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An Application to Fuss-Catalan Numbers

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Abstract
In this paper, it was investigated how many different ways an m-rung stairs can be climbed within certain rules. It was observed that the climbing numbers of the stairs have relations with the Catalan numbers. The combinatorics problem discussed in this article is different from the ones done so far and is related not only to Catalan numbers but also to some Fuss-Catalan numbers. Some results were obtained regarding the climbing numbers. It was observed that with the initial ascent being fixed, the climbing numbers of stairs with $m, m+1, m+2, m+3$, ... rungs, where $m > 1$ is an integer, are related to respectively the some Fuss-Catalan numbers.

Keywords: *Catalan Numbers, Climbing, Fuss-Catalan Numbers, Lattice Path, Stairs*

AMS Subject Classification (2020): *05C38; 06B20*

1. Introduction

Catalan numbers appear in many combinatorics problems [1–5]. Applications of these numbers are used in some engineering fields and health sciences [3, 4, 6]. In this study, it was examined the combinatorics problem of how many different ways an m-rung stairs can be climbed within certain rules. The rules that should be applied while ascent and descending the stairs are as following. The first is that no matter how many rungs we move up, our descent should be at least one rung above the beginning of the previous ascent. The second is no matter how many rungs we move down, our ascent should be at least one rung above the beginning of the previous descent. Our last rule is that when the number of rungs on the stairs is more than 1, the first move should be at least 2 rungs up. It was observed that with the initial ascent being fixed, the climbing numbers of stairs with $m, m+1, m+2, m+3$, ... rungs, where $m > 1$ is an integer, are related to respectively the Fuss-Catalan numbers.

Definition 1.1. ([7, 8]) The generalized Fuss-Catalan numbers are integer sequence defined by

$$F_i(j,k) = \frac{k}{ij+k}\binom{ij+k}{j}, \; k \geq 1, \; j \geq 1, \; i \geq 2. \tag{1.1}$$

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Specially, if we take $i = 2,\ k = 1$ in the formula (1.1), we have Catalan numbers C_n [3, 5, 6, 9, 10] as follows

$$F_2(j,1) = \frac{1}{2j+1}\binom{2j+1}{j} = \frac{1}{j+1}\binom{2j}{j} = C_j. \tag{1.2}$$

The Generalized Fuss-Catalan numbers are also called Raney numbers [8, 11, 12].

2. Problem statement and solution

The problem is to find the number of different climbings of an m-rung stairs within certain rules.

Notation 2.1. Let p^q be the representation of the ascent of a stairs from the p-th rung to the q-th rung.

Notation 2.2. Let $p^q r^s$ be the representation of the ascent of a stairs from the p-th rung to the q-th rung, the descent from the q-th rung to the r-th rung, and finally the ascent from the r-th to the s-th step.

Let's start with some numerical examples.

Example 2.1. Let's find the number of different climbing of a 1-rung stairs.

Solution. The number of climbs is 1.

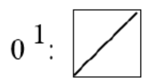

Figure 1. Climbing on a 1-rung stairs

Example 2.2. Let's find the number of different climbing of a 2-rung stairs.

Solution. The number of climbs is 1.

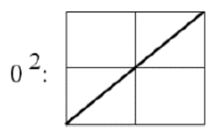

Figure 2. Climbing on a 2-rung stairs

Example 2.3. Let's find the number of different climbing of a 3-rung stairs.

Solution. The number of climbs is 2.

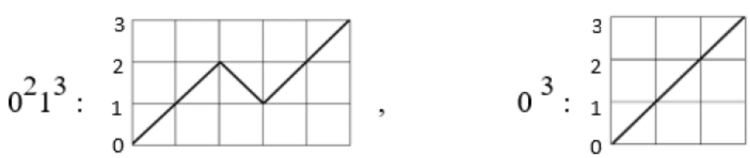

Figure 3. Climbing on a 3-rung stairs

The first one goes up from the 0-th rung to the 2-nd rung. Then go down from the 2-nd rung to the 1-st rung. So, the first climbing 0^21^3 is completed by climbing from the 1-st rung to the 3-rd rung.

