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Application of the homotopy perturbation method for weakly singular Volterra integral equations

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Abstract — In this paper, we study a weakly singular Volterra integral equation of the second kind with the kernel $K(x,t) = \left(\frac{t}{x}\right)^{\nu}\frac{1}{t}$, for some $\nu > 0$ and $x \in [0, X]$. The powerful homotopy perturbation method (HPM) is initially applied to find a solution to the integral equation for $\nu > 1$. We then consider the interesting case where $0 < \nu < 1$. Applying the homotopy perturbation method constructed by a convex homotopy or other series-related methods produces unwanted results for this case. In this study, we propose conditions to be imposed to overcome this issue. In addition, for completeness, we investigate all cases where $\nu \in \mathbb{R}$. Some numerical examples are provided to confirm the simplicity and applicability of the applied methods.

Subject Classification (2020): 45H05, 65H20

1. Introduction

Second-kind Volterra integral equations with weakly singular kernels have numerous applications in different branches of science, such as mathematical physics, engineering, electrochemistry, etc. [1–3]. In this study, we consider a weakly singular Volterra integral equation of the form:

$$\phi(x) - \int_0^x K(x,t)\phi(t)\,dt = f(x) \tag{1.1}$$

where $K(x,t) = \left(\frac{t}{x}\right)^{\nu}\frac{1}{t}$ for some $\nu > 0$ and $x \in [0, X]$. Some form of this equation arises from certain heat conduction problems with mixed boundary conditions [4,5]. An important feature distinguishing this equation from other equations is the presence of singularity at $x = 0$ as $\nu > 0$ and at $t = 0$, for all $x > 0$ as $0 < \nu < 1$. Indeed, this feature prevents the application of conventional analytical and numerical methods. This is one reason making this case harder to handle than the other cases.

A rather different perturbation technique has been proposed by He [6]. Unlike many perturbation methods, it does not require a small parameter in the equation. Instead, it combines a basic idea of the perturbation method and the homotopy concept from topology to deform a hard problem into an easy-to-solve problem. Many scientists and engineers have been working on improving and developing this technique further [7–9]. In [10], the author described an analytical technique for

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non-linear problems. This method depends on both the homotopy in topology and the Taylor series. Unlike perturbation techniques, this technique's applicability is based on not requiring small or large parameters in the equation. Some examples are provided to compare the proposed method with existing perturbation techniques. In [11], the author directly applied He's homotopy perturbation method to compute certain integrals. This technique is very simple and quite effective for evaluating certain difficult integrals.

Many classical approaches to calculating certain integrals or obtaining new formulas usually require numerical integration. Fortunately, He's homotopy perturbation method needs only basic differentiation to derive the integration formulas. In [12], the authors proposed a new method to solve a Volterra integral equation with a weakly singular kernel in the reproducing kernel space. By manipulating the equation, they first obtained a new equivalent equation and its solution is a series in the reproducing kernel space. In addition, some numerical examples are worked out to demonstrate the accuracy of the present method. In [13], the authors analyzed the discrete superconvergence properties of spline collocation solutions for some particular Volterra integral equation with the weakly singular kernel. In particular, the attainable convergence orders at the collocation points are examined for certain choices of the collocation parameters. Some examples are proposed to compare the introduced method with some existing perturbation techniques. In [14], the author considered second-kind Volterra equations with weakly singular kernels. As the kernels admit simple forms, they found analytic solution expressions and proved existence, uniqueness, and smoothness properties. In [15], the authors proposed a numerical solution using a combination of wavelets and block pulse functions. In particular, the second Chebyshev wavelets are used to construct the operational matrix for fractional integration. Then, using the disjoint property of block pulse functions, they solved a weakly singular Volterra equation, including Abel's equations. In [16], the authors used orthogonal triangular functions for constructing solutions for weakly singular Volterra integral equations. They utilized some operational matrices to bring the system to a system of algebraic equations. By solving this algebraic system, a numerical solution is obtained. In [17], the authors considered two standard techniques, namely the Adomian decomposition method (ADM) and the variational iteration method (VIM), to solve the Volterra integral equation with a weakly singular kernel in the reproducing kernel space. Both methods provide convergent series solutions for this equation as $\nu > 1$. However, when considering the case where $0 < \nu < 1$, we believe it needs more attention since the convergence of the resulting series is not obvious. We propose a method for how to overcome this issue by imposing some conditions on the small parameter p and ν.

