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Universal Orbits: Unveiling the Connection between Chaotic Dynamics, Normal Numbers, and Neurochaos Learning

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ABSTRACT This study explores the realm of chaotic dynamics, Neurochaos Learning (a brain-inspired machine learning paradigm) and Normal numbers, focusing on the introduction of a novel chaotic trajectory termed the *Universal Orbit*. The study investigates the characteristics and generation of universal orbits within two prominent chaotic maps: the Decimal Shift Map and the Gauss Map. It explores the set of points capable of forming such orbits, revealing connections with normal numbers and continued fractions. Points within the interval $(0, 1)$ can produce universal orbits under specific conditions, highlighting the intricate relationship between machine learning, chaotic dynamics and number theory. While not all points forming universal orbits are normal numbers, the trajectory of a normal number may represent a universal orbit (under certain conditions). When employing the universal orbit for feature extraction in Neurochaos Learning, the firing time feature can be interpreted by establishing an upper bound and examining its trend. Future research aims to identify sets of points producing universal orbits under various chaotic maps, intending to enhance the performance of algorithms like the Neurochaos Learning algorithm. This study contributes to advancing our understanding of chaotic systems and their applications in artificial intelligence.

KEYWORDS
Universal orbit
Decimal shift map
Gauss map
Normal numbers
Neurochaos learning
Brain-inspired machine learning

INTRODUCTION

The emergence of data accessibility and the increase in processing capacity, along with the introduction of innovative learning techniques, have resulted in significant advancements in several scientific fields. Nevertheless, the concept of computers acquiring abstract concepts, similar to how humans do, has been around since the 1950s with the development of the first neural networks (Badillo *et al.* 2020). The recently developed *Neurochaos Learning* architectures (Harikrishnan and Nagaraj 2019; Balakrishnan *et al.* 2019) for data classification are based on the concept of *chaos* that has been empirically found within the brain at several spatiotemporal scales (Faure and Korn 2001; Korn and Faure 2003; Khona and Fiete 2022) and seem to play a role in learning of symbols, and to represent thoughts, perceptions, and memory (Tsuda 2015).

Neurochaos learning (or NL) differs from traditional machine learning by integrating chaos and noise, inspired by the chaotic firing of neurons and the constructive role of noise in neuronal models (via Stochastic Resonance (Harikrishnan and Nagaraj 2021), to enable peak performance in classification tasks. NL and its variants have now been demonstrated to exhibit state-of-the-art performance for classification tasks across several datasets (Balakrishnan *et al.* 2019; Harikrishnan and Nagaraj 2020; Sethi *et al.* 2023; Harikrishnan *et al.* 2022). NL has three hyperparameters which need to be tuned and this can take a significant amount of time and computational resources, in order to optimise them for very high accuracies/F1-scores. What has been missing in literature is a deep investigation into the nature of chaotic orbits that enable NL to learn efficiently and demystify its surprisingly peak performance. In order to fill this gap, this study undertakes such an investigation.

Chaos is introduced in the Neurochaos Learning architecture by utilising the 1-dimensional chaotic skew tent map which is a type of Generalized Lüroth Series (GLS). A dynamic system f is said to be chaotic if periodic points are dense, f is transitive and f exhibits sensitive dependence on initial conditions (Devaney 2018). Dynamical systems have a rich and notable history within mathe-

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matics. Mathematical models can be categorised as deterministic or stochastic. A process that consistently produces the same result when repeated identically is referred to as deterministic. Stochastic processes produce varying results with repetition. Numerous deterministic systems display irregular, random-like behaviour (Bau *et al.* 2002). These systems are considered chaotic if they satisfy the three properties (dense periodic points, topological transitivity and sensitive dependence on initial conditions).

American meteorologist E. N. Lorenz discovered that simple differential equations might display chaotic behaviour while utilising a basic digital computer. Lorenz noted that his basic meteorological model, now known as the Lorenz system, displayed significant dependency on initial conditions. For him, this indicated that long-term weather prediction was nearly impossible and highlighted the significance of chaos theory in other scientific disciplines (Devaney 2018). Characteristics of a chaotic system include non-periodic behavior (along with periodic and quasi-periodic behaviours), sensitivity to initial conditions, chaotic motion is difficult or impossible to forecast, motion looks random and non-linear (Biswas *et al.* 2018).

One of the main properties of an orbit generated by a chaotic map $f(x)$, represented as $\{x_0, f(x_0), f^2(x_0), \ldots\}$, is its sensitive dependence on initial condition x_0 (Alligood *et al.* 1998). Decimal Shift Map (Strogatz 2018), Gauss Map (Corless 1992), Logistic Map (Alligood *et al.* 1998) are a few examples of 1-dimensional chaotic maps. Predicting the long-term behaviour of a chaotic system is not possible owing to the sensitive dependence of the chaotic system on its initial conditions (the same reason why long term prediction of weather is impossible). Deterministic chaos is crucial for the typical operation of the brain across various levels of activity. The brain is designed to maintain a consistent and manageable level of internal noise, which is mostly stable, though not completely so. Disorderly behaviour in brain facilitates quick changes in state that are crucial for processing information (Skarda and Freeman 1990).

In Neurochaos Learning architecture (Harikrishnan and Nagaraj 2019), the original input features of datasets are transformed and extracted using the neural trace generated by the skew tent map with an initial point q. Upon receiving a trigger x_{ij} (the input data value also known as *stimulus*), the chaotic map initiates firing and generates a neural trace $\{q, f(q), f^2(q), \ldots\}$. The firing will cease whenever the chaotic trajectory (trace) reaches a predefined epsilon neighbourhood of the input stimulus x_{ij}. Suppose $(x_{11}, x_{12}, \ldots x_{1n}), (x_{21}, x_{22}, \ldots x_{2n}) \ldots (x_{m1}, x_{m2}, \ldots x_{mn})$ be the dataset with m samples and n features. The NL algorithm considers each x_{ij} as the stimulus. Hence corresponding to each x_{ij}, we obtain a chaotic neural trace from which features are extracted for further processing and classification.

