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Research Article

Hankel operators between Köthe spaces

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Abstract

This paper is about the operators defined between Köthe spaces whose associated matrix is a Hankel matrix. After demonstrating how these operators are defined, the conditions for their continuity and compactness are given. It is shown that the backward and forward shift operators are mean ergodic and Cesáro bounded by establishing a relationship between the backward and forward shift operators and Hankel and Toeplitz operators on power series spaces.

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1. Introduction

A finite or infinite matrix is called a *Hankel matrix* if its entries are constant along each skew-diagonal, that is, the matrix $(a_{m,n})$ is Hankel if $a_{m_1,n_1} = a_{m_2,n_2}$ whenever $m_1 + n_1 = m_2 + n_2$. Hankel matrices are significant for several reasons and there are many applications in various fields such as signal processing, control theory, and numerical analysis. Hankel matrices also serve as functional mathematical tools with diverse applications in engineering and computer science.

Hankel and Toeplitz operators defined on the Hardy space of the unit disk, $H^2(\mathbb{D})$, can be viewed as operators with infinite Hankel and infinite Toeplitz matrices, respectively, with respect to the standard orthonormal basis of $H^2(\mathbb{D})$. A few years ago, Toeplitz operators, whose "associated" matrix is Toeplitz, were defined for more general topological vector spaces. For instance, in [4], Domański and Jasiczak developed the analogous theory for the space of $\mathcal{A}(\mathbb{R})$ real analytic functions on the real line. In [5], Jasiczak introduced and characterized the class of Toeplitz operators on the Fréchet space of all entire functions $\mathcal{O}(\mathbb{C})$.

In [5], Jasiczak defined a continuous linear operator on $\mathcal{O}(\mathbb{C})$ as a Toeplitz operator if its matrix is a Toeplitz matrix. The matrix of an operator is defined with respect to the Schauder basis $(z^n)_{n\in N_0}$. The space of all entire functions $\mathcal{O}(\mathbb{C})$ is isomorphic to the power series space of infinite type $\Lambda_\infty(n)$. By taking inspiration from Jasiczak paper [5], the author defined the Toeplitz operators on more general power series spaces of finite or infinite type. In this paper, with the same idea, we will define the operators whose associated matrix is an infinite Hankel matrix between Köthe spaces, especially power series spaces. We will construct some conditions for the continuity and compactness of

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these operators. In the final section, it is shown that the backward and forward shift operators are mean ergodic and Cesáro bounded by establishing a relationship between the backward and forward shift operators and the Hankel and Toeplitz operators on power series spaces.

2. Preliminaries

In this section, after establishing terminology and notation, we collect some basic facts and definitions that are needed in the sequel. We will use the standard terminology and notation of [7].

A complete Hausdorff locally convex space E whose topology is defined by a countable fundamental system of seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ is called a Fréchet space. A matrix $(a_{n,k})_{k,n\in\mathbb{N}}$ of non-negative numbers is called a Köthe matrix if it satisfies the following conditions:

1. For each $n\in\mathbb{N}$ there exists a $k\in\mathbb{N}$ with $a_{n,k}>0$.
2. $a_{n,k}\leq a_{n,k+1}$ for each $n,k\in\mathbb{N}$.

For a Köthe matrix $(a_{n,k})_{n,k\in\mathbb{N}}$, the space

$$K(a_{n,k})=\left\{x=(x_n)_{n\in\mathbb{N}}:\ \|x\|_k:=\sum_{n=1}^{\infty}|x_n|a_{n,k}<\infty \quad \text{for all} \quad k\in\mathbb{N}\right\}$$

is called a Köthe space. Every Köthe space is a Fréchet space. From Proposition 27.3 of [7], the dual space of a Köthe space is isomorphic with

$$(K(a_{n,k}))'=\left\{y=(y_n)_{n\in\mathbb{N}}\ \middle|\ \sup_{n\in\mathbb{N}}|y_n a_{n,k}^{-1}|<+\infty \text{ for some } k\in\mathbb{N}\right\}.$$

Grothendieck-Pietsch Criteria (Theorem 28.15 in [7]) states that a Köthe space $K(a_{n,k})$ is nuclear if and only if for every $k\in\mathbb{N}$, there exists a $l>k$ so that

$$\sum_{n=1}^{\infty}\frac{a_{n,k}}{a_{n,l}}<\infty.$$

For a nuclear Köthe space, the system $\|x\|_k=\sup_{n\in\mathbb{N}}|x_n|a_{n,k}$, $k\in\mathbb{N}$ forms an equivalent system of seminorms to the fundamental system of seminorms $\|x\|_k=\sum_{n=1}^{\infty}|x_n|a_{n,k}$, $k\in\mathbb{N}$.

