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On a Fourth Order Boundary Value Problem with Periodic and Transmission Conditions

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Highlights:

- Asymtotic expression of eigenvalues
- Asymtotic expression of eigenfunctions
- Rouche methods

Keywords:

- Fourth order problem
- Periodic boundary conditions
- Transmission conditions

ABSTRACT:

This study investigates the asymptotic expressions of eigenvalues and eigenfunctions for a fourth-order boundary value problem subject to periodic boundary conditions. It is also examined in the problem with transmission boundary conditions at zero. At $t = 0$, one of the transmission boundary conditions have jump discontinuity. Firstly, asymptotic formulas of fundamental solutions are found. The asymptotic formulas of the eigenvalues are computed by the aid of Rouche method. Finally corresponding eigenfunctions to these eigenvalues are presented.

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INTRODUCTION

Let us consider a fourth-order differential operator

$$L(u)=\begin{cases}L_1(u), t\in(-1,0)\\ L_2(u), t\in(0,1)\end{cases} \quad (1)$$

generated by differential equation

$$L_1(u_1)=u_1^{(4)}+p_1(t)u_1, \qquad L_2(u_2)=u_2^{(4)}+p_2(t)u_2, \quad (2)$$

where $p_1(x)\in C^4[-1,0)$and $p_2(x)\in C^4(0,1]$ complex valued functions. Periodic boundary conditions are considered for the boundary conditions of the operator at $t=\mp 1$:

$$U_m(u)\coloneqq u^{(m)}(-1)-u^{(m)}(1)=0, \quad m=0,1,2,3 \quad (3)$$

with transmission boundary conditions at $x=0$

$$T_m(u)\coloneqq u^{(m)}(0^-)-u^{(m)}(0^+)=0, \quad m=0,1,2 \quad (4)$$

$$T_3(u)\coloneqq u^{(3)}(0^-)-\alpha u^{(3)}(0^+)=0, \quad m=3. \quad (5)$$

where $\alpha \in \mathbb{R}-\{-1,1\}$.

Fourth-order differential equations are widely encountered in numerous fields of applied mathematics and physics such as fluid mechanical, elasticity problem and structure deformations. (Korman, 1989; Graef et all, 2003; Amster and Mariani, 2007). The study of positive solutions for fourth-order boundary value problems has been investigated in various works, including those by (Gupta, 1988; Agarwal, 1989; Yao, 2004; Li and Wang, 2023).

The linear differential operator of degree n with transmission boundary conditions is first studied by (Muravei, 1967). In (Cabri, 2019), the focus is on second-order Sturm Lioville problem with periodic and transmission conditions, where the parameter α takes on the values 1 and -1. In the case $\alpha\neq\pm1$, It is studied by (Cabri and Mamedov, 2020; Cabri and Mamedov, 2020). The periodic (antiperiodic) and transmission boundary conditions for the fourth order problem are analyzed by (Cabri, 2022). Without transmission conditions (Menken, 2010) analyzes the asymptotic properties of the problem. The case where the boundary conditions are regular but not strongly regular was studied in the work of (Kerimov and Kaya, 2013).

When $\alpha=1$, problem (1)-(5) are analyzed by (Cabri, 2022). This work aims to investigate the spectral properties of problem (1) subject to both periodic boundary conditions (3) and transmission boundary conditions (4)-(5). Condition (5) indicates a jump discontinuity in the third derivative.

In this study, to simplify the analysis, it is assumed that

$$\int_{-1}^{0} p_1(t)dt=0, \int_{0}^{1} p_2(t)dt=0. \quad (7)$$

MATERIALS AND METHODS

By (Naimark, 1967) it is well known that (2) have four linearly distinct solutions $u_r(t;s),(r=1,2\dots 4)$ within the intervals $(-1,0)$ and $(0,1)$ by

$$u_r(t,s)=e^{w_j st}\left(\sum_{m=0}^{4}\frac{v_{m,r}(t)}{(2is)^m}\right)+O(\frac{1}{s^5}) \quad (8)$$

where $v_{m,r}(x)$ satisfy following recursive diffferential equation

$$4v'_{m,r}(t)+\ 6w_r^3v''_{m-1,r}\ (t)+\ 4w_r^2\ v_{m-2,r}(t)+\ w_r v_{m-3,r}^{(4)}\ (t)+w_r p(t)v_{m-3,r}(t)=0$$

where ω_r are roots of unity and $s^4=\lambda$.