The initial neural activity (q) is one of the hyperparameters that is tuned while training/cross-validation. This tuning is done using grid-search and typically requires a significant amount of computing time to arrive at the best q that yields the maximum F1-score. If we can identify an initial trigger q (using mathematical insights and theoretical considerations) that can generate a chaotic neural trace, which approaches any given stimulus within a certain number of iterations, then we can skip the cumbersome and computationally intensive training of q. This study aims to find such initial triggers for the chaotic maps - Decimal Shift Map (a type of GLS map) and the Gauss Map. Gauss map will also act as a shift map for continued fractions. Thus, the ideas and methods developed in this study act as a stepping stone towards finding an initial trigger for a chaotic neural trace that improves the performance of Neurochaos Learning algorithm.

More than a century ago, Emile Borel developed the notion of *normality* (Émile Borel 1909), which formalised the most fundamental type of randomness for real numbers (Bailey and Crandall 2001). When you toss a coin a huge number of times, around half of the tosses will result in heads and the other half in tails. Similar claims concerning the digits in the expansion of a real number are made by normality. Numerous concerns remain unanswered, like whether π (Bailey *et al.* 2012), e, or $\sqrt{2}$ are all normal as well as Borel's conjecture that the irrational algebraic numbers are absolutely normal in any base (Borel 1950; Copeland and Erdös 1946). We will also examine the relationship between chaotic dynamics and normal numbers.

This work presents a novel and distinct chaotic orbit called the *Universal Orbit*, which has the ability to approach the immediate vicinity of any given stimulus (initial value). This research aims to analyse the initial triggers that can produce a universal orbit under the Decimal Shift map (Strogatz 2018) and Gauss map (Corless 1992), focusing specifically on normal numbers and normal continued fractions (Adler *et al.* 1981). We will also examine the specific characteristics of the firing time feature by using the Decimal Shift Map's universal orbit for feature extraction in the Neurochaos Learning Algorithm. This will allow us to determine the maximum firing time in proportion to the amounts of noise present around the stimulus.

UNIVERSAL ORBIT

Chaotic orbits, a hallmark of nonlinear dynamical systems (Devaney 2018), are trajectories that exhibit extreme sensitivity to initial conditions, leading to complex and unpredictable behavior. A fundamental property of these systems is topological transitivity (Alligood *et al.* 1998), which ensures that trajectories are densely interwoven, allowing any region of the phase space to be reached from any other. Within this context, the concept of a **universal orbit** emerges as a novel and significant idea, illustrating the pervasive reach of chaotic behavior. Universal orbits represent trajectories that can come arbitrarily close to any other orbit within the system, showcasing an extraordinary level of interconnectedness. The subsequent definition formalises this innovative concept:

Definition 1. *Let $f : X \to X$ be a chaotic map, where X is a metric space with metric d. An orbit of the map f with an initial point $x^* \in X$, $S^* = \{f^{(0)}(x^*) = x^*, f^{(1)}(x^*) = f(x^*), f^{(2)}(x^*) = f(f(x^*)), \ldots\}$ is said to be* ***universal orbit*** *if for any orbit with an initial point $x_i \in X$ and $\epsilon > 0$, there exist m and n such that $d(f^{(m)}(x^*), f^{(n)}(x_i)) \leq \epsilon$. In short, the orbit of x^* is considered as a universal orbit if for any given $\epsilon > 0$ the orbit of x^* reach the ϵ neighbourhood of x_i in a finite number of iterations.*

Throughout this study, we are defining chaotic maps (Decimal Shift Map and Gauss Map) on the metric space $((0,1), d)$ where $d(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{10^i}$, $s = 0.s_1s_2s_3\ldots$, $t = 0.t_1t_2\cdots \in (0,1)$ (Devaney 2018).

Lemma 1. *Let $s = 0.s_1s_2s_3\ldots$, $t = 0.t_1t_2\cdots \in ((0,1), d)$. If $s \neq t$ but $s_i = t_i$ for $i = 1, 2\ldots n$. Then $d(s,t) \leq 1/10^n$.*

Proof. For $i = 1, 2, \ldots, n$, $s_i = t_i$,thus

$$d(s,t) = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{10^i}.$$

But $|s_i - t_i| \leq 9$. Thus, $d(s,t) \leq \frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \ldots = 9(\frac{1}{10^{n+1}} + \frac{1}{10^{n+2}} \ldots) = \frac{9}{10^{n+1}}(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots) = \frac{1}{10^n}$. Hence proved. □

UNIVERSAL ORBIT UNDER DECIMAL SHIFT MAP

Theorem 1. *Let $((0,1), d)$ be the metric space and $f : [0,1] \to [0,1]$ be the decimal shift map (Strogatz 2018) defined by $f(x) = 10x(\text{mod } 1)$ where $x(\text{mod } 1) \equiv x - \lfloor x \rfloor$. $\lfloor x \rfloor$ is the largest integer less than or equal to x. The orbit generated by f with an initial point x^* is said to be universal if and only if its decimal expansion satisfies the property that given any natural number n should occur atleast once in the decimal expansion of x^*.*

Proof. Assume that $x^* \in (0,1)$ has a decimal expansion $0.a_0a_1a_2\ldots$ such that given any k digit natural number $n = r_1r_2\ldots r_k$, there exist an index l such that $a_{l+1}a_{l+2}\ldots a_{l+k} = r_1r_2\ldots r_k$. Let $y = 0.b_0b_1b_2\ldots \in (0,1)$. In order to show that the orbit generated by x^* is universal, it is enough to show that given any $\epsilon > 0$, the orbit of x^* will reach the ϵ neighbourhood of orbit of y.
Given $\epsilon > 0$. Choose a k such that $\frac{1}{10^k} < \epsilon$. Since any given natural number belongs to the decimal expansion of x^*, there exist an index m such that $a_m = b_0, a_{m+1} = b_1, a_{m+2} = b_2, \ldots, a_{m+k} = b_k$. *i.e*, $f^{(m)}(x^*) = 0.b_0b_1\ldots b_ka_{m+k+1}a_{m+k+2}\ldots$.Hence,

$$d(f^{(m)}(x^*), y) < \frac{1}{10^k} < \epsilon.$$

Thus the orbit of x^* under the decimal shift map is universal.
Now assume to the contrary that the initial point $x^* \in (0,1)$ has a decimal expansion $0.a_0a_1a_2\ldots$ that doesn't contain the string $n = r_1r_2\ldots r_m$ *i.e* there doesn't exist any i such that $a_{i+1}a_{i+2}\ldots a_{i+m} = r_1r_2\ldots r_m$. We have to show that the orbit of x^* under decimal shift map is not a universal orbit.
Let $y^* = 0.r_1r_2\ldots r_mr_1r_2\ldots r_m\ldots$. Clearly $y^* \in (0,1)$.
Suppose $0 < \epsilon < \frac{1}{10^{2m}}$. Consider an arbitrary element in the orbit of y^*,