Let $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ be a non-negative increasing sequence with $\lim\limits_{n\to\infty}\alpha_n=\infty$. The power series space of finite type is defined by

$$\Lambda_1(\alpha):=\left\{x=(x_n)_{n\in\mathbb{N}}:\ \|x\|_k:=\sum_{n=1}^{\infty}|x_n|\,e^{-\frac{1}{k}\alpha_n}<\infty \text{ for all } k\in\mathbb{N}\right\}$$

and the power series space of infinite type is defined by

$$\Lambda_\infty(\alpha):=\left\{x=(x_n)_{n\in\mathbb{N}}:\ \|x\|_k:=\sum_{n=1}^{\infty}|x_n|\,e^{k\alpha_n}<\infty \text{ for all } k\in\mathbb{N}\right\}.$$

The power series spaces form an important family of Köthe spaces and they contain the spaces of all holomorphic functions on $\mathbb{C}^d$ and $\mathbb{D}^d$,

$$\mathcal{O}(\mathbb{C}^d)\cong\Lambda_\infty(n^{\frac{1}{d}}) \qquad \text{and} \qquad \mathcal{O}(\mathbb{D}^d)\cong\Lambda_1(n^{\frac{1}{d}})$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ and $d\in\mathbb{N}$.

Let E and F be Fréchet spaces. A linear map $T:E\to F$ is called continuous if for every $k\in\mathbb{N}$ there exists $p\in\mathbb{N}$ and $C_{k,p}>0$ such that

$$\|Tx\|_k\leq C_{k,p}\|x\|_p$$

for all $x\in E$. A linear map $T:E\to F$ is called compact if there exists a neighborhood U of zero in E such that $T(U)$ is precompact in F.

In this paper, we fixed the symbol e_n to denote the sequence

$$(0, 0, \dots, 0, 1, 0, \dots)$$

where 1 is in the nth place and 0 is in the others.

We will use the following Lemma to determine the continuity and compactness of operators defined between Köthe spaces.

Lemma 2.1. *Let $K(a_{n,k})$ and $K(b_{n,k})$ be Köthe spaces.*

a. *$T : K(a_{n,k}) \to K(b_{n,k})$ is a linear continuous operator if and only if for each k there exists m such that*

$$\sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty.$$

b. *If $K(b_{n,k})$ is Montel, then $T : K(a_{n,k}) \to K(b_{n,k})$ is a compact operator if and only if there exists m such that for all k*

$$\sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty.$$

Proof. Lemma 2.1 of [1]. □

A Fréchet space E is Montel if each bounded set in E is relatively compact. Every power series space is Montel, since for every subsequence α_{j_k}, the limit $\lim\limits_{k\to\infty} e^{(r_k - r_m)\alpha_{j_k}}$ is zero with $r_k = -\frac{1}{k}$ in the case of finite-type power series spaces and $r_k = k$ in the case of infinite-type power series spaces for every $k \in \mathbb{N}$ and $m > k$, see Theorem 27.9 of [7].

The next proposition says that the continuity condition is sufficient to ensure that linear operators defined only on the basis elements are well-defined.

Proposition 2.2. *Let $K(a_{n,k})$, $K(b_{n,k})$ be Köthe spaces and $(a_n)_{n\in\mathbb{N}} \in K(b_{n,k})$ be a sequence. Let us define a linear map $T : K(a_{n,k}) \to K(b_{n,k})$ such as*

$$Te_n = a_n \qquad \text{and} \qquad Tx = \sum_{n=1}^{\infty} x_n Te_n$$

for every $x = \sum_{n=1}^{\infty} x_n e_n$ and $n \in \mathbb{N}$. If the continuity condition

$$\forall k \in \mathbb{N} \qquad \exists m \in \mathbb{N} \qquad \sup_{n\in\mathbb{N}} \frac{\|Te_n\|_k}{\|e_n\|_m} < \infty$$

holds, then T is well-defined and continuous operator.

Proof. Proposition 2.2 of [3]. □

In this paper, we will call an operator which is defined between Köthe spaces as a Hankel operator if its matrix is a Hankel matrix defined with respect to the Schauder basis $(e_n)_{n\in\mathbb{N}}$. We will concentrate on Hankel operators defined between power series spaces and determine the conditions that give us the continuity and compactness of these operators.

3. Hankel operator defined between Köthe spaces

Let $\theta = (\theta_n)_{n\in\mathbb{N}_0}$ be any sequence. The Hankel matrix defined by θ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 & \cdots \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 & \cdots \\ \theta_2 & \theta_3 & \theta_4 & \theta_5 & \cdots \\ \theta_3 & \theta_4 & \theta_5 & \theta_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We aim to define an operator $H_\theta : K(a_{n,k}) \to K(b_{n,k})$ by taking $H_\theta e_n$ as the nth column of the above matrix, that is,

$$H_\theta e_n = (\theta_{n-1}, \theta_n, \theta_{n+1}, \cdots) = \sum_{j=1}^{\infty} \theta_{j+n-2} e_j$$

provided that $H_\theta e_n \in K(b_{n,k})$ for every $n = 1, 2, \ldots$. Therefore, for every $x = \sum_{n=1}^{\infty} x_n e_n \in K(a_{n,k})$, the operator H_θ can be written as

$$H_\theta x = \sum_{n=1}^{\infty} x_n H_\theta e_n. \tag{3.1}$$

Actually, we cannot confirm that the operator $H_\theta : K(a_{n,k}) \to K(b_{n,k})$ is well-defined as we cannot guarantee that the series $\sum_{n=1}^{\infty} x_n H_\theta e_n$ converges in $K(b_{n,k})$ for every $x \in K(a_{n,k})$. Proposition 3.1 asserts that H_θ will be well-defined, provided that the condition for its continuity is satisfied. In this section, we will share some conditions under which this operator can be appropriately defined between power series spaces and in those instances, we will analyze its continuity and compactness.