As it is noted in [14] as $\nu \leq 0$, to have a meaningful integration, a solution of (1.1) must, together with its certain derivatives, vanish at $t = 0$. Therefore, this requirement can be used to reduce the case of $\nu \leq 0$ to that of $\nu > 0$. This article aims to investigate all possible choices of ν. As stated in the previous section, the case $\nu \leq 0$ can be reduced to the case $\nu > 0$ under some mild conditions. Thus, we consider only the case where $\nu > 0$ and focus mainly on the case where $0 < \nu < 1$. In addition to the aforesaid studies, interested readers are recommended to read [3, 18–23].

The rest of the paper is organized as follows: In section 2, the homotopy perturbation method is reviewed, and important points that make the equation more useful are pointed out. Section 3 is organized in a way that all possible cases for ν, which means for all real numbers $\mathbb{R}$, are analyzed. The obtained results from this study and the results from the literature for other cases are stated. The prime focus is on the case where $0 < \nu < 1$. In section 4, a pair of numerical examples is investigated thoroughly. The examples are taken from the literature to be able to compare and discuss the results that are obtained using different methods. Section 5 is the final section of the article and it provides a conclusion of the study.

2. Preliminaries

This section provides the following theorem, which contributes to understanding the rest of the paper and is cited in some parts.

Theorem 2.1. [14] Assume that $\nu > 1$. if the function f belongs to $C[0, X]$, then the integral equation

$$\phi(x) + \int_0^x K(x,t)\phi(t)\, dt = f(x), \quad x \in (0, X]$$

where $K(x,t) = \dfrac{1}{\sqrt{\pi}} \dfrac{1}{\sqrt{\ln(x/t)}} \left(\dfrac{t}{x}\right)^{\nu} \dfrac{1}{x}$ has a unique solution $\phi \in C[0, X]$. Moreover, if for an integer $m \geq 1$, $f \in C^m[0, X]$, then $\phi \in C^m[0, X]$.

On the other hand, for any $f \in C[0, X]$, the integral equation

$$\phi(x) - \int_0^x K(x,t)\phi(t)\, dt = f(x), \quad x \in (0, X]$$

where $K(x,t) = \left(\dfrac{t}{x}\right)^{\nu} \dfrac{1}{t}$ has a unique solution $\phi \in C[0, X]$. Furthermore, if for an integer $m \geq 1$, $f \in C^m[0, X]$, then $\phi \in C^m[0, X]$.

For $X > 0$ and m a nonnegative integer, $C^m[0, X]$ represents the space of the real-valued continuous functions whose derivatives of order up to m are continuously extendable to the endpoints $x = 0$ and X. For convenience, $C^0[0, X]$ is denoted by $C[0, X]$.

3. Review of the Homotopy Perturbation Method (HPM)

We plan to reserve this section for reviewing the HPM. This technique was originally introduced by J.J.He [6, 24, 25] and was further investigated by many scientists. We illustrate the basic idea through an integral equation, which is what we needed throughout the article. In most basic terms, the HPM could be described as a combination of the traditional perturbation method and homotopy technique in topology. The basic idea is to successfully deform a hard problem into an easy-to-solve problem. With HPM, this is usually achieved by obtaining a rapidly convergent series at the end of the process. Otherwise, the series representation is used to gain the approximate solutions. This powerful combination has been successfully applied to obtain analytical or numerical solutions for many problems arising from different branches of science [26–29]. To explain the basic idea of the HPM, consider a general integral equation as

$$I(u) = 0 \tag{3.1}$$

where I is an integral operator. Then a convex homotopy with an embedding parameter $p \in [0, 1]$ could be defined by

$$H(v, p) = (1 - p)F(v) + pI(v), \quad p \in [0, 1]$$

where F(v) is a functional operator with a known solution, say v_0. As it can be easily observed that

$$H(v, p) = 0 \tag{3.2}$$

implies

$$H(v, 0) = F(v) \text{ and } H(v, 1) = I(v)$$

Notice that as the embedding parameter monotonically increases from 0 to 1, then the trivial problem (H(v,0)=F(v)=0) deforms to the original problem (I(v)=0) [6, 25]. The parameter p can also be considered as an expanding parameter since it is used to obtain