$$f^{(k)}(y^*) = 0.r_ir_{i+1}\ldots r_mr_1r_2\ldots r_m\ldots.$$

Similarly, consider an element in the orbit of x^*,

$$f^{(k')}(x^*) = 0.a_{j_1}a_{j_2}a_{j_3}\ldots.$$

But by our assumption on initial point x^*, there doesn't exist an s such that
$a_{j_s} = r_1, a_{j_s+1} = r_2, \ldots, a_{j_s+m} = r_m$. Thus, the following are the possible decimal expansions of $f^{(k')}(x^*)$:
Case (i) :

$$f^{(k')}(x^*) = 0.a_{j_1}a_{j_2}a_{j_3}\ldots \text{ where } a_{j_1} \neq r_i.$$

Then clearly,$d(f^{(k)}(x^*), f^{(k')}(y^*)) > \frac{1}{10^{2m}}$.
Case (ii) :

$$f^{(k')}(x^*) = 0.a_{j_1}a_{j_2}a_{j_3}\ldots \text{ where } a_{j_1} = r_i.$$

In this case, the decimal expansion of $f^{(k')}(x^*)$ closest to $f^{(k)}(y^*)$ is $0.r_ir_{i+1}\ldots r_mr_1\ldots r_{m-1}a_m\ldots$, where $r_i \neq r_1$ and $a_m \neq r_m$. Then

$$d(f^{(k)}(x^*), f^{(k')}(y^*)) = \frac{1}{10^{2m-1-i}}.$$

Hence for any k, k',

$$d(f^{(k)}(x^*), f^{(k')}(y^*)) > \frac{1}{10^{2m}}.$$

i.e,

$$d(f^{(k)}(x^*), f^{(k')}(y^*)) > \epsilon.$$

for any $0 < \epsilon < \frac{1}{10^{2m}}$. Thus the orbit generated by x^* without the string $n = r_1r_2\ldots r_m$ is not universal. □

The Theorem 1 provides a characterization of the points within the interval $(0,1)$ that can form a universal orbit when subjected to the decimal shift map. For example, consider $x^* = 0.12345678910111213\ldots.$ The number x^* is constructed by concatenating all the successive natural numbers together. The orbit of x^* under decimal shift map is $\{0.123456789\ldots, 0.234567891011\ldots, 0.3456789101112\ldots, \ldots\}$. Hence, by Theorem 1,the orbit of x^* under decimal shift map is universal. The orbit generated by decimal shift map after 50 iterations with initial point q as Champernowne's Constant (truncated upto the number 100) is shown in Figure 1.

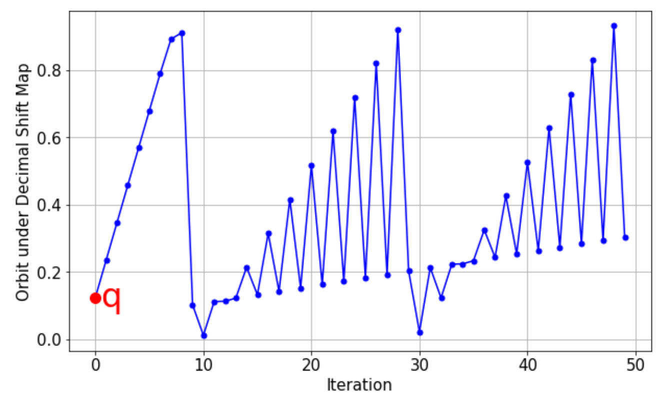

Figure 1 Orbit(first 50 iterations) under decimal shift map with initial value set to the truncated Champernowne's constant $q = 0.1234567\ldots 99100$.

There are also other points in $(0,1)$ that can generate universal orbit. Another such example is $x' = 0.11121314151617181920\ldots.$ x' is obtained by concatenating successive natural numbers from 11. x' also contains the natural numbers from 0 to 9 along with other natural numbers. Hence x' can create universal orbit.Similarly, any k^{th} element in the orbit of x^*, $f^{(k)}(x^*)$, can generate universal orbit. Other examples are $0.24681012141618\ldots$(obtained by concatenating even numbers), $0.100200300400500600700\ldots$(obtained by concatenating the multiples of 100).

Remark 1. *The points in $(0,1)$ that generate universal orbit under decimal shift maps are irrational numbers. Their decimal expansion is non terminating and non recurring.*

Remark 2. *The decimal expansion of a rational number is either terminating or periodic. Thus the orbit generated by rational numbers are not chaotic and universal.*

Remark 3. *The orbit generated by an irrational number in $(0,1)$ need not be a universal orbit under decimal shift map.*

For example, consider the orbit of $y^* = 0.12112111211112\ldots.$ Clearly y^* is an irrational number. But given a natural number $n = 3456789$ doesn't occur in the decimal expansion of y^*. Hence by Theorem 1, the orbit of y^* under decimal shift map,

$$S^* = \{0.121121112\ldots, 0.21121112\ldots, 0.1121112\ldots, \ldots\},$$

is not a universal orbit.

Next, our objective is to determine the properties of the set of points that can produce a universal orbit under the decimal shift map. We shall prove that the set is both uncountable and dense in the interval (0,1).

Theorem 2. *The set of points in* $(0,1)$ *that generate universal orbit under decimal shift map is uncountable.*

Proof. Let $S \subset (0,1)$ be the set of all points whose orbit under decimal shift map f is universal. We need to prove that the set S is uncountable. Assume to the contrary that S is countable. Let $S = \{0.r_{11}r_{12}r_{13}\ldots, 0.r_{21}r_{22}r_{23}\ldots, 0.r_{31}r_{32}r_{33}\ldots, \ldots\}$ Consider

$$x_0 = 0.1\,\overline{r_{12}}\,2\,\overline{r_{24}}\,3\,\overline{r_{36}}\,\ldots$$

where $\overline{r_{12}} \neq r_{12}$, $\overline{r_{24}} \neq r_{24}$, $\overline{r_{36}} \neq r_{36}$,.... Clearly, $x_0 \notin S$. Since x_0 is constructed in such a way that its decimal expansion contains all natural numbers, by Theorem 1, the orbit of x_0 under the decimal shift map is universal and $x_0 \notin S$. This contradicts our assumption that S is countable. Hence, S is uncountable.