As a direct consequence of Proposition 2.2, we have the following:

Proposition 3.1. *The operator $H_\theta : K(a_{n,k}) \to K(b_{n,k})$ is well-defined and continuous if and only if $H_\theta e_n \in K(b_{n,k})$ for every $n \in \mathbb{N}$ and the continuity condition*

$$\forall k \in \mathbb{N} \qquad \exists m \in \mathbb{N} \qquad \sup_{n\in\mathbb{N}} \frac{\|H_\theta e_n\|_k}{\|e_n\|_m} < \infty$$

holds.

Remark 3.2. When H_θ defines a continuous linear operator, it can be especially said that

$$H_\theta e_1 = (\theta_0, \theta_1, \theta_2, \cdots) = \theta$$

is always in $K(b_{n,k})$.

As mentioned in Remark 3.2 the sequence θ lies in the range space of Hankel operator H_θ. In the proposition below we will demonstrate that the sequence θ should be in the dual space of the domain space of Hankel operator H_θ.

Proposition 3.3. *Let $K(a_{n,k})$, $K(b_{n,k})$ be Köthe spaces. Assume that $H_\theta : K(a_{n,k}) \to K(b_{n,k})$ is a continuous linear operator whose associated matrix is a Hankel matrix given by a sequence θ, then $\theta \in (K(a_{n,k}))'$.*

Proof. Let $H_\theta : K(a_{n,k}) \to K(b_{n,k})$ be a continuous linear operator whose associated matrix is a Hankel matrix given by a sequence θ. Then we have the formula

$$H_\theta e_n = (\theta_{n-1}, \theta_n, \theta_{n+1}, \cdots) = \sum_{j=1}^{\infty} \theta_{j+n-2} e_j.$$

for every $n \in \mathbb{N}$. By Lemma 2.1, for all $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\|H_\theta e_n\|_k = \sum_{j=1}^{\infty} |\theta_{j+n-2}| b_{j,k} \leq C\|e_n\|_m = C a_{n,m} \qquad \forall n \in \mathbb{N}.$$

Then, we have that for all $n, j \in \mathbb{N}$

$$|\theta_{j+n-2}| b_{j,k} \leq C a_{n,m}. \tag{3.2}$$

Since $(b_{n,k})_{n,k\in\mathbb{N}}$ is Köthe matrix, there exists a $k_0 \in \mathbb{N}$ such that $b_{1,k_0} \neq 0$. Hence there exist $m_0 \in \mathbb{N}$ and $C_0 > 0$ such that

$$|\theta_{n-1}| \leq \frac{C_0}{b_{1,k_0}} a_{n,m_0} \qquad \forall n \in \mathbb{N}.$$

This says that $\theta \in (K(a_{n,k}))'$. □

3.1. H_θ between power series spaces

In this subsection, we examine the continuity and compactness of Hankel operators from a Köthe space to a power series space. Initially, we will start with the case that the range space is a power series space of infinite type $\Lambda_\infty(\beta)$.

Proposition 3.4. *Let $K(a_{n,k})$ be a Köthe space and $\theta \in \Lambda_\infty(\beta)$. If the condition*

$$\exists m_0 \in \mathbb{N},\ C > 0 \qquad a_{n,m_0} \geq C \qquad \forall n \in \mathbb{N} \tag{3.3}$$

holds, then $H_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is well-defined, continuous and compact.

Proof. Let us assume that $\theta \in \Lambda_\infty(\beta)$ and the condition (3.3) holds. This gives us that for every $k \in \mathbb{N}$ there exists a $D > 0$ such that

$$\begin{aligned}\|H_\theta e_n\|_k = \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{k\beta_j} &\leq \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{k\beta_{j+n-1}} \\ &\leq \|\theta\|_k \leq \frac{1}{C}\|\theta\|_k a_{n,m_0} \leq D a_{n,m_0}\end{aligned}$$

for every $n \in \mathbb{N}$. This says that $H_\theta e_n \in K(b_{n,k})$ for every $n \in \mathbb{N}$ and for every $k \in \mathbb{N}$

$$\sup_{n\in\mathbb{N}} \frac{\|H_\theta e_n\|_k}{\|e_n\|_{m_0}} < \infty.$$

Then H_θ is well-defined and continuous from Proposition 3.1. Since every power series space is a Montel space and m_0 does not depend on k, H_θ is also compact from Lemma 2.1. □

By a direct consequence of Proposition 3.4 we can give the following theorem:

Theorem 3.5. *For every $\theta \in \Lambda_\infty(\beta)$, the Hankel operator H_θ from any infinite type power series space $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$ is continous and compact.*

Now we want to write a weaker condition on the matrix of $K(a_{n,k})$ in Proposition 3.4.