Utilizing Equation (8) and the results from (Menken, 2010), the functions $u_{m,r}(x)$ in both intervals are expressed as follows. Within interval $(-1,0)$, $v^1_{m,r}(t)$ functions of $u^1_r(t,s)$ become

$$v^1_{0,r}(x)=1,\qquad v^1_{1,r}(x)=v^1_{2,r}(t)=0,\ v^1_{3,r}(x)=-\frac{1}{\omega_n^3}\int_0^{-1}p_1(z)dz$$
$$v^1_{4,r}(t)=\frac{3}{8}\big(p_1(t)-p_1(0^-)\big) \tag{9}$$

Within the interval $(0,1)$, $v^2_{m,n}(x)$ functions of $u^2_n(t,s)$ become

$$v^2_{0,r}(t)=1, v_{1,r}(t)=v^2_{2,r}(t)=0, v^2_{3,r}(t)=-\frac{1}{\omega_n^3}\int_0^{1}p_2(z)dz$$
$$v^2_{4,r}(t)=\frac{3}{8}\big(p_2(t)-p_2(0^+)\big) \tag{10}$$

Within the interval $(-1,0)$, $v^{(k)}_{m,r}(t)$ of $u^1_r(t,s)$ obtained by

$$v^{(k)}_{0,r}(t)=1,\qquad v^{(k)}_{1,r}(t)=v^{(k)}_{2,r}(t)=0,\qquad v^{(k)}_{(3,r)}(t)=-\frac{1}{\omega_n^3}\int_0^{-1}p_1(z)dz$$
$$u^{(1)}_{4,r}(t)=\frac{1}{8}p_1(t)-\frac{3}{8}p_1(0^-),\ u^{(2)}_{4,r}(t)=-\frac{1}{8}p_1(t)-\frac{3}{8}p_1(0^-), \tag{11}$$
$$u^{(3)}_{4,r}(t)=-\frac{3}{8}p_1(t)-\frac{3}{8}p_1(0^-)$$

Within the interval $(0,1)$, $v^{(k)}_{m,n}(t)$ of $u^2_r(t,s)$ are found by

$$v^{(k)}_{0,r}(t)=1,\qquad v^{(k)}_{1,r}(t)=v^{(k)}_{2,r}(t)=0,\ v^{(k)}_{(3,n)}(t)=-\frac{1}{\omega_n^3}\int_0^{1}p_2(z)dz$$
$$v^{(1)}_{4,r}(t)=\frac{1}{8}p_2(t)-\frac{3}{8}p_2(+0),\qquad v^{(2)}_{4,r}(t)=-\frac{1}{8}p_2(0^+)-\frac{3}{8}p_2(0^+) \tag{12}$$
$$v^{(3)}_{4,r}(t)=-\frac{3}{8}p_2(0^+)-\frac{3}{8}p_2(0^+).$$

RESULTS AND DISCUSSION

Theorem 1: Let $p_1(t)\in C^4[-1,0)$and $p_2(t)\in C^4(0,1]$ complex valued functions. Then, the eigenvalue problem (1)-(5) has two infinite sequences of eigenvalues, $\lambda_{k,1}$ and $\lambda_{k,2}$ $(k=M,M+1,\dots)$ where M is a large integer, and these sequences have the following expressions:

$$\lambda_{k,1}=(k\pi)^4+\frac{\pi^3(1-\alpha)(b+c)}{2i(1+\alpha)k}+O\left(\frac{1}{k^2}\right),$$

$$\lambda_{k,2}=(k\pi)^4-\frac{\pi^3(1-a)(b+c)}{2i(1+\alpha)k}+O\left(\frac{1}{k^2}\right).$$

where $b=p_1(-1)-p_2(1), c=p_1(0^-)-p_2(0^+)$.