□

Theorem 3. *The set of points in (0,1) that generate universal orbit under decimal shift map is dense in (0,1).*

Proof. Let S be the set of points in (0,1) that generate universal orbit. Let $a, b \in (0,1)$ where $a < b$. We will show that there exist an element $c \in S$ such that $a < c < b$.
Choose a k such that $\frac{1}{10^k} < \frac{b-a}{2}$. Now choose a rational number $0.q_1 \ldots q_k \in (a, \frac{a+b}{2})$. Consider $c = 0.q_1 \ldots q_k 123\ldots$(c is obtained by concatenating natural numbers to $0.q_1 \ldots q_k$). By Theorem 1, c can generate a universal orbit. Thus, $c \in S$. Now it remains to prove $c \in (a,b)$.
We have $a < 0.q_1 \ldots q_k < 0.q_1 \ldots q_k 123456789101112\ldots$. That is, $a < c$. Hence, it is enough to prove $c < b$.
Clearly, $0.q_1q_2 \ldots q_k 1234567891011\ldots < 0.q_1q_2 \ldots q_k 999999999\ldots$.

$$\begin{aligned} 0.q_1q_2\ldots q_k 9999\ldots &< 0.q_1q_2\ldots q_k + \frac{9}{10^{k+1}} + \frac{9}{10^{k+2}} + \ldots \\ &= 0.q_1q_2\ldots q_k + \frac{9}{10^{k+1}}(1 + \frac{1}{10} + \ldots) \\ &= 0.q_1q_2\ldots q_k + \frac{9}{10^{k+1}}(\frac{10}{9}) \\ &= 0.q_1q_2\ldots q_k + \frac{1}{10^k} \\ &< \frac{a+b}{2} + \frac{b-a}{2} \\ &< b \end{aligned}$$

Thus,

$$0.q_1q_2\ldots q_k 1234567891011\ldots < 0.q_1q_2\ldots q_k 99999999\ldots < b$$

.
That is, $c < b$. Hence proved. □

Remark 4. *The set of all irrational numbers in (0,1) is dense in (0,1). The set* S*, which is a subset of irrational numbers in (0,1), is also dense in (0,1).*

Universal orbit with normal numbers as initial triggers

In this section we will look for the universal orbits under decimal shift maps with normal numbers as initial point.

Definition 2. *(Khoshnevisan 2006) and (Champernowne 1933) Let* $b \geq 2$ *be a positive integer. A real number* α *is* ***normal in base b*** *if for every* $k \geq 1$ *we have* $f_{c_1c_2\ldots c_k}(\alpha, b) = \frac{1}{b^k}$*, where* $f_{c_1c_2\ldots c_k}(\alpha, b)$ *is the frequency of the string* $c_1c_2\ldots c_k$ *of length k appearing in the decimal expansion of* α *in base b. Let* $\alpha = \lfloor\alpha\rfloor + \{\alpha\}$ *where* $\lfloor\alpha\rfloor$ *is the integer part and* $\{\alpha\}$ *is the fractional part of* α *denoted by* $0.a_0a_1a_2\ldots a_n\ldots$ *Then we can define the frequency as follows:*

$$f_{c_1c_2\ldots c_k}(\alpha, b) = \lim_{n\to\infty} \frac{\#\{i \leq n-k+1 : a_i = c_1, a_{i+1} = c_2, \ldots, a_{i+k-1} = c_k\}}{n}$$

In other words, a normal number in base b, contains every possible combination of digits, but each combination occurs with equal likelihood to other combinations of that length. The numbers 0.1234567891011...(Champernowne number (Champernowne 1933), 0.2357111317...(Copeland-Erdos constant (Fan 1946) obtained by concatenating prime numbers) *etc* are all examples of normal numbers.

Theorem 4. *Numbers that generate universal orbit under decimal shift maps need not be normal.*

Proof. Consider (0,1) with the metric $d(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{10^i}$. Let S be the set of all numbers in (0,1) that generate universal orbit under decimal shift map. Consider $t = 0.001002000300004\ldots\underbrace{000\ldots00}_{N times}N\ldots$. The orbit of t, T is

$$\{0.0010020003\ldots, 0.010020003\ldots, 0.10020003\ldots, 0.0020003\ldots, \ldots\}$$

First we have to show that T is a universal orbit.
By the construction of t, any given natural number will occur in the decimal expansion of t. Hence by Theorem 1, the orbit T is universal and $t \in S$.
Now need to show t is not a normal number. By the definition of normal number, every possible combination of digits should occur with same frequency. Consider $t' = 0.0010020003\ldots\underbrace{000\ldots00}_{(10^N-1) times}\underbrace{999\ldots99}_{N times}$, which has the decimal expansion of t upto the number $10^N - 1 = \underbrace{999\ldots99}_{N times}$.
Here t' is constructed by concatenating numbers from 0 to $\underbrace{999\ldots99}_{N times}$ and zeroes are added in between them such that one zero before the digit 1, two zeroes before the digit 2, ..., $10^N - 1$ zeroes before the digit $\underbrace{999\ldots99}_{N times}$.
The number of occurrences of a digit i (where $i = 0, 1, 2, \ldots, 9$) in the string

$$0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\ldots\underbrace{999\ldots99}_{N times}$$

is given by $N10^{N-1}$(Pomstra 2018). Hence the number of occurrences of digit $i \neq 0$ in the decimal expansion of t' is given by $N10^{N-1}$. But the number of occurrences of 0 in the decimal expansion of t' is given by $N10^{N-1} + \frac{(10^N-1)10^N}{2}$. Thus,

$$f_0(t', 10) = \frac{N10^{N-1} + \frac{(10^N-1)10^N}{2}}{10N10^{N-1} + \frac{(10^N-1)10^N}{2}}$$

$$f_i(t',10) = \frac{N10^{N-1}}{\frac{(10^N-1)10^N}{2}}$$

where $i \neq 0$ As $N \to \infty$ the decimal expansion of t' tends to t. Therefore,

$$f_0(t,10) = \lim_{N\to\infty} \frac{(N\times 10^{N-1}) + \frac{(10^N-1)10^N}{2}}{10N10^{N-1} + \frac{(10^N-1)10^N}{2}}$$

$$f_i(t,10) = \lim_{N\to\infty} \frac{(N\times 10^{N-1})}{10N10^{N-1} + \frac{(10^N-1)10^N}{2}}$$

Now consider

$$\begin{aligned} \frac{f_0(t,10)}{f_i(t,10)} &= \lim_{N\to\infty} \frac{(N\times 10^{N-1}) + \frac{(10^N-1)10^N}{2}}{(N\times 10^{N-1})} \\ &= \lim_{N\to\infty} 1 + \frac{\frac{(10^N-1)10^N}{2}}{(N\times 10^{N-1})} \\ &> 1 + \lim_{N\to\infty} \frac{(10^N-1)10^N}{2N\times 10^{N-1}} \\ &= 1 + \lim_{N\to\infty} \frac{5(10^N-1)}{N} > 1 \end{aligned}$$

$f_0(t,10) > f_i(t,10)$ where i=1,2,...9.