Proposition 3.6. *Let $K(a_{n,k})$ be a Köthe space and $\theta \in \Lambda_\infty(\beta)$. Assume that the following condition holds:*

$$\forall k \in \mathbb{N} \qquad \exists m \in \mathbb{N}, C > 0 \qquad e^{-k\beta_n} \leq C a_{n,m} \qquad \forall n \in \mathbb{N}. \tag{3.4}$$

Then $H_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is well-defined and continuous.

Proof. Let $\theta \in \Lambda_\infty(\beta)$. Since β is increasing, $\max\{\beta_j, \beta_n\} \leq \beta_{j+n-1}$ and $\beta_j + \beta_n \leq 2\beta_{j+n-1}$ for all $j, n \in \mathbb{N}$. By using the condition (3.4), for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned}\|H_\theta e_n\|_k = \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{k\beta_j} &\leq \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{2k\beta_{j+n-1}} e^{k(\beta_j - 2\beta_{j+n-1})} \\ &\leq \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{2k\beta_{j+n-1}} e^{-k\beta_n} = \|\theta\|_k e^{-k\beta_n} \leq C\|\theta\|_k a_{m,n}\end{aligned}$$

and then $H_\theta e_n \in \Lambda_\infty(\beta)$ for every $n \in \mathbb{N}$ and

$$\sup_{n\in\mathbb{N}} \frac{\|H_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$

Proposition 3.1 says that H_θ is well-defined and continuous. □

Proposition 3.7. *Let $K(a_{n,k})$ be a Köthe space and $\theta \in \Lambda_\infty(\beta)$. Assume that the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{-k\beta_n} \leq C a_{n,m} \qquad \forall n \in \mathbb{N}. \tag{3.5}$$

Then $H_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is compact.

Proof. We can use the same idea in Proposition 3.6. Let $\theta \in \Lambda_\infty(\beta)$. Since β is increasing, $\max\{\beta_j, \beta_n\} \leq \beta_{j+n-1}$ and $\beta_j + \beta_n \leq 2\beta_{j+n-1}$ for all $j, n \in \mathbb{N}$. By using the condition (3.5), there exists an $m \in \mathbb{N}$ so that for every $k \in \mathbb{N}$, there exists a $C > 0$ such that

$$\begin{aligned} \|H_\theta e_n\|_k &= \sum_{j=1}^\infty |\theta_{j+n-2}| e^{k\beta_j} \leq \sum_{j=1}^\infty |\theta_{j+n-2}| e^{2k\beta_{j+n-1}} e^{k(\beta_j - 2\beta_{j+n-1})} \\ &\leq \sum_{j=1}^\infty |\theta_{j+n-2}| e^{2k\beta_{j+n-1}} e^{-k\beta_n} = \|\theta\|_k e^{-k\beta_n} \leq C\|\theta\|_k a_{m,n} \end{aligned}$$

and then we have

$$\sup_{n\in\mathbb{N}} \frac{\|H_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$

Lemma 2.1 says that H_θ is compact. □

As a direct consequence of Proposition 3.6 and Proposition 3.7 we can give the following theorem:

Theorem 3.8. *Let β, α be two nonnegative increasing sequences that tend to infinity. Assume that there exist $A, B > 0$ such that*

$$\alpha_n \leq A\beta_n + B \tag{3.6}$$

for all $n \in \mathbb{N}$. Then, for every $\theta \in \Lambda_\infty(\beta)$, the Hankel operator $H_\theta : \Lambda_1(\alpha) \to \Lambda_\infty(\beta)$ is well-defined, continous and compact.

Proof. Let us assume that there exist $A, B > 0$ satisfying $\alpha_n \leq A\beta_n + B$ for all $n \in \mathbb{N}$. Then for all $m, k \in \mathbb{N}$ we write

$$\frac{1}{mA}\alpha_n - \frac{B}{A} \leq \frac{1}{A}\alpha_n - \frac{B}{A} \leq \beta_n \leq k\beta_n$$

and

$$-k\beta_n \leq \frac{B}{A} - \frac{1}{mA}\alpha_n$$

for all $n \in \mathbb{N}$. Then for all $\tilde{m} \in \mathbb{N}$ satisfying $\tilde{m} > mA$ and for all $k \in \mathbb{N}$, there exists a $C > 0$ such that

$$e^{-k\beta_n} \leq Ce^{-\frac{1}{\tilde{m}}\alpha_n}$$

for every $n \in \mathbb{N}$. This says that the conditions in (3.4) and (3.5) are satisfied. From Proposition 3.6 and 3.7, $H_\theta : \Lambda_1(\alpha) \to \Lambda_\infty(\beta)$ is well-defined, continuous and compact. □

Now, we will explore the continuity and compactness of the Hankel operator H_θ, when the range space is a power series space of finite type $\Lambda_1(\beta)$. To this, we require the stability condition on the sequence β. A sequence β is called stable if

$$\sup_{n\in\mathbb{N}} \frac{\beta_{2n}}{\beta_n} < \infty. \tag{3.7}$$

Proposition 3.9. *Let β be a stable sequence, $K(a_{n,k})$ be a Köthe space and $\theta \in \Lambda_1(\beta)$. Assume that the following condition holds:*

$$\forall k \in \mathbb{N} \qquad \exists m \in \mathbb{N}, C > 0 \qquad e^{\frac{1}{k}\beta_n} \leq Ca_{m,n} \qquad \forall n \in \mathbb{N}. \tag{3.8}$$

Then $H_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is well-defined and continuous.