The second one 0^3 is completed by climbing from the 0th rung to the 3-rd rung.
In this case, according to the rules of the problem, there is no other climbing position.

Example 2.4. Let's find the number of different climbing of one 4-rung stairs.

Solution. The number of climbs is 5.

$$0^21^32^4,\ \ 0^21^4,\ \ 0^31^4,\ \ 0^32^4,\ \ 0^4.$$

Let's show two particular climbs with figures, the others can be done similarly.

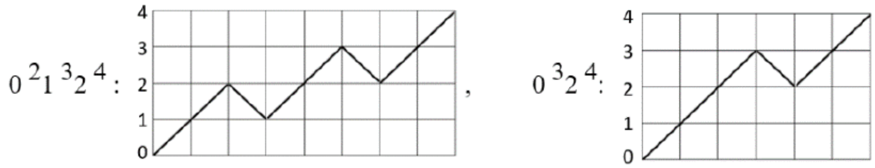

Figure 4. Two of the five climbings on a 4-rung stairs

Example 2.5. Let's find the number of different climbing of one 5-rung stairs.

Solution. The number of climbs is 14.

$$
\begin{array}{lllll}
0^2 1^3 2^4 3^5, & 0^2 1^3 2^5, & 0^2 1^4 2^5, & 0^2 1^4 3^5, & 0^2 1^5, \\
0^3 1^4 2^5, & 0^3 1^4 3^5, & 0^3 1^5, & 0^3 2^4 3^5, & 0^3 2^5, \\
0^4 1^5, & 0^4 2^5, & 0^4 3^5, & & \\
0^5. & & & &
\end{array}
$$

Let's show two particular climbs with figures, the others can be done similarly.

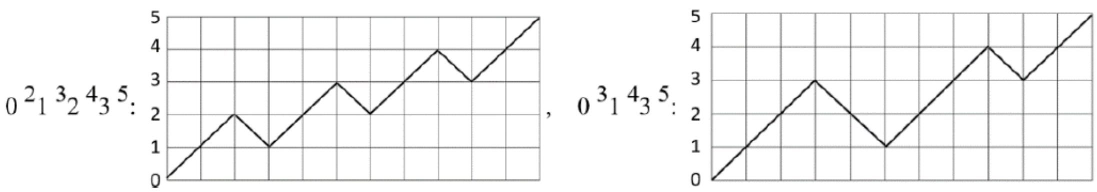

Figure 5. Two of the fourteen climbings on a 5-rung stairs

Example 2.6. Let's find the number of different climbing of a 6-rung stairs using the number of climbing of one 5-rung stairs.

Solution. Let's write the climbings of a 6-rung stairs starting at 1^3 and a 5-rung stairs starting at 0^2, respectively.

$$
\begin{array}{llllll}
1^3 \to & 1^3 2^4 3^5 4^6, & 1^3 2^4 3^6, & 1^3 2^5 3^6, & 1^3 2^5 4^6, & 1^3 2^6 \\
0^2 \to & 0^2 1^3 2^4 3^5, & 0^2 1^3 2^5, & 0^2 1^4 2^5, & 0^2 1^4 3^5, & 0^2 1^5
\end{array}
$$

Note that here the number of different climbings starting with 1^3 in a 6-rung stairs and the number of climbings starting with 0^2 in a 5-rung stairs are equal to each other. Because in both cases, after the start the number of rungs remaining to the top of the stairs are equal. This result can be used also for other situations. We can see all the situations in the Table 1 below.

