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On the Diophantine Equation $(8r^2+1)^x+(r^2-1)^y=(3r)^z$ Regarding Terai's Conjecture

Tuba Çokoksen[1]*, Murat Alan[2]

Abstract
This study establishes that the sole positive integer solution to the exponential Diophantine equation $(8r^2+1)^x+(r^2-1)^y=(3r)^z$ is $(x,y,z)=(1,1,2)$ for all $r>1$. The proof employs elementary techniques from number theory, a classification method, and Zsigmondy's Primitive Divisor Theorem.

Keywords: Diophantine equations, Primitive divisor theorem, Terai's conjecture
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[1] *Department of Mathematics, Yildiz Technical University, İstanbul, Türkiye, tuba.cokoksen@std.yildiz.edu.tr, ORCID: 0009-0004-3164-1211*
[2] *Department of Mathematics, Yildiz Technical University, İstanbul, Türkiye, alan@yildiz.edu.tr, ORCID: 0000-0003-2031-2725*
***Corresponding author**
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1. Introduction

Let p, q, and r be coprime positive integers greater than 1 and let us consider exponential Diophantine equation

$$p^x+q^y=r^z$$

with $x,y,z\in\mathbb{N}$. In 1956, Sierpiński demonstrated that by reformulating the Pythagorean theorem with exponential expressions as variables, the exponential Diophantine equation $3^x+4^y=5^z$ has a unique solution, $(x,y,z)=(2,2,2)$ [1]. Subsequently, Jeśmanowicz extended this idea to general Pythagorean triples, proposing that for positive integers a, b, and c satisfying the exponential Diophantine equation, the only solution remains $(2,2,2)$ [2].
In 1994, Terai extended this framework by considering the equation $p^x+q^y=r^z$ for positive integers p,q,r with $p,q,r\geq 2$. He conjectured that while multiple solutions may exist for some triples (p,q,r), only a few specific sets of such triples yield exceptions [3]. This conjecture has been verified for numerous specific cases, including particular forms of Diophantine equations

$$(ar^2+1)^x+(br^2-1)^y=(cr)^z. \tag{1.1}$$

In this study, the following exponential Diophantine equation equation is examined

$$(8r^2+1)^x+(r^2-1)^y=(3r)^z. \tag{1.2}$$

It is important to observe that equation (1.2) serves as a special case of equation (1.1), where the condition $a+b=c^2$ is fulfilled. This research was initiated based on Terai's conjecture. Expanding on this conjecture, various specific cases of equation (1.1) have been examined, resulting in the validation of Terai's conjecture in these instances.

- [4] $(4r^2+1)^p+(5r^2-1)^q=(3r)^t$
- [5] $(r^2+1)^p+(yr^2-1)^q=(zr)^t,\ 1+y=z^2$
- [6] $(12r^2+1)^p+(13r^2-1)^q=(5r)^t$
- [7] $(xr^2+1)^p+(yr^2-1)^q=(zr)^t\ , z|r$
- [8] $(xr^2+1)^p+(yr^2-1)^q=(zr)^t\ ,\ r=\pm 1 \pmod z$
- [9] $(18r^2+1)^p+(7r^2-1)^q=(5r)^t$
- [10] $((x+1)r^2+1)^p+(xr^2-1)^q=(zr)^t,\ 2x+1=z^2$
- [11] $(3xr^2-1)^p+(x(x-3)r^2+1)^q=(xr)^t$
- [12] $(4r^2+1)^p+(21r^2-1)^q=(5r)^t$
- [13] $(5xr^2-1)^p+(x(x-5)r^2+1)^q=(xr)^t$
- [14] $(3r^2+1)^p+(yr^2-1)^q=(zr)^t$
- [15] $(4r^2+1)^p+(45r^2-1)^q=(7r)^t$
- [16] $(6r^2+1)^p+(3r^2-1)^q=(3r)^t$
- [17] $(x(x-l)r^2+1)^p+(xlr^2-1)^q=(xr)^t$
- [18] $(44r^2+1)^p+(5r^2-1)^q=(7r)^t$
- [19] $(9r^2+1)^p+(16r^2-1)^q=(5r)^t$

For the Diophantine equations related to Recurrence sequences see [20], [21] and [22]. The exponential Diophantine equation (1.2), where r denotes a positive integer, is analyzed, and the following theorem is established.