Proof. Let $\theta \in \Lambda_1(\beta)$. Since β is stable, there exists an $M \in \mathbb{N}$, $M > 1$ such that

$$\beta_{2n} \leq M\beta_n \qquad \forall n \in \mathbb{N}.$$

Since β is increasing we have the following: Let $j, n \in \mathbb{N}$. If $j+n-1 = 2t$ or $j+n-1 = 2t-1$ for some $t \in \mathbb{N}$, then

$$\beta_{j+n-1} \leq \beta_{2t} \leq M\beta_t \leq M\max\{\beta_n, \beta_j\} \leq M(\beta_n + \beta_j)$$

since $t \le \max\{n, j\}$ and β is increasing. Therefore, we have

$$\beta_{j+n-1} \le M(\beta_n + \beta_j) \tag{3.9}$$

for all $j, n \in \mathbb{N}$. By using the condition in (3.8), we can write that for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \|H_\theta e_n\|_k &= \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{-\frac{1}{k}\beta_j} \le \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{-\frac{1}{Mk}\beta_{j+n-1}} e^{\frac{1}{k}(\frac{1}{M}\beta_{j+n-1}-\beta_j)} \\ &\le e^{\frac{1}{k}\beta_n} \sum_{j=1}^{\infty} |\theta_{j+n-2}| e^{-\frac{1}{Mk}\beta_{j+n-1}} \le e^{\frac{1}{k}\beta_n} \|\theta\|_{Mk} \le C a_{m,n} \end{aligned}$$

and then $H_\theta e_n \in \Lambda_1(\beta)$ for every $n \in \mathbb{N}$ and

$$\sup_{n \in \mathbb{N}} \frac{\|H_\theta e_n\|_k}{\|e_n\|_m} < \infty.$$

From Proposition 3.1, H_θ is well-defined and continuous. □

By modifying the condition in (3.8), we can establish a condition for the compactness of the operator $H_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$.

Proposition 3.10. *Let β be a stable sequence, $K(a_{n,k})$ be a Köthe space and $\theta \in \Lambda_1(\beta)$. Assume that the following condition holds:*

$$\exists m \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \exists C > 0 \qquad e^{\frac{1}{k}\beta_n} \le C a_{m,n} \qquad \forall n \in \mathbb{N}. \tag{3.10}$$

Then $H_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is compact.

Proof. It follows similar steps to the proof of Proposition 3.9. □

As a consequence of Proposition 3.6 and Proposition 3.7 we can give the following theorem:

Theorem 3.11. *Let β, α be two nonnegative increasing sequences that tend to infinity. Assume that β is stable and there exist $A, B > 0$ such that*

$$\beta_n \le A\alpha_n + B \tag{3.11}$$

for all $n \in \mathbb{N}$. Then, for every $\theta \in \Lambda_1(\beta)$, the Hankel operator $H_\theta : \Lambda_\infty(\alpha) \to \Lambda_1(\beta)$ is well-defined, continous and compact.

Proof. Let assume that there exist $A, B > 0$ satisfying $\beta_n \le A\alpha_n + B$ for all $n \in \mathbb{N}$. Then for all $m, k \in \mathbb{N}$ we write

$$\frac{1}{k}\beta_n \le \beta_n \le A\alpha_n + B \le mA\alpha_n + B$$

for all $n \in \mathbb{N}$. Then for all $\tilde{m} \in \mathbb{N}$ satisfying $\tilde{m} > mA$ and for all $k \in \mathbb{N}$, there exists a $C > 0$ such that

$$e^{\frac{1}{k}\beta_n} \le C e^{\tilde{m}\alpha_n}$$

for every $n \in \mathbb{N}$. This says that the conditions in (3.8) and (3.10) are satisfied. From Proposition 3.9 and 3.10, $H_\theta : \Lambda_1(\alpha) \to \Lambda_\infty(\beta)$ is well-defined, continuous and compact. □

Now we will consider the operators $H_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$ for $\theta \in (\Lambda_1(\alpha))'$.

Proposition 3.12. *Let $\theta \in (\Lambda_1(\alpha))'$. Assume that the following conditions hold:*

$$\forall m \in \mathbb{N} \qquad e^{-\frac{1}{m}\alpha_n} \in K(b_{n,k}), \tag{3.12}$$

$$\forall m \in \mathbb{N} \qquad \exists s \in \mathbb{N}, C > 0 \qquad e^{-\frac{1}{m}\alpha_n} \le C b_{s,n} \qquad \forall n \in \mathbb{N}. \tag{3.13}$$

Then $H_\theta : \Lambda_1(\alpha) \to K(b_{n,k})$ is well-defined and continuous. If $K(b_{n,k})$ is Montel, then H_θ is compact.