Starting Position of 5-Rung Stairs	# of climbing of 5-Rung Stairs	# of climbing of 6-Rung Stairs	Starting Position of 6-Rung Stairs
$\mathbf{0^2}$	5	5	$0^2\mathbf{1^3}$
$\mathbf{0^3}$	5	5	$0^2\mathbf{1^4}$
		5	$0^3\mathbf{1^4}$
$\mathbf{0^4}$	3	3	$0^2\mathbf{1^5}$
		3	$0^3\mathbf{1^5}$
		3	$0^4\mathbf{1^5}$

$\mathbf{0^5}$	1	1	$0^2\mathbf{1^6}$
		1	$0^3\mathbf{1^6}$
		1	$0^4\mathbf{1^6}$
$\mathbf{1^3}$	2	2	$0^3\mathbf{2^4}$
$\mathbf{1^4}$	2	2	$0^3\mathbf{2^5}$
		2	$0^4\mathbf{2^5}$
$\mathbf{1^5}$	1	1	$0^3\mathbf{2^6}$
		1	$0^4\mathbf{2^6}$
$\mathbf{2^4}$	1	1	$0^4\mathbf{3^5}$
$\mathbf{2^5}$	1	1	$0^4\mathbf{3^6}$
		1	$0^5\mathbf{3^6}$
$\mathbf{0^5}$	1	1	$0^5\mathbf{1^6}$
$\mathbf{1^5}$	1	1	$0^5\mathbf{2^6}$
$\mathbf{3^5}$	1	1	$0^5\mathbf{4^6}$
		1	0^6
		42	**Total**

Table 1. Obtaining the number of climbing of a 6-rung stairs by using the number of climbing of a 5-rung stairs

Therefore the number of different climbings of a 6-rung stairs is 42.

Remark 2.1. In general, the number of different climbings of a stairs with m-rungs starting from p^q is equal to the number of different climbings of a stairs with $(m+1)$- rungs starting from $(p+1)^{q+1}$.

Similarly, the number of different climbings of a 7-step stairs can be found using the number of different climbings of a 6-step stairs. By continuing like this, the number of different climbings up to a 13-step stairs was found and shown in the Table 2 below. Using the Table 2 we can write the following results.

	Number of Rungs of the Stairs											
Beginings	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\mathbf{0^2}$	1	1	2	5	14	42	132	429	1430	4862	16796	58786
$\mathbf{0^3}$	0	1	2	5	14	42	132	429	1430	4862	16796	58786
$\mathbf{0^4}$	0	0	1	3	9	28	90	297	1001	3432	11934	41990
$\mathbf{0^5}$	0	0	0	1	4	14	48	165	572	2002	7072	25194
$\mathbf{0^6}$	0	0	0	0	1	5	20	75	275	1001	3640	13260
$\mathbf{0^7}$	0	0	0	0	0	1	6	27	110	429	1638	6188
$\mathbf{0^8}$	0	0	0	0	0	0	1	7	35	154	637	2548
$\mathbf{0^9}$	0	0	0	0	0	0	0	1	8	44	208	910
$\mathbf{0^{10}}$	0	0	0	0	0	0	0	0	1	9	54	273
$\mathbf{0^{11}}$	0	0	0	0	0	0	0	0	0	1	10	65
Total	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

Table 2. All climbing numbers from 2-rung stairs to 13-rung stairs

Notation 2.3. Let $N(m, 0^n)$ denotes the number of different climbs of a m-rung stairs with 0^n ascents.

For example, from the Table 2 it can be seen that $N(8, 0^6) = 20,\ N(12, 0^7) = 1638$.
Generally, denote by $N(m, k^n)$ the number of different climbs of a m-rungs stairs starting with k^n. Note that for m, n positive integers,

$$N(m, 0^n) = \begin{cases} 1, & n = m \\ 0, & n > m \end{cases}$$

Corollary 2.1. For m positive integers, $m \geq 3$, $N(m, 0^{m-1}) = m - 2$.

Proof.

$$\underbrace{0^{m-1}1^m,\ 0^{m-1}2^m,\ 0^{m-1}3^m, ..., 0^{m-1}(m-2)^m}_{(m-2)\text{ terms}}$$

Corollary 2.2. For $m, (m \geq 4)$ positive integers, $N(m, 0^{m-2}) = \frac{(m-2)(m-1)}{2} - 1$.