By applying L'Hôpital's rule, $\lim_{N\to\infty} \frac{5(10^N-1)}{N} = \infty$. Thus t is not a normal number but an element in S. □

Conjecture 1. *The orbit of a normal number under decimal shift map is universal.*

UNIVERSAL ORBIT UNDER GAUSS MAP

In the previous section we have discussed about Decimal Shift Map and its universal orbit. The Decimal Shift map shifts the digits of the decimal representation to the left.Now our aim is to study about the universal orbit generated by Gauss map (Continued Fraction Map) (Dajani and Kraaikamp 2002; Corless 1992).

The Gauss map $G : [0,1) \to [0,1)$ is defined as follows:

$$\begin{aligned} G(x) &= 0 \; if \; x = 0 \\ &= \frac{1}{x} \; mod \; 1 \; if \; 0 < x \leq 1 \end{aligned}$$

The Gauss map is an excellent example of a chaotic discrete dynamical system. There is a significant relationship between the Gauss map and continued fractions. An expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}} \tag{1}$$

is called a continued fraction. In general, the numbers $a_0, a_1, a_2, a_3, \ldots, b_0, b_1, b_2, \ldots$ may be any real or complex numbers and the number of terms may be finite or infinite (Olds 1963). A simple continued fraction is of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2}$$

where a_i are all positive integers except a_0, which may be zero or negative (Corless 1992). We will denote equation 2 as

$$[a_0; a_1, a_2, \ldots, a_n, \ldots] \tag{3}$$

The k^{th} convergent of a continued fraction (Dajani and Kraaikamp 2002) $[a_0; a_1, a_2, \ldots, a_n, \ldots]$ is given as follows :

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_k}}}} = [a_0; a_1, a_2, \ldots, a_k] \tag{4}$$

Continued Fraction (CF) expansion corresponding to $x \in (0,1)$ is represented as $[0; a_1, a_2, \ldots]$, where $a_i \geq 1$. Since Gauss map is defined on $[0,1)$, in this section we are focusing on Simple Continued Fractions. All rational numbers admit finite continued fraction expansion while every irrational number $x \in (0,1)$ can be expressed through a (unique) infinite continued fraction (Dajani and Kraaikamp 2002).

The Gauss map acts on the digits of the CF expansion as the one-sided shift. *i.e*, $G(x) = [0; a_2, a_3, \ldots]$, $G^2(x) = [0; a_3, a_4, \ldots]$, $G^3(x) = [0; a_4, a_5, \ldots]$, ... (Bates *et al.* 2005).

Now we will try to characterise points in $(0,1)$ that can generate a universal orbit under Gauss Map. Table 1 shows that as the coefficients in continued fraction become same, their decimal expansion will also converge.

Table 1 Continued Fraction expansions and their decimal expansion

Continued Fraction	Decimal Expansion
$[0;1,2]$	0.6666666666666667
$[0;1,2,3]$	0.7
$[0;1,2,3,4]$	0.6976744186046512
$[0;1,2,3,4,5]$	0.6977777777777778
$[0;1,2,3,4,5,6]$	0.6977745872218234
$[0;1,2,3,4,5,6,7]$	0.6977746591820369
$[0;1,2,3,4,5,6,7,8]$	0.6977746579475622
$[0;1,2,3,4,5,6,7,8,9]$	0.6977746579641866

Lemma 2. *Let $a = [0; a_1, a_2, \ldots] \in (0,1)$ and $c_k = [0; a_1, a_2, \ldots, a_k]$ be the k^{th} convergent of a. Then given any $n \in \mathbb{N}$, there exist a N such that $\forall r \geq N$ the first n digits of decimal expansions of c_r and c_{r+m} are same,where $m > 0$.*

Proof. Let $c_{r+m} = [0; a_1, a_2, \ldots, a_{r+m}]$ and $c_r = [0; a_1, a_2, \ldots, a_r]$ be the $r + m^{th}$ and r^{th} convergents of a. Let $0 < \epsilon < \frac{1}{10^{n+1}}$.Since $\lim_{r\to\infty} c_r = a$, there exist N such that for $r \geq N$,$|c_r - a| < \frac{\epsilon}{2}$. Then, by triangular inequality,

$$\begin{aligned}|c_r - c_{r+m}| &= |c_r - a + a - c_{r+m}| \\ &\leq |c_r - a| + |a - c_{r+m}| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \\ &< \frac{1}{10^{n+1}}.\end{aligned}$$

Thus the the first n digits of decimal expansions of c_r and c_{r-1} are same $\forall r \geq N$. □

Next theorem provides a characterization of the continued fraction representation of points in the interval (0,1) that can construct a universal orbit under the Gauss Map.