$$v = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots \tag{3.3}$$

As $p \to 1$, (3.3) becomes an approximate solution of (3.1). That is,

$$u = \lim_{p \to 1} v = u_0 + u_1 + u_2 + u_3 + \dots \tag{3.4}$$

For most of the cases, (3.4) will be a convergent series and the rate of convergence will be based on $I(u)$ [10]. For a more thorough treatment of this method, the reader is referred to [6, 24, 25, 30].

4. Main Section

This section presents for cases.

4.1. Case: $\nu \leq 0$

This case can be reduced to the case $\nu > 0$. In [14], it's noted that to make the integration valid, a solution of (1.1) with its certain derivatives must vanish at $t = 0$. This condition in turn could be used to reduce the case of $v \leq 0$ to the case of $v > 0$. Hence, we will not investigate this case further and focus on the remaining cases.

4.2. Case: $0 < \nu < 1$

In this section, we consider the important case where $0 < \nu < 1$. As stated before, the application of the homotopy perturbation method constructed by a convex homotopy or other series-related methods produces unwanted results (divergent series, etc.). In this study, we propose conditions to be imposed to overcome this issue.

Theorem 4.1.

$$\text{If } 0 < \nu < 1 \text{ and } p = 1 - \sqrt{1 - \nu}, \text{ then } 0 < p < \nu$$

The motivation here is that when constructing the convex homotopy, we impose a condition on the small parameter p so that we overcome the issue we faced before. Notice that it is still true that $0 < p < 1$. To be more precise, we construct the homotopy with p being $1 - \sqrt{1 - \nu}$.

We consider the following the Volterra integral equation with a weakly singular kernel

$$u(x) = f(x) + \int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\, dt, \quad 0 < \nu < 1, \quad x \in [0, X] \tag{4.1}$$

where $f(x) = 1 + x$.

Following the theorem stated above, a homotopy can be readily formed as follows:

$$H(u, p) = (1 - p)F(u) + pL(u) = 0$$

$$(1 - p)\left(u(x) - f(x)\right) + p\left(u(x) - f(x) - \int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\, dt\right) = 0 \tag{4.2}$$

where $p = 1 - \sqrt{1 - \nu}$.

As p traces the path along $1 - \sqrt{1 - \nu}$, $0 < \nu < 1$, (4.2) takes following form

$$\lim_{p \to 0^+} H(u, p) = u(x) - f(x) = 0, \text{(initial approximation)}$$

$$\lim_{p \to 1^-} H(u, p) = u(x) - f(x) - \int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\, dt = 0, \text{(approximate solution)}.$$

This is simplified as

$$u(x) - f(x) - p\int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\,dt = 0 \tag{4.3}$$

If we use the embedding parameter p as an expanding parameter, i.e.,

$$u(x) = v_0 + pv_1 + p^2 v_2 + \dots \tag{4.4}$$

we can replace $u(x)$ with (4.4) and take the limit as

$$\lim_{p\to 1^-} u(x) = v_0 + v_1 + v_2 + \dots .$$

The obtained series converges most of the cases.

Substituting (4.4) into (4.3), we obtain

$$v_0 + pv_1 + p^2 v_2 + \dots - f(x) - p\left(\int_0^x \frac{t^{\nu-1}}{x^\nu}\left(v_0 + pv_1 + p^2 v_2 + \dots\right) dt\right) = 0$$

Combining the like terms and setting them equal to zero yield

$$\begin{aligned}
p^0 &: v_0 = f(x) = 1 + x,\\
p^1 &: v_1 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_0(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}(1+t)\,dt = \frac{1}{\nu} + \frac{1}{\nu+1}x\\
p^2 &: v_2 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_1(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu} + \frac{1}{\nu+1}t\right) dt = \frac{1}{\nu^2} + \frac{1}{(\nu+1)^2}x\\
&\vdots\\
p^{k+1} &: v_{k+1} = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_k(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu^k} + \frac{1}{(\nu+1)^k}t\right) dt = \frac{1}{\nu^{k+1}} + \frac{1}{(\nu+1)^{k+1}}x
\end{aligned}$$