Proof.

$$\underbrace{0^{m-2}1^{m-1}2^m,\ 0^{m-2}1^{m-1}3^m, ..., 0^{m-2}1^{m-1}(m-2)^m,\ 0^{m-2}1^m}_{(m-2)\text{ terms}}$$

$$\underbrace{0^{m-2}2^{m-1}3^m,\ 0^{m-2}2^{m-1}4^m, ...,\ 0^{m-2}2^{m-1}(m-2)^m,\ 0^{m-2}2^m}_{(m-3)\text{ terms}}$$

$$\underbrace{0^{m-2}3^{m-1}4^m,\ 0^{m-2}3^{m-1}5^m, ...,\ 0^{m-2}3^{m-1}(m-2)^m,\ 0^{m-2}3^m}_{(m-4)\text{ terms}}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\underbrace{0^{m-2}(m-3)^{m-1}(m-2)^m,\ 0^{m-2}(m-3)^{m-1}(m-1)^m,\ 0^{m-2}(m-2)^m}_{3\text{ terms}}$$

$$\underbrace{0^{m-2}(m-3)^{m-1}(m-2)^m,\ 0^{m-2}(m-3)^m}_{2\text{ terms}}$$

Therefore,
$N(m, 0^{m-2}) = (m-2) + (m-3) + (m-4) + ... + 3 + 2 = \frac{(m-2)(m-1)}{2} - 1.$

Corollary 2.3. For m positive integers, $m \geq 3$, $N(m, 0^2) = N(m, 0^3)$.

Proof. Since $m \geq 3$, the movements starting with 0^3 and 0^2 are as follows:

(i) Movements starting with 0^3 are in the form of $0^31^4\ldots,\ 0^31^5\ldots,\ 0^31^6\ldots,\ \ldots,\ 0^31^{m-1}\ldots,\ 0^31^m\ldots$ and $0^32^4\ldots,\ 0^32^5\ldots,\ 0^32^6\ldots,\ \ldots,\ 0^32^{m-1}\ldots,\ 0^32^m\ldots$

(ii) Movements starting with 0^2 are in the form of $0^21^32^4\ldots,\ 0^21^32^5\ldots,\ 0^21^32^6\ldots,\ \ldots,\ 0^21^32^{m-1}\ldots,$ $0^21^32^m\ldots$ and $0^21^4\ldots,\ 0^21^5\ldots,\ 0^21^6\ldots,\ \ldots,\ 0^21^{m-1}\ldots,\ 0^21^m\ldots.$

Comparing (i) and (ii), it is not difficult to see that $N(m, 0^31^4) = N(m, 0^21^4),\ N(m, 0^31^5) = N(m, 0^21^5),\ \ldots$ and $N(m, 0^32^4) = N(m, 0^21^32^4),\ N(m, 0^32^5) = N(m, 0^21^32^5), \ldots$.
Therefore we obtain

$$\begin{aligned} N(m, 0^3) =& N(m, 0^31^4) + N(m, 0^31^5) + \ldots + N(m, 0^31^{m-1}) + N(m, 0^31^m) \\ &+ N(m, 0^32^4) + N(m, 0^32^5) + \ldots + N(m, 0^32^{m-1}) + N(m, 0^32^m) \\ =& N(m, 0^21^4) + N(m, 0^21^5) + \ldots + N(m, 0^21^{m-1}) + N(m, 0^21^m) \\ &+ N(m, 0^21^32^4) + N(m, 0^21^32^5) + \ldots + N(m, 0^21^32^{m-1}) + N(m, 0^21^32^m) \\ =& N(m, 0^2) \end{aligned}$$

Hence the equality $N(m, 0^2) = N(m, 0^3)$ is valid for all $m \geq 3$.

Corollary 2.4. The number of possible climbings of a stairs with m-rungs is equal to the number of climbings of a stairs with $(m+1)$-rungs starting from 0^2.