Proof. Since α is increasing, $\max\{\alpha_j, \alpha_n\} \leq \alpha_{j+n-1}$ and $\alpha_j + \alpha_n \leq 2\alpha_{j+n-1}$ for all $j, n \in \mathbb{N}$. Let $\theta \in (\Lambda_1(\alpha))'$. Then there exist $m_0 \in \mathbb{N}$ and $C_1 > 0$ such that

$$|\theta_{n-1}| \leq C_1 e^{-\frac{1}{m_0}\alpha_n}$$

for every $n \in \mathbb{N}$. (3.13) gives us that there exist and $s \in \mathbb{N}$ and a $C_2 > 0$ such that

$$e^{-\frac{1}{m_0}\alpha_n} \leq C_2 b_{s,n}$$

for every $n \in \mathbb{N}$. By using these and the conditions (3.12), we can write that for every $k \in \mathbb{N}$ there exists a $C_3 > 0$ such that

$$\begin{aligned} \|H_\theta e_n\|_k &= \sum_{j=1}^{\infty} |\theta_{j+n-2}| b_{n,k} \leq C_1 \sum_{j=1}^{\infty} e^{-\frac{1}{m_0}\alpha_{j+n-1}} b_{k,j} \\ &\leq C_1 e^{-\frac{1}{2m_0}\alpha_n} \sum_{j=1}^{\infty} e^{-\frac{1}{2m_0}\alpha_j} b_{k,j} = C_1 C_2 b_{s,n} \|e^{-\frac{1}{2m_0}\alpha_n}\|_k = C_3 \|e_n\|_s \end{aligned}$$

and then

$$\sup_{n\in\mathbb{N}} \frac{\|H_\theta e_n\|_k}{\|e_n\|_s} < \infty.$$

H_θ is well-defined and continuous by Proposition 2.2. Since s does not depend on k, H_θ is compact provided that $K(b_{n,k})$ is Montel. □

Proposition 3.12 gives us the following result:

Theorem 3.13. *Let $\Lambda_1(\beta)$ be a nuclear power series spaces of finite type. Assume that there exist $A, B > 0$ such that*

$$\beta_n \leq A\alpha_n + B \tag{3.14}$$

for all $n \in \mathbb{N}$. For every $\theta \in (\Lambda_1(\alpha))'$, the Hankel operator $H_\theta : \Lambda_1(\alpha) \to \Lambda_1(\beta)$ is well-defined, continous and compact.

Proof. Let assume that there exist $A, B > 0$ satisfying $\beta_n \leq A\alpha_n + B$ for all $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we write

$$\frac{1}{mA}\beta_n \leq \frac{1}{m}\alpha_n + \frac{B}{mA}$$

for all $n \in \mathbb{N}$. If we choose an $s \in \mathbb{N}$ satisfying $s > mA$, then there exists a $C > 0$ such that

$$e^{-\frac{1}{m}\alpha_n} \leq C e^{-\frac{1}{s}\beta_n}$$

for every $n \in \mathbb{N}$. Then (3.13) is satisfied. Furthermore we have

$$\sup_{n\in\mathbb{N}} e^{-\frac{1}{m}\alpha_n} e^{-\frac{1}{k}\beta_n} < +\infty$$

for every $m, k \in \mathbb{N}$, then $e^{-\frac{1}{m}\alpha_n} \in \Lambda_1(\beta)$ for every $m \in \mathbb{N}$. Proposition 3.12 says that $H_\theta : \Lambda_1(\alpha) \to \Lambda_1(\beta)$ is well-defined, continuous and compact for every $\theta \in (\Lambda_1(\alpha))'$. □

3.2. S-tameness of the family of Hankel operators

A grading on a Fréchet space E consists of a sequence of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ that are increasing, which means that for every $x \in E$, the inequalities $\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \ldots$ hold. This sequence also determines the topology of the space. Every Fréchet space can be given a grading, and a graded Fréchet space is simply a Fréchet space equipped with such a grading. In this paper, we will assume that all Fréchet spaces discussed are graded.

A pair of graded Fréchet spaces (E, F) is said to be tame if there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that for any continuous linear operator $T : E \to F$, there exists an $N \in \mathbb{N}$ and $C > 0$ satisfying $\|Tx\|_n \leq C\|x\|_{\sigma(n)}$ for all $x \in E$ and $n \geq N$. A Fréchet space E is considered tame if the pair (E, E) is tame. The concept of tameness provides a way to control the continuity of operators. Dubinsky and Vogt introduced the tame Fréchet spaces in [2] and used it to identify a basis in complemented subspaces of certain infinite-dimensional power series spaces.