Theorem 5. *Let $f : (0,1) \to (0,1)$ be the Gauss map on $((0,1), d)$ .Then the orbit of $x^* = [0; x_1, x_2, \ldots] \in (0,1)$ under f is universal if and only if continued fraction expansion of x^* contains the finite continued fraction expansion of any given rational number in $(0,1)$.*

Proof. Let $x^* \in (0,1)$ have a continued fraction expansion $[0; x_1, x_2, \ldots]$ that contains the continued fraction expansion of any given rational number in $(0,1)$. Now to prove the orbit of x^* is universal. Let $y \in (0,1)$ have a continued fraction expansion $[0; a_1, a_2 \ldots]$. Given $\epsilon > 0$ and $n \in \mathbb{N}$ such that $\frac{1}{10^n} < \epsilon$. By Lemma 2, Choose a sufficiently large r such that decimal expansion of the convergents of $f^{(k)}(y) = [0; a_k, a_{k+1}, \ldots]$: c_{r-1}, c_r, c_{r+1} have first n decimal places as same. *i.e,*

$$c_{r-1} = [0; a_k, a_{k+1} \ldots a_{k+(r-1)}] = 0.d_1 d_2 \ldots d_n d'_{n+1} \ldots d'_l,$$

$$c_r = [0; a_k, a_{k+1} \ldots a_{k+(r)}] = 0.d_1 d_2 \ldots d_n d''_{n+1} \ldots d''_p,$$

$$c_{r+1} = [0; a_k, a_{k+r+1} \ldots a_{k+(r+1)}] = 0.d_1 d_2 \ldots d_n d'''_{n+1} \ldots d'''_q.$$

Thus,

$$f^{(k)}(y) = 0.d_1 d_2 \ldots d_n d_{n+1} \ldots$$

Now, by the construction of x^*, there exist a k' such that $f^{(k')}(x^*) = [0; a_k, a_{k+1}, \ldots a_{k+(r-1)}, x_i, x_{i+1}, \ldots]$. Since c_{r-1}, c_r, c_{r+1} are also the convergents continued fraction of $f^{(k')}(x^*)$,

$$f^{(k')}(x^*) = 0.d_1 d_2 \ldots d_n d^*_{n+1} d^*_{n+2} \cdots$$

Now by Lemma 1,

$$d(f^{(k)}(y), f^{(k')}(x^*)) < \frac{1}{10^n} < \epsilon.$$

Thus, the orbit of x^* is universal.
Now to prove the converse part by contradiction. Suppose that continued fraction of $x^* = [0; x_1, x_2, \ldots]$ doesn't contain the sequence $b_1, b_2 \ldots b_r$ *i.e*, there doesn't exist an index i such that $x_i = b_1, x_{i+1} = b_2, \ldots, x_{i+r} = b_r$. Consider $y = [0; b_1, b_2, \ldots, b_r, b_1, b_2, \ldots, b_r, \ldots]$. Clearly $y \in (0,1)$. Thus, the longest common pattern that can be there in the continued fraction of y and x^* is $b_2, b_3, \ldots, b_r, b_1, \ldots, b_{r-1}$. Then there exist a point in the orbit of x^* such that $f^{(n^*)}(x^*) = [0; b_2, b_3, \ldots, b_r, b_1, \ldots, b_{r-1}, x_i, \ldots]$ where $x_i \neq b_r$. Also $f(y) = [0; b_2, b_3, \ldots, b_r, b_1, \ldots, b_{r-1}, b_r, b_1 \ldots]$. Thus there exist $\epsilon > 0$ such that $d(f(y), f^{(n^*)}(x^*)) > \epsilon$. Hence, the orbit of x^* is not universal. □

The continued fraction expansion obtained by concatenating the continued fraction of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, ..., $[0; 2, 3, 1, 2, 4, 2, 1, 3, \ldots]$ is normal with respect to continued fraction partition (Adler *et al.* 1981) and can also generate universal orbit with respect to Gauss Map by Theorem 5. The continued fraction expansion of $\frac{1}{2}$ is $[0; 2]$, $\frac{1}{3}$ is $[0; 3]$, $\frac{2}{3}$ is $[0; 1, 2]$, $\frac{1}{4}$ is $[0; 4]$, $\frac{2}{4}$ is $[0; 2] \ldots$. Thus, by concatenating the continued fraction of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3} \ldots$, $\frac{8}{10}$, $\frac{9}{10}$, we get $q = [0; 2, 3, 1, 2, 4, 2, 1, 3, \ldots, 1, 4, 1, 9]$. The decimal expansion of $q = [0; 2, 3, 1, 2, 4, 2, 1, 3, \ldots, 1, 4, 1, 9]$ is 0.4403388262519711 $(approximately)$. Figure 2 shows the orbit generated by Gauss Map with this initial point $q = 0.4403388262519711$.

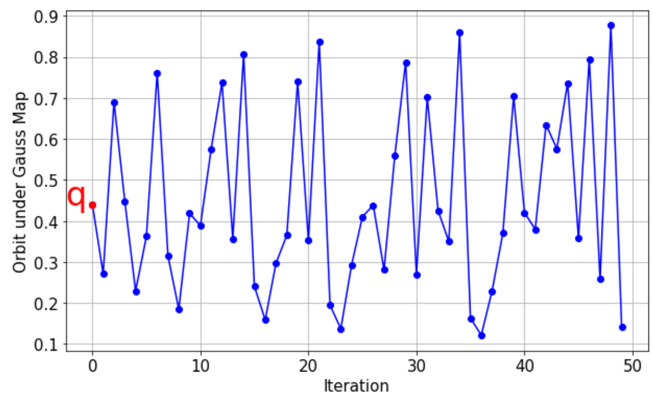

Figure 2 Orbit (first 50 iterations) under Gauss map with initial value q having continued fraction expansion $[0; 2, 3, 1, 2, 4, 2, 1, 3, \ldots, 1, 4, 1, 9]$, constructed by concatenating the continued fraction of $\frac{1}{2}, \frac{1}{3}, \frac{2}{3} \ldots, \frac{8}{10}, \frac{9}{10}$.

In 1770, Lagrange proved that any quadratic irrational $x \in (0,1)$ has a continued fraction expansion which is periodic after a certain stage, *i.e,* $x = [0; a_0, a_1, \ldots, a_i, n_1, n_2, \ldots, n_k, n_1, n_2, \ldots, n_k, n_1, \ldots]$. Hence by Theorem 5 the orbit of a quadratic irrational is not universal under Gauss Map and is not continued fraction normal (Becher and Yuhjtman 2019).

Remark 5. *Quadratic Irrationals in $(0,1)$ will not generate universal orbits under Gauss Map.*

Remark 6. *Rational numbers will not generate universal orbit under Gauss Map or Decimal Shift map.*

Remark 7. *The points that generate universal orbit under decimal shift map may or may not be universal under Gauss Map.*

Consider the Champernowne's number $c = 0.1234567891011\ldots$. By Theorem 1, the orbit of c under decimal shift map is universal. But the continued fraction expansion of $c = [0; c_1, c_2, c_3 \ldots]$ is as follows : [0;8,9,1,149083,...]. Not only is the fourth term of the continued fraction of the Champernowne constant huge, but there are also other terms that are similarly significant in magnitude (Pomstra 2018). Table 2 illustrates this. c_{18} has 166 digits and c_{40} has 2504 digits (Pomstra 2018). Therefore, it is uncertain whether any given finite continued fraction will appear in the continued fraction expansion of the Champernowne Constant. Hence by Theorem 5, the orbit generated by c may or may not be universal with respect to Gauss Map.