Proof: To begin proving the theorem, we first need to compute the characteristic determinant. It can be derived by the aid of asymptotic expression of the fundamental solution. That is

$$\Delta(s)=\begin{vmatrix} u_{1,1}(-1) & u_{2,1}(-1) & \dots & u_{4,1}(-1) & u_{1,2}(1) & u_{2,2}(1) & \dots & u_{4,2}(1) \\ u_{1,1}^{(1)}(-1) & u_{2,1}^{(1)}(-1) & \dots & u_{4,1}^{(1)}(-1) & u_{1,2}^{(1)}(1) & u_{2,2}^{(1)}(1) & \dots & u_{4,2}^{(1)}(1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{1,1}^{(3)}(-1) & u_{2,1}^{(3)}(-1) & \dots & u_{4,1}^{(3)}(-1) & u_{1,2}^{(1)}(1) & u_{2,2}^{(1)}(1) & \dots & u_{4,2}^{(3)}(1) \\ u_{1,1}(0^-) & u_{2,1}(0^-) & \dots & u_{4,1}(0^-) & u_{1,2}(0^+) & u_{2,2}(0^+) & \dots & u_{4,2}(0^+) \\ u_{1,1}^{(1)}(0^-) & u_{2,1}^{(1)}(0^-) & \dots & u_{4,1}^{(1)}(0^-) & u_{1,2}^{(1)}(0^+) & u_{2,2}^{(1)}(0^+) & \dots & u_{4,2}^{(1)}(0^+) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{1,1}^{(3)}(0^-) & u_{2,1}^{(3)}(0^-) & \dots & u_{4,1}^{(3)}(0^-) & u_{1,2}^{(1)}(0^+) & u_{2,2}^{(1)}(0^+) & \dots & u_{4,2}^{(3)}(0^+) \end{vmatrix}$$

For simplicity, let us represent b and c by $b=p_1(-1)-p_2(1), c=p_1(-0)-p_2(0^+)$. By simplification of chacteristic determinant, we have following equation

$$\frac{-\Delta(s)}{128s^9}=\left(\left(e^{-2is}+e^{2is}\right)\left[1+\frac{(1-\alpha)b}{16(1+\alpha)s^4}-\frac{(27+21\alpha)c}{16(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right)\right]-2\left[1+O\left(\frac{1}{s^5}\right)\right]\right)$$
$$\times\left(\left(e^{-2s}+e^{2s}\right)\left[1+\frac{(1-\alpha)b}{16(1+\alpha)s^4}-\frac{(27+21\alpha)c}{16(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right)\right]\right)$$
$$-2\left[1-\frac{(1-\alpha)b}{8(1+\alpha)s^4}-\frac{(13+11\alpha)c}{8(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right)\right] \tag{13}$$

Multiplying equation by

$$e^{2is}\left[1+\frac{(1-\alpha)b}{16(1+\alpha)s^4}-\frac{(27+21\alpha)c}{16(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right)\right]$$
$$\times e^{2s}\left[1-\frac{(1-\alpha)b}{16(1+\alpha)s^4}-\frac{(27+21\alpha)c}{16(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right)\right], \tag{14}$$

we get

$$\frac{\Delta(s)}{128s^9}=\left[(e^{2s}-A)^2-1+A^2\right]\times\left[\left(e^{2is}-A\right)-1+A^2\right). \tag{15}$$

where

$$A=1-\frac{(1-\alpha)b}{16(1+\alpha)s^4}-\frac{(1-\alpha)c}{16(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right).$$

When $|\ s\ |$ becomes substantially large, roots of $\Delta(s)$ fulfill following conditions

$$e^{2is}=1+\frac{(1-a)(b+c)}{4(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right),$$

$$e^{2is}=1+\frac{(1-a)(b+c)}{4(1+\alpha)s^4}+O\left(\frac{1}{s^5}\right). \tag{16}$$

By applying Rouche's theorem in (16) and expressing as $s=n\pi+\delta_n$ and $s=n\pi i+\delta_n$ asymptotic forms of eigenvalues are found by $\lambda_{n_1},\lambda_{n_2}$.