Therefore, the solution with embedding parameter p admits the following form:

$$\begin{aligned}
u(x) &= v_0 + pv_1^2 + p^2 v_2 + \dots,\\
&= 1 + x + p\left(\frac{1}{\nu} + \frac{1}{\nu+1}x\right) + p^2\left(\frac{1}{\nu^2} + \frac{1}{(\nu+1)^2}x\right) + \dots\\
&= \left(1 + \frac{p}{\nu} + \frac{p^2}{\nu^2} + \dots\right) + \left(1 + \frac{p}{\nu+1} + \frac{p^2}{(\nu+1)^2} + \dots\right)x\\
&= \frac{\nu}{\nu - p} + \frac{\nu+1}{\nu+1-p}x
\end{aligned}$$

Note that the last step follows from that geometric series converges since the following inequalities

$$\left|\frac{p}{\nu}\right| < 1 \text{ and } \left|\frac{p}{\nu+1}\right| < 1$$

guaranteed by the theorem given above. Thus, taking the limit as $p \to 1$, we have

$$\lim_{p\to 1^-} u(x) = \frac{\nu}{\nu-1} + \frac{\nu+1}{\nu}x$$

This is indeed the exact solution of (4.1).

4.3. Case: $\nu = 1$

In this case, we consider

$$u(x) = f(x) + \int_0^x \frac{t^{\nu-1}}{x^{\nu}} u(t)\, dt \tag{4.5}$$

becomes

$$u(x) = f(x) + \int_0^x \frac{1}{x} u(t)\, dt$$

for some function f. The question of existence and uniqueness boils down to that of a second kind Volterra integral equation. For the sake of completeness, let us restate the following theorem.

Theorem 4.2. [14] Suppose that $\nu = 1$. Suppose also that $f \in C^1[0, T]$ and $f(0) = 0$. Then the solutions of (4.5)

$$u(x) = \alpha_0 + f(x) + \int_0^x \frac{1}{t} f(t)\, dt$$

where $\alpha_0 \in \mathbb{R}$.

We also want to note that a necessary condition for the existence of a solution to (4.5) is that $f(0) = 0$.

4.4. Case: $\nu > 1$

This is the other main case we consider in this study. We first construct a homotopy and then follow the steps described in HPM. The following example is worked out and it will be seen that HPM works out perfectly and produces exact solutions for many examples.

Example 4.3. [12,17] We first consider finding the solutions of the following Volterra integral equation with a weakly singular kernel

$$u(x) = f(x) + \int_0^x \frac{t^{\nu-1}}{x^{\nu}} u(t)\, dt, \quad \nu > 1, \quad x \in [0, X] \tag{4.6}$$

where $f(x) = 1 + x$.

A homotopy can be readily formed as follows:

$$H(u, p) = (1 - p)F(u) + pL(u) = 0$$

or

$$(1 - p)\left(u(x) - f(x)\right) + p\left(u(x) - f(x) - \int_0^x \frac{t^{\nu-1}}{x^{\nu}} u(t)\, dt\right) = 0$$

This is simplified as

$$u(x) - f(x) - p\int_0^x \frac{t^{\nu-1}}{x^{\nu}} u(t)\, dt = 0 \tag{4.7}$$

If we use the embedding parameter p as an expanding parameter, i.e.,

$$u(x) = v_0 + pv_1 + p^2 v_2 + \ldots \tag{4.8}$$

we can replace $u(x)$ with (4.8) and take the limit as

$$\lim_{p\to 1} u(x) = v_0 + v_1 + v_2 + \ldots .$$

The obtained series converges most of the cases.