The author focused specifically on a subset of operators rather than considering all operators defined on a Fréchet space and gave the definition of the S-tameness in [3] as follows:

Definition 3.14. Let $S : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. A family of linear continuous operators $\mathcal{A} \subseteq L(E, F)$ is called S-tame if for every operator $T \in \mathcal{A}$ there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that

$$\|Tx\|_k \leq C\|x\|_{S(k)} \qquad \forall x \in E, k \geq k_0.$$

We would like to note that if a family $\mathcal{A}$ of linear, continuous operators is S_1-tame and $S_1(n) \leq S_2(n)$ for sufficiently large $n \in \mathbb{N}$, then it is obvious that the family $\mathcal{A}$ is also S_2-tame.

The author characterized the S-tameness of a family of operators defined by Toeplitz matrices between power series spaces in [3]. Here, we will discuss the S-tameness of a family of operators defined by Hankel matrices between power series spaces.

Firstly we want to emphasize that a family of compact operators is I-compact where $I : \mathbb{N} \to \mathbb{N}$ is the identity. Let us assume that $\mathcal{A}$ is a family of compact operators from $K(a_{k,n})$ to $K(b_{k,n})$. Then for every $T \in \mathcal{A}$, there exists a $m \in \mathbb{N}$ such that for all $k \geq m$ there exists a $C > 0$ such that

$$\|Tx\|_k \leq C\|x\|_m \leq C\|x\|_k$$

for all $x \in K(a_{k,n})$. This means that the family A is I-tame.

Now we want to address the I-tameness of the family of operators defined by Hankel matrices between power series spaces.

1. $H_\theta : \Lambda_\infty(\alpha) \to \Lambda_\infty(\beta)$ is compact for every $\theta \in \Lambda_\infty(\beta)$ by Theorem 3.5. Then the family
$$\mathcal{A} = \{H_\theta : \Lambda_\infty(\alpha) \to \Lambda_\infty(\beta) \mid \theta \in \Lambda_\infty(\beta)\}$$
is I-tame.
2. $H_\theta : \Lambda_1(\alpha) \to \Lambda_\infty(\beta)$ is compact for every β, α satisfying the condition (3.6) by Theorem 3.8. Then, the family
$$\mathcal{B} = \{H_\theta : \Lambda_1(\alpha) \to \Lambda_\infty(\beta) \mid \theta \in \Lambda_\infty(\beta)\}$$
is I-tame.
3. $H_\theta : \Lambda_\infty(\alpha) \to \Lambda_1(\beta)$ is compact for every α and stable β satisfying the condition (3.11) by Theorem 3.11. Then, the family
$$\mathcal{C} = \{H_\theta : \Lambda_\infty(\alpha) \to \Lambda_1(\beta) \mid \theta \in \Lambda_1(\beta)\}$$
is I-tame.
4. $H_\theta : \Lambda_1(\alpha) \to \Lambda_1(\beta)$ is compact for every β, α satisfying the condition (3.14) and $\theta \in (\Lambda_1(\alpha))'$ provided that $\Lambda_1(\beta)$ is a nuclear power series space of finite type by Theorem 3.13. Then, the family
$$\mathcal{D} = \{H_\theta : \Lambda_1(\alpha) \to \Lambda_1(\beta) \mid \theta \in (\Lambda_1(\alpha))'\}$$
is I-tame.

4. The interactions of Toeplitz and Hankel operators with shift operators

In section 3, a Hankel operator $H_\theta : K(a_{n,k}) \to K(b_{n,k})$, which we associated with a Hankel matrix

$$\begin{pmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 & \cdots \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 & \cdots \\ \theta_2 & \theta_3 & \theta_4 & \theta_5 & \cdots \\ \theta_3 & \theta_4 & \theta_5 & \theta_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

corresponding to a sequence $\theta = (\theta_n)_{n\in\mathbb{N}_0}$ was defined as

$$H_\theta e_n = (\theta_{n-1}, \theta_n, \theta_{n+1}, \cdots) = \sum_{j=1}^{\infty} \theta_{j+n-2} e_j$$

for all $n \in \mathbb{N}$. We discussed the necessary conditions for such an operator to be well-defined between power series spaces in section 3.

Similarly, in [3], a Toeplitz operator $\widehat{T}_\theta : K(a_{n,k}) \to K(b_{n,k})$ whose associate matrix is a lower triangular Toeplitz matrix

$$\begin{pmatrix} \theta_0 & 0 & 0 & 0 & \cdots \\ \theta_1 & \theta_0 & 0 & 0 & \cdots \\ \theta_2 & \theta_1 & \theta_0 & 0 & \cdots \\ \theta_3 & \theta_2 & \theta_1 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

corresponding to a sequence $\theta = (\theta_n)_{n\in\mathbb{N}_0}$ was defined as

$$\widehat{T}_\theta e_n = (0, \cdots, 0, \theta_0, \theta_1, \theta_2, \cdots) = \sum_{j=n}^{\infty} \theta_{j-n} e_j$$

for all $n \in \mathbb{N}$. In [3], the following theorems were given.