Table 2 The initial 40 terms of the continued fraction expansion of the Champernowne constant.

n	c_n	n	c_n	n	c_n
1	8	15	1	29	1
2	9	16	1	30	7
3	1	17	15	31	2
4	149083	18	457...987	32	1
5	1	19	6	33	83
6	1	20	1	34	1
7	1	21	1	35	156
8	4	22	21	36	4
9	1	23	1	37	58
10	1	24	9	38	8
11	1	25	1	39	54
12	3	26	1	40	44...23
13	4	27	2		
14	1	28	3		

NEUROCHAOS LEARNING AND UNIVERSAL ORBITS

In the realm of Neurochaos Learning (Harikrishnan and Nagaraj 2019; Balakrishnan *et al.* 2019), the quest for efficient feature extraction and classification has led to the development of innovative algorithms. One such breakthrough is the concept of *Universal Orbits*, which has the potential to revolutionize the field. In this section, we delve into the connection between universal orbits and Neurochaos Learning, exploring the possibilities of harnessing chaotic maps to uncover hidden patterns in data.

In the Neurochaos Learning algorithm (Harikrishnan and Nagaraj 2019), chaotic orbits generated by the skew tent map are employed for feature extraction in classification tasks. The number of chaotic neurons in the input layer corresponds to the number of input attributes in the dataset. When the input attributes are received, each neuron begins firing independently with an initial neural activity of q units. The neural trace of these chaotic neurons stops once the stimulus is recognized. From this neural trace, features such as firing time, firing rate, energy, and entropy are extracted. This discussion will particularly examine the unique characteristics of the firing time feature when using the universal orbit generated by the Decimal Shift Map for feature extraction.

Firing Time and noise level around stimulus

The objective of this section is to examine the link between firing time of Universal Orbit and the level of noise present surrounding the stimulus in the context of Neurochaos Learning.

Let f be a decimal shift map on the metric space $((0,1), d)$. The number of iterations taken by the decimal shift map $f:(0,1)\to(0,1)$ with an initial point q to reach any given ϵ neighbourhood of a point x in (0,1) is called the firing time (Harikrishnan and Nagaraj 2019) corresponding to x, denoted by $n(\epsilon)$. Here, ϵ is the level of noise around the stimulus x and q is the initial neural activity.

Consider the initial point for decimal shift map as $q = 0.q_0q_1q_2q_3\ldots$ and $x = 0.a_1a_2\ldots \in (0,1)$. Let $\epsilon, \epsilon' > 0$ and $\epsilon < \epsilon'$. Suppose the orbit generated by q, $\{0.q_0q_1q_2q_3\ldots, 0.q_1q_2q_3q_4\ldots, 0.q_2q_3q_4q_5\ldots, \ldots\}$, is a universal orbit. Then, by Theorem 1, there exists an index n_1 in the decimal expansion of q such that $q_{n_1}q_{n_1+1}\ldots q_{n_1+k} = a_0a_1\ldots a_k$($k$ depends on the value of ϵ). So f takes n_1 iterations to reach the ϵ neighbourhood of $f^{(0)}(x) = x$ with initial point q *i.e*, the n_1^{th} element in the orbit of q is $f^{(n_1)}(q) = 0.a_0a_1a_2\ldots a_kq_{k+1}q_{k+2}\ldots$. *i.e*, $d(f^{(n_1)}(q), f^{(0)}(x)) < \epsilon$ and hence the firing time corresponding to ϵ, $n(\epsilon) = n_1$. Since $\epsilon < \epsilon'$,

$$d(f^{(n_1)}(x), f^{(0)}(x)) < \epsilon < \epsilon',$$
$$d(f^{(n_1)}(x), f^{(0)}(x)) < \epsilon'.$$

That is, f can reach the ϵ' neighbourhood of x either with n_1 or less than n_1 iterations. Thus, the number of iterations taken by the orbit of q to reach the ϵ' neighbourhood of x is less than or equal to that of ϵ. *i.e*, $n(\epsilon') \leq n(\epsilon)$. As the noise($\epsilon$) around x decreases, firing time increases.

Firing time with Champernowne constant as initial neural activity

In this section, we will consider the Champernowne Constant as initial neural activity, $q = 0.123456789101112\ldots$ for the orbit generated by the decimal shift map. Note that Champernowne number is obtained by concatenating natural numbers in order. Thus by Theorem 1, it is a universal orbit.

Suppose the stimulus is $x = 0.a_1a_2\ldots a_s$ and the level of noise is $\epsilon > 0$. Since the orbit under q is universal there exists a least integer m such that $d(f^{(m)}(q), x) < \epsilon$. Thus, m is the firing time corresponding to x. Here, our aim is to find an upper bound for firing time, m.

Prior to that, we must establish the following lemma regarding the count of digits preceding a particular natural number in the decimal representation of the Champernowne number.

Lemma 3. *Let $c = 0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\ldots N(N+1)(N+2)\ldots$ be the Champernowne constant and $c_N = 0.1234567891011\ldots(N-2)(N-1)$ be the truncated portion of c, where $N = a_1a_2\ldots a_d$, $d \geq 2$. Then the number of digits after the decimal point in c_N is given by*

$$dN - (10 + 10^2 + \ldots + 10^{d-1}) - 1. \quad (5)$$

In other words, the number N will occur in the decimal expansion of c after $dN - (10 + 10^2 + \ldots + 10^{d-1}) - 1$ digits from the decimal point.

Proof. We proceed using mathematical induction.
Let $P(N)$ be the mathematical statement.
$P(N)$: The number of digits after the decimal point in c_N is given by ,where $N = a_1a_2\ldots a_d$,$d \geq 2$.

Base Case: Given that N should at least be a two-digit number as $d \geq 2$. Consider $N = 10$.We have $c_{10} = 0.123456789$. Then the number of digits after the decimal point in c_{10}, $P(10) = 9$.
Also by formula 5, $P(10) = 2 \times 10 - 10 - 1 = 9$.
Thus, base case is verified.