Substituting (4.8) into (4.7), we obtain

$$v_0 + pv_1 + p^2 v_2 + \ldots - f(x) - p\left(\int_0^x \frac{t^{\nu-1}}{x^\nu}\left(v_0 + pv_1 + p^2 v_2 + \ldots\right) dt\right) = 0$$

Combining the like terms and setting them equal to zero yield

$$\begin{aligned}
p^0 &: v_0 = f(x) = 1 + x \\
p^1 &: v_1 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_0(t)\, dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}(1+t)\, dt = \frac{1}{\nu} + \frac{1}{\nu+1}x \\
p^2 &: v_2 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_1(t)\, dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu} + \frac{1}{\nu+1}t\right) dt = \frac{1}{\nu^2} + \frac{1}{(\nu+1)^2}x \\
&\vdots \\
p^{k+1} &: v_{k+1} = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_k(t)\, dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu^k} + \frac{1}{(\nu+1)^k}t\right) dt = \frac{1}{\nu^{k+1}} + \frac{1}{(\nu+1)^{k+1}}x
\end{aligned}$$

Therefore, the solution with embedding parameter p admits the following form:

$$\begin{aligned}
u(x) &= v_0 + pv_1^2 + p^2 v_2 + \ldots \\
&= 1 + x + p\left(\frac{1}{\nu} + \frac{1}{\nu+1}x\right) + p^2\left(\frac{1}{\nu^2} + \frac{1}{(\nu+1)^2}x\right) + \ldots \\
&= \left(1 + \frac{p}{\nu} + \frac{p^2}{\nu^2} + \ldots\right) + \left(1 + \frac{p}{\nu+1} + \frac{p^2}{(\nu+1)^2} + \ldots\right) x \\
&= \frac{\nu}{\nu-p} + \frac{\nu+1}{\nu+1-p}x
\end{aligned}$$

Note that the last step follows from that geometric series converges since both

$$\left|\frac{p}{\nu}\right| < 1 \text{ and } \left|\frac{p}{\nu+1}\right| < 1$$

Taking the limit as $p \to 1$, we have

$$\lim_{p\to1} u(x) = \frac{\nu}{\nu-1} + \frac{\nu+1}{\nu}x$$

This is indeed the exact solution of (4.6).

5. Numerical Examples

In this section, we elaborate on examples of various values of ν. To be able to compare the results we choose the examples from the literature.

Example 5.1. [17] We first consider the following Volterra integral equation with a weakly singular kernel

$$u(x) = f(x) + \int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\, dt, \quad x \in [0, X] \tag{5.1}$$

where $f(x) = 1 + \frac{(\nu+1)^2}{\nu(\nu+2)}x^2$.

For the sake of brevity, we start with the case where $\nu > 1$. A homotopy can be formed as follows:

$$H(u,p) = (1-p)F(u) + pL(u) = 0$$

or

$$(1-p)\left(u(x) - f(x)\right) + p\left(u(x) - f(x) - \int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\,dt\right) = 0$$

This is simplified as

$$u(x) - f(x) - p\int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\,dt = 0 \tag{5.2}$$

If we use the embedding parameter p as an expanding parameter, i.e.,

$$u(x) = v_0 + pv_1 + p^2 v_2 + \dots \tag{5.3}$$

we can replace $u(x)$ with (5.3) and take the limit as

$$\lim_{p\to 1} u(x) = v_0 + v_1 + v_2 + \dots$$

The obtained series converges most of the cases.

Substituting (5.3) into (5.2), we obtain

$$v_0 + pv_1 + p^2 v_2 + \dots - f(x) - p\left(\int_0^x \frac{t^{\nu-1}}{x^\nu}\left(v_0 + pv_1 + p^2 v_2 + \dots\right)dt\right) = 0$$

Combining the like terms and setting them equal to zero yield

$$p^0 : v_0 = 1 + \frac{(\nu+1)^2}{\nu(\nu+2)} x^2$$

$$p^1 : v_1 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_0(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(1 + \frac{(\nu+1)^2}{\nu(\nu+2)} t^2\right)dt = \frac{1}{\nu} + \frac{(\nu+1)^2}{\nu(\nu+2)^2} x^2$$

$$p^2 : v_2 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_1(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu} + \frac{(\nu+1)^2}{\nu(\nu+2)^2} t^2\right)dt = \frac{1}{\nu^2} + \frac{(\nu+1)^2}{\nu(\nu+2)^3} x^2$$

$$\vdots$$

$$p^{k+1} : v_{k+1} = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_k(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu^k} + \frac{(\nu+1)^2}{\nu(\nu+2)^{(k+1)}} t^2\right)dt = \frac{1}{\nu^{k+1}} + \frac{(\nu+1)^2}{\nu(\nu+2)^{k+2}} x^2$$