Proof. It is sufficient to show the equality $N(m+1,0^2) = N(m,0^2) + N(m,0^3) + \ldots + N(m,0^{m-1}) + N(m,0^m)$. Using Remark 2.1, we have

$$\begin{aligned} N(m+1,0^2) =& N(m+1,0^2 1^3) + N(m+1,0^2 1^4) + \ldots + N(m+1,0^2 1^m) + N(m+1,0^2 1^{m+1}) \\ &+ N(m+1,1^3) + N(m+1,1^4) + \ldots + N(m+1,1^m) + N(m+1,1^{m+1}) \\ =& N(m,0^2) + N(m,0^3) + \ldots + N(m,0^{m-1}) + N(m,0^m). \end{aligned}$$

Hence $N(m+1,0^2) = N(m,0^2) + N(m,0^3) + \ldots + N(m,0^{m-1}) + N(m,0^m)$.

Corollary 2.5. Let m,n be positive integers with $m \geq 6,\ n \geq 3$, then the equality

$$N(m-1,0^{n-1}) + N(m,0^{n+1}) = N(m,0^n)$$

holds.

Proof. Firstly, let's write number of the climbs of a stairs with m-rungs starting at 0^n:

$$\begin{aligned} N(m,0^n) =& N(m,0^n 1^{n+1} 2^{n+2}) + N(m,0^n 1^{n+1} 2^{n+3}) + \ldots + N(m,0^n 1^{n+1} 2^m) \\ &+ N(m,0^n 1^{n+1} 3^{n+2}) + N(m,0^n 1^{n+1} 3^{n+3}) + \ldots + N(m,0^n 1^{n+1} 3^m) \\ &+ \ldots \\ &+ N(m,0^n 1^{n+1} n^{n+2}) + N(m,0^n 1^{n+1} n^{n+3}) + \ldots + N(m,0^n 1^{n+1} n^m) \\ &+ N(m,0^n 1^{n+2}) + N(m,0^n 1^{n+3}) + \ldots + N(m,0^n 1^m) \\ &+ N(m,0^n 2^{n+1}) + N(m,0^n 2^{n+2}) + \ldots + N(m,0^n 2^m) \\ &+ N(m,0^n 3^{n+1}) + N(m,0^n 3^{n+2}) + \ldots + N(m,0^n 3^m) \\ &+ \ldots \\ &+ N(m,0^n (n-1)^{n+1}) + N(m,0^n (n-1)^{n+2}) + \ldots + N(m,0^n (n-1)^m). \end{aligned}$$

Secondly, let's write number of the climbs of a stairs with m-rungs starting at 0^{n+1}:

$$\begin{aligned} N(m,0^{n+1}) =& N(m,0^{n+1} 1^{n+2}) + N(m,0^{n+1} 1^{n+3}) + \ldots + N(m,0^{n+1} 1^m) \\ &+ N(m,0^{n+1} 2^{n+2}) + N(m,0^{n+1} 2^{n+3}) + \ldots + N(m,0^{n+1} 2^m) \\ &+ N(m,0^{n+1} 3^{n+2}) + N(m,0^{n+1} 3^{n+3}) + \ldots + N(m,0^{n+1} 3^m) \\ &+ \ldots \\ &+ N(m,0^{n+1} n^{n+2}) + N(m,0^{n+1} n^{n+3}) + \ldots + N(m,0^{n+1} n^m). \end{aligned}$$

Finally, let's write number of the climbs of a stairs with $(m-1)$-rungs starting at 0^{n-1}:

$$\begin{aligned} N(m-1,0^{n-1}) =& N(m-1,0^{n-1} 1^n) + N(m-1,0^{n-1} 1^{n+1}) + \ldots + N(m-1,0^{n-1} 1^{m-1}) \\ &+ N(m-1,0^{n-1} 2^n) + N(m-1,0^{n-1} 2^{n+1}) + \ldots + N(m-1,0^{n-1} 2^{m-1}) \\ &+ N(m-1,0^{n-1} 3^n) + N(m-1,0^{n-1} 3^{n+1}) + \ldots + N(m-1,0^{n-1} 3^{m-1}) \\ &+ \ldots \\ &+ N(m-1,0^{n-1} (n-2)^n) + N(m-1,0^{n-1} (n-2)^{n+1}) + \ldots \\ &+ N(m-1,0^{n-1} (n-2)^{m-1}). \end{aligned}$$