Theorem 2: For boundary value problem (1)-(5), Asymptotic formulas of eigenfunctions are

$$u_{n_1}(t)=sin(\mathrm{n}\pi t)+O\left(\frac{1}{n}\right),\qquad t\in[-1,0)\cup(0,1],$$
$$u_{n_2}(t)=cos(\mathrm{n}\pi t)+O\left(\frac{1}{n}\right),\qquad t\in[-1,0)\cup(0,1] \tag{17}$$

Proof: If $U_r(u_n)$ and $T_r\left(u_n\left(t, s_{n,1}\right)\right)$ $(r = 0,1,2,3)$ are computed up to order $O(s^{-5})$, hence first component $u_{1,n}^1(x)$ of $u_{n,1}(x)$ determined by

$$u_{k,1}^1(t) = \begin{vmatrix} u_{k,1}(t) & u_{k,2}(t) & \dots & u_{k,4}(t) & 0 & 0 & \dots & 0 \\ u_{1,1}(-1) & u_{2,1}(-1) & \dots & u_{4,1}(-1) & u_{1,2}(1) & u_{2,2}(1) & \dots & u_{4,2}(1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{1,1}^{(3)}(-1) & u_{2,1}^{(3)}(-1) & \dots & u_{4,1}^{(3)}(-1) & u_{1,2}^{(2)}(1) & u_{2,2}^{(2)}(1) & \dots & u_{4,2}^{(2)}(1) \\ u_{1,1}(0^-) & u_{2,1}(0^-) & \dots & u_{4,1}(0^-) & u_{1,2}(0^+) & u_{2,2}(0^+) & \dots & u_{4,2}(0^+) \\ u_{1,1}^{(1)}(0^-) & u_{2,1}^{(1)}(0^-) & \dots & u_{4,1}^{(1)}(0^-) & u_{1,2}^{(1)}(0^+) & u_{2,2}^{(1)}(0^+) & \dots & u_{4,2}^{(1)}(0^+) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{1,1}^{(3)}(0^-) & u_{2,1}^{(3)}(0^-) & \dots & u_{4,1}^{(3)}(0^-) & u_{1,2}^{(1)}(0^+) & u_{2,2}^{(1)}(0^+) & \dots & u_{4,2}^{(3)}(0^+) \end{vmatrix}$$

This determinant yields

$$u_{k,1}^1(t) = \frac{16(b-c)\left(1 - sinhs_{n,1}\right)}{s_{n,1}^5}\left(\frac{e^{is_{n,1}t} - e^{is_{n,1}t}}{2i}\right) + O\left(\frac{1}{s^6}\right) \tag{18}$$

By normalizing the result (18), The eigenfunction corresponding to λ_{n_1} can be written as

$$u_{n,1}^1(t) = sin(\mathrm{n\pi}t) + O\left(\frac{1}{n}\right), t \in [-1,0) \tag{19}$$

In a similar manner, second part $u_{n,1}^2(t)$ of $u_{n,1}(t)$ can be determined by

$$u_{n,1}^2(t) = sin(\mathrm{n\pi}x) + O\left(\frac{1}{n}\right), t \in (0,1] \tag{20}$$

Similarly, first part $u_{(n,2)}^1(t)$ of $u_{n,2}^1(t)$ is found by

$$u_{n,1}^2(t) = \begin{vmatrix} u_{n,1}(t) & u_{n,2}(t) & \dots & u_{n,4}(t) & 0 & 0 & \dots & 0 \\ u_{1,1}^{(1)}(-1) & u_{2,1}^{(1)}(-1) & \dots & u_{4,1}^{(1)}(-1) & u_{1,2}^{(1)}(1) & u_{2,2}^{(1)}(1) & \dots & u_{4,2}^{(1)}(1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{1,1}^{(3)}(-1) & u_{2,1}^{(3)}(-1) & \dots & u_{4,1}^{(3)}(-1) & u_{1,2}^{(1)}(1) & u_{2,2}^{(1)}(1) & \dots & u_{4,2}^{(3)}(1) \\ u_{1,1}(0^-) & u_{2,1}(0^-) & \dots & u_{4,1}(0^-) & u_{1,2}(0^+) & u_{2,2}(0^+) & \dots & u_{4,2}(0^+) \\ u_{1,1}^{(1)}(0^-) & u_{2,1}^{(1)}(0^-) & \dots & u_{4,1}^{(1)}(0^-) & u_{1,2}^{(1)}(0^+) & u_{2,2}^{(1)}(0^+) & \dots & u_{4,2}^{(1)}(0^+) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_{1,1}^{(3)}(0^-) & u_{2,1}^{(3)}(0^-) & \dots & u_{4,1}^{(3)}(0^-) & u_{1,2}^{(1)}(0^+) & u_{2,2}^{(1)}(0^+) & \dots & u_{4,2}^{(3)}(0^+) \end{vmatrix}$$