Therefore, the solution with embedding parameter p admits the following form:

$$\begin{aligned} u(x) &= v_0 + pv_1^2 + p^2 v_2 + \dots \\ &= 1 + \frac{(\nu+1)^2}{\nu(\nu+2)} x^2 + p\left(\frac{1}{\nu} + \frac{(\nu+1)^2}{\nu(\nu+2)^2} x^2\right) + p^2\left(\frac{1}{\nu^2} + \frac{(\nu+1)^2}{\nu(\nu+2)^3} x^2\right) + \dots \\ &= \left(1 + \frac{p}{\nu} + \frac{p^2}{\nu^2} + \dots\right) + \frac{(\nu+1)^2}{\nu(\nu+2)}\left(1 + \frac{p}{\nu+2} + \frac{p^2}{(\nu+2)^2} + \dots\right) \\ &= \frac{\nu}{\nu-p} + \frac{(\nu+1)^2}{\nu(\nu+2)} \frac{(\nu+2)}{(\nu-p+2)} x^2 \end{aligned} \tag{5.4}$$

Note that the last step follows from that geometric series converges since both

$$\left|\frac{p}{\nu}\right| < 1 \text{ and } \left|\frac{p}{\nu+2}\right| < 1$$

Taking the limit as $p \to 1$, we have

$$\lim_{p\to 1} u(x) = \frac{\nu}{\nu-1} + \frac{\nu+1}{\nu}x^2$$

This is indeed the exact solution of (5.1).

For the case where $0 < \nu < 1$, we follow the steps proposed in 3.2 Case above.

The algorithm produces the same results up to (5.4). The only issue that one needs to pay special attention is that whether the geometric series that produced as a result is convergent or not. The construction of the homotopy gives a convergent geometric series since the following inequalities

$$\left|\frac{p}{\nu}\right| < 1 \text{ and } \left|\frac{p}{\nu+2}\right| < 1$$

still hold.

Thus as we pass to the limit;

$$u(x) = \lim_{p\to 1^-} \frac{\nu}{\nu-p} + \frac{(\nu+1)^2}{\nu(\nu+2)}\frac{(\nu+2)}{(\nu-p+2)}x^2 = \frac{\nu}{\nu-1} + \frac{\nu+1}{\nu}x^2$$

For the case $\nu = 1$, a necessary condition for existence of a solution fails since $f(0) = 1$.

Example 5.2. [17] We first consider the following Volterra integral equation with a weakly singular kernel

$$u(x) = f(x) + \int_0^x \frac{t^{\nu-1}}{x^\nu} u(t)\,dt, \quad x \in [0, X] \tag{5.5}$$

where $f(x) = x^\alpha(1+x)$ and α is any constant satisfying both $\nu + \alpha > 0$ and $\nu + \alpha - 1 \neq 0$.

A homotopy can be readily formed as follows:

$$H(u,p) = (1-p)F(u) + pL(u) = 0$$

or

$$(1-p)\left(u(x) - f(x)\right) + p\left(u(x) - f(x) - \int_0^x \frac{t^{\nu-1}}{x^\nu}u(t)\,dt\right) = 0$$

This is simplified as

$$u(x) - f(x) - p\int_0^x \frac{t^{\nu-1}}{x^\nu}u(t)\,dt = 0 \tag{5.6}$$

If we use the embedding parameter p as expanding parameter, i.e.,

$$u(x) = v_0 + pv_1 + p^2v_2 + \dots \tag{5.7}$$

we can replace $u(x)$ with (5.7) and take the limit as

$$\lim_{p\to 1} u(x) = v_0 + v_1 + v_2 + \dots$$

The obtained series converges most of the cases.