Theorem 4.1 ([3], Theorem 3.3)**.** *Let $K(a_{n,k})$ be a Köthe space, $\Lambda_1(\beta)$ be a power series space of finite type and assume that θ be a sequence and $s_0 = \min\{t : \theta_t \neq 0\}$. $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_1(\beta)$ is well-defined and continuous if and only if $\theta \in \Lambda_1(\beta)$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad e^{-\frac{1}{k}\beta_{n+s_0}} \leq C a_{n,m} \qquad \forall n \in \mathbb{N}. \tag{4.1}$$

Theorem 4.2 ([3], Theorem 3.6)**.** *Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a stable sequence. $\widehat{T}_\theta : K(a_{n,k}) \to \Lambda_\infty(\beta)$ is well-defined and continuous if and only if $\theta \in \Lambda_\infty(\beta)$ and the following condition holds:*

$$\forall k \in \mathbb{N} \quad \exists m \in \mathbb{N}, C > 0 \qquad e^{k\beta_n} \leq C a_{n,m} \qquad \forall n \in \mathbb{N}. \tag{4.2}$$

As a direct consequence of these theorems, we can write the following corollary:

Corollary 4.3. *Let α be a stable sequence and $\theta \in \Lambda_r(\alpha)$, $r \in \{1, \infty\}$. Then the operator $\widehat{T}_\theta : \Lambda_r(\alpha) \to \Lambda_r(\alpha)$, $r \in 1, \infty$ is well-defined and continuous.*

In [3], you can find the conditions for the compactness and tameness of the operators $\widehat{T}_\theta$, as well as the results obtained regarding operators $\check{T}_\theta$ defined by upper triangular Toeplitz matrices.

In this section, some properties of the shift operators defined between power series spaces will be discussed using the Hankel and Toeplitz operators.

The backward shift operator is defined as

$$B : \Lambda_r(\alpha) \to \Lambda_r(\alpha), \qquad B(\theta) = (\theta_{n+1})_{n\in\mathbb{N}},$$

and the forward shift operator is defined as

$$F : \Lambda_r(\alpha) \to \Lambda_r(\alpha), \qquad F(\theta) = (\theta_{n-1})_{n\in\mathbb{N}}$$

where we assume that $\theta_{-n} = 0$ for all $n \in \mathbb{N}$ and here $r \in \{1, \infty\}$. These operators are well-defined and continuous in the case that α is a weakly-stable exponent sequence, that is, $\limsup_{n\in\mathbb{N}} \frac{\alpha_{n+1}}{\alpha_n} < \infty$. We recommend [6] for more detailed information about the shift operators on Köthe spaces.

We will proceed assuming that the sequence α is stable, but in some cases, this assumption is not necessary. We have the following relation operators B and F with $\widehat{T}_\theta$ and H_θ

$$F^n(\theta) = (0, \cdots, 0, \theta_0, \theta_1, \theta_2, \cdots) = \widehat{T}_\theta(e_{n+1}) \tag{4.3}$$

and

$$B^n(\theta) = (\theta_{n-1}, \theta_n, \theta_{n+1}, \cdots) = H_\theta(e_{n+1}) \tag{4.4}$$

for $n \in \mathbb{N}$.

Definition 4.4. Let T be a continuous linear operator on a Fréchet space E. The n-th Cesáro mean is

$$T^{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m.$$

T is said to be **mean ergodic** if the limits $\lim_{n\to\infty} T^{[n]}x$, $x \in E$, exists in E. T is said to be **Cesáro bounded** if the family $\{T^{[n]} : n \in \mathbb{N}\}$ is an equicontinuous subset of $L(E)$.

If E be a Montel Fréchet space, then T is mean ergodic if and only if T is Cesàro bounded and $\lim_{n\to\infty} \frac{1}{n}T^n x = 0$ for every $x \in E$ by Theorem 2.5 of [6].

By using equations 4.3 and 4.4, we have that

$$\begin{aligned}\lim_{n\to\infty} F^{[n]}(\theta) &= \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} F^m(\theta) = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \widehat{T}_\theta(e_{m+1}) \\ &= \lim_{n\to\infty} \widehat{T}_\theta\Big(\sum_{m=1}^{n} \frac{1}{n} e_m\Big) = \widehat{T}_\theta(0) = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} B^{[n]}(\theta) &= \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} B^m(\theta) = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} H_\theta(e_{m+1}) \\ &= \lim_{n\to\infty} H_\theta\Big(\sum_{m=1}^{n} \frac{1}{n} e_m\Big) = H_\theta(0) = 0\end{aligned}$$

for all $\theta \in \Lambda_r(\alpha)$, $r = 1, \infty$. This means that F and B are mean ergodic and hence F and B are Cesáro bounded for all $\theta \in \Lambda_r(\alpha)$, $r = 1, \infty$.

Proposition 4.5. *Let α be a stable, increasing sequence tending to infinity. Forward shift operator F and Backward shift operator B defined on $\Lambda_r(\alpha)$, $r = 1, \infty$ are mean ergodic and Cesáro bounded.*

We again recommend [6] for the mean ergodicity of weighted shift operators on Köthe spaces.

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