Induction Hypothesis: Assume that $P(K)$ is true for some positive integer $K > 10$. That means the number of digits after the decimal point in c_K is given by

$$dK - (10 + 10^2 + \ldots + 10^{d-1}) - 1,$$

where $K = a_1a_2 \ldots a_d, d \geq 2$.

Induction Step: We will now show that $P(K+1)$ is true.
Consider the following cases :

Case(i) K is a d digit number and $K \neq \underbrace{999\ldots 99}_{d\ times}$

Then $K+1$ will always have d digits. That is, we have to prove :

$$P(K+1) = d(K+1) - (10 + 10^2 + \ldots + 10^{d-1}) - 1.$$

We have $P(K+1)$ is the number of digits after the decimal point in $c_{K+1} = 0.12345678910\ldots(K-1)K$.
By our assumption, the number of digits upto $K-1$ in c_{K+1} is $dK - (10 + 10^2 + \ldots + 10^{d-1}) - 1$. Now K has d digits. Therefore

$$\begin{aligned} P(K+1) &= dK - (10 + 10^2 + \ldots + 10^{d-1}) - 1 + d \\ &= d(K+1) - (10 + 10^2 + \ldots + 10^{d-1}) - 1. \end{aligned}$$

Hence, proved.

Case(ii) $K = \underbrace{999\ldots 99}_{d\ times}$

Then $K+1 = 10^d$, which has (d+1) digits. That is, we have to prove :

$$P(K+1) = (d+1)(K+1) - (10 + 10^2 + \ldots + 10^d) - 1.$$

By our assumption, the number of digits upto $K-1$ in c_{K+1} is $dK - (10 + 10^2 + \ldots + 10^{d-1}) - 1$. Now K has d digits. Therefore, the number of digits after the decimal point in $c_{K+1} = 0.12345678910\ldots(K-1)K(K+1)$,

$$\begin{aligned} P(K+1) &= dK - (10 + 10^2 + \ldots + 10^{d-1}) - 1 + d \\ &= dK - (10 + 10^2 + \ldots + 10^{d-1}) - 1 + d + 10^d - 10^d \\ &(\textit{Adding and subtracting } 10^d) \\ &= dK + 10^d - (10 + 10^2 + \ldots + 10^{d-1} + 10^d) - 1 + d \\ &= d(K+1) + 10^d - (10 + 10^2 + \ldots + 10^{d-1} + 10^d) - 1 \\ &= d(K+1) + (K+1) - (10 + 10^2 + \ldots + 10^{d-1} + 10^d) - 1 \\ &(\textit{Since, } K+1 = 10^d) \\ &= (d+1)(K+1) - (10 + 10^2 + \ldots + 10^{d-1} + 10^d) - 1. \end{aligned}$$

Hence proved. □

Remark 8. *If $d = 1$, that is, if N is a one-digit number, the number of digits before N in c is given by $N-1$.*

Now we can use the above lemma to find an upper bound for the firing time, m, corresponding to a stimulus $x = a_1a_2 \ldots a_s$ with respect to the universal orbit generated by Champernowne constant $c = 0.12345678910\ldots$. Let $\epsilon > 0$ be the level of noise around stimulus. Choose a positive integer $d > 1$ ($d \leq s$, number of digits in x) such that $\frac{1}{10^d} < \epsilon$. Let $N = a_1a_2 \ldots a_d$.
Thus, by Lemma 3 the number N will occur in the decimal expansion of q after $m^* = dN - (10 + 10^2 + \ldots + 10^{d-1}) - 1$ digits. i.e,

$$f^{(m^*)}(q) = 0.N(N+1)(N+2)\ldots$$

Since $x = 0.a_1a_2 \ldots a_s$, $N = a_1a_2 \ldots a_d$ and $k \leq s$

$$d(f^{(m^*)}(q), x) = \frac{1}{10^d} < \epsilon.$$

Thus, the firing time, $m \leq m^*$.

For example, let $x = 0.316$ and $\epsilon = 0.01$. We have $f^{(16)}(q) = 0.31415161718192021\ldots$ and $f^{(2\times 31-10-1)}(q) = f^{(51)}(q) = 0.31\ 32\ 33\ 34\ldots$. Thus, the firing time is 16, which is less than 51.

In short, if we are considering Champernowne's constant as initial point for the orbit generated by decimal shift map,then the orbit will surely reach any given neighbourhood of a stimulus within m^* iterations. Thus by employing universal orbits in Neurochaos Learning algorithm, one of its extracted features, firing time can be interpreted by setting an upperbound and analysing its trend depending on the noise around stimulus.

CONCLUSION

In this study, we introduced a new type of chaotic trajectory known as the *Universal Orbit*. We specifically examined the universal orbit produced by two chaotic maps: the Decimal Shift Map and the Gauss Map. Additionally, we discussed the set of points capable of producing a universal orbit under these maps. This research also investigated the chaotic orbit produced by normal numbers under both the Gauss Map and Decimal Shift Map. We specifically analysed the unique characteristics of the firing time feature when the universal orbit generated by the decimal shift map is used for feature extraction, aiming to establish an upper bound for firing time in relation to noise levels surrounding the stimulus.

A point in the open interval (0,1) can form a universal orbit under the decimal map if and only if its decimal expansion contains all possible natural numbers. Similarly, a point in (0,1) can produce a universal orbit under the Gauss map only if its continued fraction expansion includes any given finite continued fraction expansion. We have demonstrated that the set of points in the interval (0,1) that may produce a universal orbit under the decimal shift map is both uncountable and densely distributed inside the interval (0,1). Furthermore, the set of points in (0,1) that generate a universal orbit under the decimal shift map is a proper subset of the set of irrational numbers. Upon analysing the orbit produced by normal numbers as starting triggers, it is evident that not all points that form a universal orbit are themselves normal numbers. However, we conjecture that the trajectory of a normal number is a universal orbit under the decimal shift map. In conclusion, our research reveals a remarkable connection between chaotic dynamics and number theory. When utilizing universal orbits in Neurochaos Learning (a brain-inspired machine learning architecture), the firing time exhibits an inverse relationship with the noise level surrounding the stimulus, remaining bounded within a specific range determined by the noise intensity. This is a new result which was unknown in the NL literature.

In the future, our intention is to identify the set of points that produce a universal orbit under various chaotic maps. We will then truncate these points and use them as starting triggers in the Neurochaos Learning Algorithm to enhance its classification performance.

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Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

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