Substituting (5.7) into (5.6), we obtain

$$v_0 + pv_1 + p^2v_2 + \dots - f(x) - p\left(\int_0^x \frac{t^{\nu-1}}{x^\nu}\left(v_0 + pv_1 + p^2v_2 + \dots\right)dt\right) = 0$$

Combining the like terms and setting them equal to zero yield

$$p^0 : v_0 = x^\alpha(1+x)$$

$$p^1 : v_1 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_0(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(1 + \frac{(\nu+1)^2}{\nu(\nu+2)}t^2\right)dt = \frac{1}{\nu} + \frac{(\nu+1)^2}{\nu(\nu+2)^2}x^2$$

$$p^2 : v_2 = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_1(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu} + \frac{(\nu+1)^2}{\nu(\nu+2)^2}t^2\right)dt = \frac{1}{\nu^2} + \frac{(\nu+1)^2}{\nu(\nu+2)^3}x^2$$

$$\vdots$$

$$p^{k+1} : v_{k+1} = \int_0^x \frac{t^{\nu-1}}{x^\nu} v_k(t)\,dt = \int_0^x \frac{t^{\nu-1}}{x^\nu}\left(\frac{1}{\nu^k} + \frac{(\nu+1)^2}{\nu(\nu+2)^{(k+1)}}t^2\right)dt = \frac{1}{\nu^{k+1}} + \frac{(\nu+1)^2}{\nu(\nu+2)^{k+2}}x^2$$

Therefore, the solution with embedding parameter p admits the following form:

$$\begin{aligned} u(x) &= v_0 + pv_1^2 + p^2v_2 + \dots, \\ &= 1 + \frac{(\nu+1)^2}{\nu(\nu+2)}x^2 + p\left(\frac{1}{\nu} + \frac{(\nu+1)^2}{\nu(\nu+2)^2}x^2\right) + p^2\left(\frac{1}{\nu^2} + \frac{(\nu+1)^2}{\nu(\nu+2)^3}x^2\right) + \dots \\ &= \left(1 + \frac{p}{\nu} + \frac{p^2}{\nu^2} + \dots\right) + \frac{(\nu+1)^2}{\nu(\nu+2)}\left(1 + \frac{p}{\nu+2} + \frac{p^2}{(\nu+2)^2} + \dots\right) \\ &= \frac{\nu}{\nu-p} + \frac{(\nu+1)^2}{\nu(\nu+2)}\frac{(\nu+2)}{(\nu-p+2)}x^2 \end{aligned} \tag{5.8}$$

Note that the last step follows from that geometric series converges since both

$$\left|\frac{p}{\nu}\right| < 1 \text{ and } \left|\frac{p}{\nu+2}\right| < 1$$

Taking the limit as $p \to 1$, we have

$$\lim_{p\to1} u(x) = \frac{\nu}{\nu-1} + \frac{\nu+1}{\nu}x^2$$

This is indeed the exact solution of (5.5).

For the case where $0 < \nu < 1$, we follow similar steps as explained in Case 3.2 above.

$$|\frac{p}{\nu}| < 1 \quad \text{and} \quad |\frac{p}{\nu+2}| < 1$$

still hold.

For the case $\nu = 1$, a necessary condition for existence of a solution fails as $\alpha \leq 0$. i.e., $f(0) \neq 0$.

As $\alpha > 0$, it follows from the Theorem 4.2 that

$$u(x) = \alpha_0 + f(x) + \int_0^x \frac{1}{t} f(t)\,dt$$

$$= \alpha_0 + \frac{\alpha(\alpha+2)x^{\alpha+1} + (\alpha+1)^2x^\alpha}{\alpha(\alpha+1)}$$

where $\alpha_0 \in \mathbb{R}$.

6. Conclusion

Singular Volterra integral equations and in particular weakly singular Volterra integral equations have appeared in many applications from various areas. Unlike weakly singular Volterra equations, numerous methods exist for applying singular Volterra integral equations. In this article. We aim to contribute to filling this gap. We initially use the homotopy perturbation method (HPM) when $\nu > 1$ in the weakly singular Volterra integral equation. We then propose adjusting HPM to overcome an issue raised as $0 < \nu < 1$. For completeness, we explain and describe what to do for the remaining cases for ν (i.e., $\nu \leq 0$ and $\nu = 1$). We elaborate on some examples to show the simplicity and efficiency of the proposed algorithm. Lastly, we want to emphasize that the idea and methodology presented in this article demonstrate that constructing an effective homotopy equation is the most crucial aspect of HPM. For future studies, modifications of HPM can be explored more systematically to develop variations that yield better results than HPM when applied to different types of linear and nonlinear integral equations, including all kinds of Fredholm and Volterra integral equations.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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