein it is thus, by Theorem 3.8, enough to show that $\operatorname{Hom}_R(V, \boldsymbol{X}), \operatorname{Hom}_R(\boldsymbol{X}, W)$ are exact for any $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Let $V \in \mathcal{V}$ and $n \in \mathbb{Z}$. Notice that $\Sigma\mathrm{D}^1_0(V) = \mathrm{D}^{N-1}_{N-1}(V)$ and as $\boldsymbol{X}$ is exact, Lemma 2.2 and [29, Lemma 2.2(vii)] combine with [31, Proposition 3.5] to yield

$$\begin{aligned} \mathrm{H}^1_n\left(\operatorname{Hom}_R(V, \Theta^n\boldsymbol{X})\right) &\cong \operatorname{Ext}^1_{\mathcal{C}_N(R)}\left(\Sigma\mathrm{D}^1_0(V), \boldsymbol{X}\right) \\ &\cong \operatorname{Ext}^1_{\mathcal{C}_N(R)}\left(\mathrm{D}^{N-1}_{N-1}(V), \boldsymbol{X}\right) \\ &\cong \operatorname{Ext}^1_R\left(V, \mathrm{Z}^{N-1}_{N-1}(\boldsymbol{X})\right) = 0. \end{aligned}$$

Thus, Lemma 2.1 implies that $\operatorname{Hom}_R(V, \boldsymbol{X})$ is exact. Given a $W \in \mathcal{W}$ and an $n \in \mathbb{Z}$. As $\boldsymbol{X}$ is exact, it follows from [10, Proposition 3.2(ii)] that $\Sigma\boldsymbol{X}$ is also exact. This yields

$$\begin{aligned} \mathrm{H}^1_n\left(\operatorname{Hom}_R(\boldsymbol{X}, \Theta^n\mathrm{D}^1_0(W))\right) &\cong \operatorname{Ext}^1_{\mathcal{C}_N(R)}\left(\Sigma\boldsymbol{X}, \mathrm{D}^1_0(W)\right) \\ &\cong \operatorname{Ext}^1_R\left((\Sigma\boldsymbol{X})_0/\mathrm{B}^1_0(\Sigma\boldsymbol{X}), W\right) \\ &\cong \operatorname{Ext}^1_R\left(\mathrm{Z}^1_{1-N}(\Sigma\boldsymbol{X}), W\right). \end{aligned}$$

In this sequence, the first isomorphism comes from Lemma 2.2 and [31, Proposition 3.5]. The second isomorphism is due to [29, Lemma 2.2(viii)] and the third isomorphism is an immediate consequence of the exactness of $\boldsymbol{X}$. The proof [15, Lemma 3.5] shows that

$$\mathrm{Z}^1_{1-N}(\Sigma\boldsymbol{X}) = \mathrm{B}^1_{-N}(\boldsymbol{X}) \oplus \mathrm{B}^2_{-N-1}(\boldsymbol{X}) \oplus \cdots \oplus \mathrm{B}^{N-1}_{2-2N}(\boldsymbol{X}).$$

Since $\boldsymbol{X}$ is N-exact, we conclude that

$$\mathrm{Z}^1_{1-N}(\Sigma\boldsymbol{X}) = \mathrm{Z}^{N-1}_{-N}(\boldsymbol{X}) \oplus \mathrm{Z}^{N-2}_{-N-1}(\boldsymbol{X}) \oplus \cdots \oplus \mathrm{Z}^1_{2-2N}(\boldsymbol{X}),$$

and so $\mathrm{Z}^1_{1-N}(\Sigma\boldsymbol{X}) \in \mathcal{G}(\mathcal{V}\mathcal{W})$ by assumption. Thus, $\mathrm{Ext}^1_R(\mathrm{Z}^1_{1-N}(\Sigma\boldsymbol{X}), W) = 0$ by [31, Proposition 3.5]. From the isomorphism above we deduce that

$$\mathrm{H}^1_n\left(\mathrm{Hom}_R(\boldsymbol{X}, \Theta^n\mathrm{D}^1_0(W))\right) = 0,$$

which yields that $\mathrm{Hom}_R(\boldsymbol{X}, W)$ is N-exact. This completes the proof. □

Finally, we outline the consequences of Theorems 3.8 and 3.9 for the examples of Remark 3.2.

Corollary 3.10. *Let $\boldsymbol{X}$ be an N-complex. Then $\boldsymbol{X}$ is strongly Gorenstein projective if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}^t_n(\boldsymbol{X})$ is a Gorenstein projective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N-1$.*

Proof. Take $\mathcal{V} = \mathcal{W} = \mathcal{P}(R)$. Then $\mathcal{V}\mathcal{W}$-Gorenstein R-modules are exactly Gorenstein projective R-modules, strongly $\mathcal{V}\mathcal{W}$-Gorenstein N-complexes are the so called strongly Gorenstein projective N-complexes by Remark 3.2. If $\boldsymbol{X}$ is a strongly Gorenstein projective N-complex, then it follows from Theorem 3.8 that $\boldsymbol{X} \cong \mathrm{Hom}_R(R, \boldsymbol{X})$ is exact. Now, apply Theorem 3.9. □

Corollary 3.11. ([19, Theorem 1.1]) *Let $\boldsymbol{X}$ be a complex. Then $\boldsymbol{X}$ is strongly Gorenstein projective if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n(\boldsymbol{X})$ is a Gorenstein projective module for each $n \in \mathbb{Z}$.*

Proof. This follows from [30, Theorem 2.2] and Theorem 3.8, Corollary 3.10 by taking $N = 2$. □

The proofs of the next two results are dual to the previous two.