From this determinant, we can derive

$$u_{n,2}^1(t) = \frac{16(b-c)\left(1 - sinhs_{n,2}\right)}{s_{k,2}^5}\left(\frac{e^{is_{n,2}t} + e^{is_{n,2}t}}{2i}\right) + O\left(\frac{1}{s^6}\right) \tag{21}$$

Hence eigenfunction corresponding to λ_{k_2}can be expressed as

$$u_{n,2}^1(t) = cos(\mathrm{n\pi}t) + O\left(\frac{1}{n}\right), \qquad t \in [-1,0) \tag{22}$$

Second part $u_{n,2}^2(t)$of $u_{n,2}(t)$is obtained by

$$u_{n,2}^2(t) = cos(\mathrm{n\pi}t) + O\left(\frac{1}{n}\right), \qquad t \in (0,1] \tag{23}$$

This completes the proof.

CONCLUSION

This work explores the asymptotic formulations of eigenvalues and eigenfunctions of problem (1)-(5). This approach departs from the framework of strong regularity employed by (Muravei 1967). When α =1, periodic(antiperiodic) boundary conditions and transmission conditions for fourth-order problem are analyzed by (Cabri, 2022). By the using same method, asymptotic formulas of eigen,values and eigenfunctions are provided when $\alpha \in \mathbb{R} - \{-1,1\}$.

Conflict of Interest

The article authors declare that there is no conflict of interest between them

REFERENCES

Agarwal, R. P**.** (1989). On fourth order boundary value problems arising in beam analysis. *Differential and Integral Equations*, 2(1), 91-110.

Amster, P., & Mariani, M. C**.** (2007). Oscillating solutions of a nonlinear fourth order ordinary differential equation. *Journal of Mathematical Analysis and Applications*, 325, 1133-1141.

Cabri, O. (2019). On the Riesz basis property of the root functions of a discontinuous boundary problem. *Mathematical Methods in Applied Sciences*, 6733-6740.

Cabri, O., & Mamedov, Kh. R. (2020). Riesz basisness of root functions of a Sturm-Liouville operator with transmission conditions. *Lobachevskii Journal of Mathematics*, 41(1), 1-6.

Cabri, O., & Mamedov, Kh. R. (2020). On the Riesz basisness of root functions of a Sturm–Liouville operator with transmission conditions. *Lobachevskii Journal of Mathematics*, 41(9), 1784-1790.

Graef, J. R., Qian, C., & Yang, B**.** (2003). A three point boundary value problem for nonlinear fourth order differential equations. *Journal of Mathematical Analysis and Applications*, 287(1), 217-233.

Gupta, C. (1988). Solvability of a fourth order boundary value problem with periodic boundary conditions. *International Journal of Mathematics and Mathematical Sciences*, 11(2), 275-284.

Kerimov, N. B., & Kaya, U. (2013). Spectral asymptotics and basis properties of fourth order differential operators with regular boundary conditions. *Mathematical Methods in the Applied Sciences*, 36. https://doi.org/10.1002/mma.2827.

Korman, P. (1989). A maximum principle for fourth-order ordinary differential equations. *Applied Analysis*, 33, 267-373.

Li, Y., & Wang, D. (2023). An existence result of positive solutions for the bending elastic beam equations. *Symmetry*, 15(2), 405. DOI:10.3390/sym15020405.

Menken, H. (2010). Accurate asymptotic formulas for eigenvalues and eigenfunctions of a boundary value problem of fourth order. *Boundary Value Problems*.

Muravei, L. A. (1967). Riesz bases in L2(−1; 1). *Proceedings of the Steklov Institute of Mathematics*, 91, 113-131.

Naimark, M. A. (1967). *Linear differential operators, Part I*. New York: Frederick Ungar.

Yao, Q. (2004). Positive solutions for eigenvalue problems of fourth-order elastic beam equations. *Applied Mathematics Letters*, 17, 237-243.