Since

$$
\begin{aligned}
&N(m,0^{n}1^{n+1}2^{n+k}) = N(m,0^{n+1}2^{n+k}), \text{ for } k=2,\ 3,\ \ldots\ ,\ (m-n),\\
&N(m,0^{n}1^{n+1}3^{n+k}) = N(m,0^{n+1}3^{n+k}), \text{ for } k=2,\ 3,\ \ldots\ ,\ (m-n),\\
&\ldots\ldots\ldots\ldots\\
&N(m,0^{n}1^{n+1}n^{n+k}) = N(m,0^{n+1}n^{n+k}), \text{ for } k=2,\ 3,\ \ldots\ ,\ (m-n),\\
&N(m,0^{n}1^{n+k}) = N(m,0^{n+1}1^{n+k}), \text{ for } k=2,\ 3,\ \ldots\ ,\ (m-n),\\
&N(m,0^{n}2^{n+1+k}) = N(m-1,0^{n-1}1^{n+k}), \text{ for } k=0,\ 1,\ \ldots\ ,\ (m-n-1),\\
&N(m,0^{n}3^{n+1+k}) = N(m-1,0^{n-1}2^{n+k}), \text{ for } k=0,\ 1,\ \ldots\ ,\ (m-n-1),\\
&\ldots\ldots\ldots\ldots\\
&N(m,0^{n}(n-1)^{n+1+k}) = N(m-1,0^{n-1}(n-2)^{n+k}), \text{ for } k=0,\ 1,\ \ldots\ ,\ (m-n-1),
\end{aligned}
$$

then we have

$$
\begin{aligned}
N(m,0^{n}) =& N(m,0^{n}1^{n+1}2^{n+k}) + N(m,0^{n}1^{n+1}2^{n+3}) + \ldots + N(m,0^{n}1^{n+1}2^{m})\\
&+N(m,0^{n}1^{n+1}3^{n+2}) + N(m,0^{n}1^{n+1}3^{n+3}) + \ldots + N(m,0^{n}1^{n+1}3^{m})\\
&+\ldots\\
&+N(m,0^{n}1^{n+1}n^{n+2}) + N(m,0^{n}1^{n+1}n^{n+3}) + \ldots + N(m,0^{n}1^{n+1}n^{m})\\
&+N(m,0^{n}1^{n+2}) + N(m,0^{n}1^{n+3}) + \ldots + N(m,0^{n}1^{m})\\
&+N(m,0^{n}2^{n+1}) + N(m,0^{n}2^{n+2}) + \ldots + N(m,0^{n}2^{m})\\
&+N(m,0^{n}3^{n+1}) + N(m,0^{n}3^{n+2}) + \ldots + N(m,0^{n}3^{m})\\
&+\ldots\\
&+N(m,0^{n}(n-1)^{n+1}) + N(m,0^{n}(n-1)^{n+2}) + \ldots + N(m,0^{n}(n-1)^{m})\\
=& N(m,0^{n+1}2^{n+2}) + N(m,0^{n+1}2^{n+3}) + \ldots + N(m,0^{n+1}2^{m})\\
&+N(m,0^{n+1}3^{n+2}) + N(m,0^{n+1}3^{n+3}) + \ldots + N(m,0^{n+1}3^{m})\\
&+\ldots\\
&+N(m,0^{n+1}n^{n+2}) + N(m,0^{n+1}n^{n+3}) + \ldots + N(m,0^{n+1}n^{m})\\
&+N(m,0^{n+1}1^{n+2}) + N(m,0^{n+1}1^{n+3}) + \ldots + N(m,0^{n+1}1^{m})\\
&+N(m-1,0^{n-1}1^{n}) + N(m-1,0^{n-1}1^{n+1}) + \ldots + N(m-1,0^{n-1}1^{m-1})\\
&+N(m-1,0^{n-1}2^{n}) + N(m-1,0^{n-1}2^{n+1}) + \ldots + N(m-1,0^{n-1}2^{m-1})\\
&+\ldots\\
&+N(m-1,0^{n-1}(n-2)^{n}) + N(m-1,0^{n-1}(n-2)^{n+1}) + \ldots + N(m-1,0^{n-1}(n-2)^{m-1})\\
=& \{N(m,0^{n+1}1^{n+2}) + N(m,0^{n+1}1^{n+3}) + \ldots + N(m,0^{n+1}1^{m})\\
&+N(m,0^{n+1}2^{n+2}) + N(m,0^{n+1}2^{n+3}) + \ldots + N(m,0^{n+1}2^{m})\\
&+N(m,0^{n+1}3^{n+2}) + N(m,0^{n+1}3^{n+3}) + \ldots + N(m,0^{n+1}3^{m})\\
&+\ldots\\
&+N(m,0^{n+1}n^{n+2}) + N(m,0^{n+1}n^{n+3}) + \ldots + N(m,0^{n+1}n^{m})\}\\
&+\{N(m-1,0^{n-1}1^{n}) + N(m-1,0^{n-1}1^{n+1}) + \ldots + N(m-1,0^{n-1}1^{m-1})\\
&+N(m-1,0^{n-1}2^{n}) + N(m-1,0^{n-1}2^{n+1}) + \ldots + N(m-1,0^{n-1}2^{m-1})\\
&+\ldots
\end{aligned}
$$