Corollary 3.12. *Let $\boldsymbol{X}$ be an N-complex. Then $\boldsymbol{X}$ is strongly Gorenstein injective if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}^t_n(\boldsymbol{X})$ is a Gorenstein injective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N-1$.*

Corollary 3.13. ([13, Proposition 4.6]) *Let $\boldsymbol{X}$ be a complex. Then $\boldsymbol{X}$ is strongly Gorenstein injective if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n(\boldsymbol{X})$ is a Gorenstein injective module for each $n \in \mathbb{Z}$.*

Corollary 3.14. *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{X}$ an N-complex. Then $\boldsymbol{X}$ is strongly G_C-projective if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n^t(\boldsymbol{X})$ is a G_C-projective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N-1$.*

Proof. Take $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{P}_C(R)$. Then $\mathcal{VW}$-Gorenstein R-modules are precisely G_C-projective R-modules, while strongly $\mathcal{VW}$-Gorenstein N-complexes are the so called strongly G_C-projective N-complexes by Remark 3.2. From [24, Proposition 2.6] we conclude that projective R-modules and C-projective R-modules are G_C-projective R-modules. The subcategory $\mathcal{P}_C(R)$ is self-orthogonal by [5, Remark 2.3]. Assume that $\boldsymbol{X}$ is strongly G_C-projective, then Theorem 3.8 yields that $\boldsymbol{X} \cong \mathrm{Hom}_R(R, \boldsymbol{X})$ is an exact N-complex. The result now follows from Theorem 3.9. □

Set $N = 2$ in Corollary 3.14, one gets:

Corollary 3.15. *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{X}$ an R-complex. Then $\boldsymbol{X}$ is strongly G_C-projective if and only if $\boldsymbol{X}$ is an exact complex and $\mathrm{Z}_n(\boldsymbol{X})$ is a G_C-projective R-module for each $n \in \mathbb{Z}$.*

Dually, we have the following result.

Corollary 3.16. *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{X}$ an N-complex. Then $\boldsymbol{X}$ is strongly G_C-injective if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n^t(\boldsymbol{X})$ is a G_C-injective R-module for any $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N-1$.*

It follows from [9, Lemma 6.1, Theorems 2 and 6.1] that

$$\mathcal{A}_C(R) = \mathcal{G}(\mathcal{P}(R)\mathcal{I}_C(R)), \;\; \mathcal{B}_C(R) = \mathcal{G}(\mathcal{P}_C(R)\mathcal{I}(R)).$$

Note that $\mathcal{P}(R), \mathcal{I}_C(R) \subseteq \mathcal{A}_C(R)$ and $\mathcal{P}_C(R), \mathcal{I}(R) \subseteq \mathcal{B}_C(R)$ by [9, Lemma 4.1 and Corollary 6.1]. As another application of Theorem 3.9, we have the following result.

Corollary 3.17. *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{X}$ an N-complex. Then the following statements hold:*

(1) *$\boldsymbol{X}$ is a strongly $\mathcal{P}(R)\mathcal{I}_C(R)$-Gorenstein N-complex if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n^t(\boldsymbol{X}) \in \mathcal{A}_C(R)$ for any $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N$.*
(2) *$\boldsymbol{X}$ is a strongly $\mathcal{P}_C(R)\mathcal{I}(R)$-Gorenstein N-complex if and only if $\boldsymbol{X}$ is exact and $\mathrm{Z}_n^t(\boldsymbol{X}) \in \mathcal{B}_C(R)$ for any $n \in \mathbb{Z}$ and $t = 1, 2, \ldots, N$.*

In particular, set $N = 2$, we have:

Corollary 3.18. *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{X}$ an R-complex.*

(1) $\boldsymbol{X}$ *is a strongly* $\mathcal{P}(R)\mathcal{I}_C(R)$*-Gorenstein complex if and only if* $\boldsymbol{X}$ *is exact and* $\mathrm{Z}_n(\boldsymbol{X}) \in \mathcal{A}_C(R)$ *for any* $n \in \mathbb{Z}$.

(2) (2) $\boldsymbol{X}$ *is a strongly* $\mathcal{P}_C(R)\mathcal{I}(R)$*-Gorenstein complex if and only if* $\boldsymbol{X}$ *is exact and* $\mathrm{Z}_n(\boldsymbol{X}) \in \mathcal{B}_C(R)$ *for each* $n \in \mathbb{Z}$.

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

Disclosure statement. The authors report there are no competing interests to declare.

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