$$+N(m-1,0^{n-1}(n-2)^{n})+N(m-1,0^{n-1}(n-2)^{n+1})+\ldots+N(m-1,0^{n-1}(n-2)^{m-1})\}$$
$$=N(m,0^{n+1})+N(m-1,0^{n-1})$$

Therefore the equality $N(m-1,0^{n-1})+N(m,0^{n+1})=N(m,0^{n})$ holds for $m\geq 6,\ n\geq 3$.

3. Relation of the problem to some Fuss-Catalan numbers

Formula (1.2) in Definition 1.1 gives the sequence of Catalan numbers
$(C_j)_{j\geq 0}=\big(1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ \ldots\big)$, which is the sequence $\big(N(m,0^2)\big)_{m\geq 2}$ in the Table 2.
If we take now $i=2,\ k=2$ in the formula (1.1), we have

$$F_2(j,2)=\frac{1}{j+1}\binom{2j+2}{j},\tag{3.1}$$

which are the Catalan numbers $(C_{j+1})_{j\geq 0}=\big(1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ \ldots\big)$.
The sequence $(C_{j+1})_{j\geq 0}$ is the sequence $\big(N(m,0^3)\big)_{m\geq 3}$ in the Table 2.
If we take now $i=2,\ k=3$ in the formula (1.1), we obtain

$$F_2(j,3)=\frac{3}{2j+3}\binom{2j+3}{j},\tag{3.2}$$

so the sequence $\big(F_2(j,3)\big)_{j\geq 0}=\big(1,\ 3,\ 9,\ 28,\ 90,\ 297,\ 1001,\ \ldots\big)$ is the sequence $\big(N(m,0^4)\big)_{m\geq 4}$ in the Table 2.
If we take now $i=2,\ k=4$ in the formula (1.1), we get

$$F_2(j,4)=\frac{2}{j+2}\binom{2j+4}{j},\tag{3.3}$$

that means $\big(F_2(j,4)\big)_{j\geq 0}=\big(1,\ 4,\ 14,\ 48,\ 165,\ 572,\ 2002,\ \ldots\big)$ which is the sequence $\big(N(m,0^5)\big)_{m\geq 5}$ in the Table 2 and so on.
Generally, the sequence $\big(F_2(j,k)\big)_{j\geq 0}$ is equal to the sequence $\big(N(m,0^{k+1})\big)_{m\geq k+1},\ k\geq 1$, in the Table 2, i.e. $\big(F_2(j,k)\big)_{j\geq 0}=\big(N(m,0^{k+1})\big)_{m\geq k+1},\ k\geq 1$.

4. Conclusion

In this study, we give an application of Fuss-Catalan numbers and Catalan numbers. This application was formulated with a problem. By solving this problem some formulas related to Fuss-Catalan numbers are proved. According to that, other applications related to Fuss-Catalan numbers can be done in future studies. We believe that other formulas will be obtained with the help of this problem we presented.

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