Theorem 1.1. *Let r be a positive integer. The equation* (1.2) *possesses a single positive integer solution* $(x,y,z)=(1,1,2)$ *for any* $r>1$.

The theorem's proof relies on two approaches. The initial method, leveraging [23, 24], enables the derivation of additional potential solutions for the Diophantine equations $M^2+WN^2=q^K$ and $aM^2+bN^2=q^K$ from established solutions, subject to certain conditions [25, 26]. The second method draws upon an earlier rendition of the Primitive Divisor Theorem attributed to Zsigmondy [27].

2. Preliminaries

Consider a positive integer W. The notation $h(-4W)$ denotes the class number of positive binary quadratic forms with discriminant $-4W$.

Lemma 2.1. *([28], Theorems 11.4.3, 12.10.1 and 12.14.3])*

$$h(-4W)<\frac{4}{\pi}\sqrt{W}\log(2e\sqrt{W}).$$

Let W,W_1,W_2,q *be positive integers such that* $\min\{W,W_1,W_2\}>1$, $gcd(W_1,W_2)=1$, $2\nmid q$ *and* $gcd(W,q)=gcd(W_1,W_2,q)=1$.

Lemma 2.2. *[23] Given fixed relatively prime positive integers W and q, with $W>1$ and q being an odd integer, the equation is considered*

$$M^2+WN^2=q^K,$$

where $M,N,K\in\mathbb{Z}$, $K>0$ and $\gcd(M,N)=1$, has solutions (M,N,K) then any solution to the aforementioned equation can be represented as follows

$$M+N\sqrt{-W}=\lambda_1(M_1+\lambda_2 N_1\sqrt{-W})^t,\quad K=K_1t\quad \lambda_1,\lambda_2\in\{\pm1\}$$

M_1,N_1,K_1 are positive integers satisfying $M_1^2+WN_1^2=q^{K_1}$, $\gcd(M_1,N_1)=1$ and $h(-4W)\equiv 0 \pmod{K_1}$.

Lemma 2.3. *[23] Consider relatively prime positive integers W_1 and W_2, both greater than 1. Let (M,N,K) denote a fixed solution of the equation*

$$W_1M^2+W_2N^2=q^K. \tag{2.1}$$

Given that $K>0$, $\gcd(M,N)=1$, $2\nmid q$ and $M,N,K\in\mathbb{Z}$, there also exists a unique positive integer s such that

$$s=W_1\alpha M+W_2\beta N,\qquad 0<t<q$$

where α and β are integers such that $\beta M-\alpha N=1$ [[23], Lemma 1]. The positive integer s is referred to as the characteristic number of the specific solution (M,N,K) and is denoted by $<M,N,K>$. When $<M,N,K>=s$, it implies that $W_1M\equiv -sN \pmod q$ [[23], Lemma 6]. Let (M_0,N_0,K_0) be a solution to (2.1) with $<M_0,N_0,K_0>=s_0$. Therefore, the set of all solutions (M,N,K) with $<M,N,K>\equiv \pm s_0 \pmod q$ is termed a solution class of (2.1), expressed as $S(s_0)$.

Lemma 2.4. *[23] For each solution class $S(s_0)$ of (2.1), a unique solution exists $(M_1,N_1,K_1)\in S(s_0)$ such that M_1 and N_1 are positive, and $K_1\geq K$ for all solutions $(M,N,K)\in S(s_0)$, where K spans all possible solutions. This particular solution (M_1,N_1,K_1) is referred to as the least solution of $S(s_0)$. If (M,N,K) is a solution in the set $S(s_0)$ then*

$$K=K_1t,\ \ 2\nmid t,\ \ t\in\mathbb{N},$$

$$M\sqrt{W_1}+N\sqrt{W_2}=\lambda_1\left(M_1\sqrt{W_1}+\lambda_2N_1\sqrt{-W_2}\right)^t,\ \lambda_1,\lambda_2\in\{1,-1\}.$$

Lemma 2.5. *[24] Let (M_1,N_1,K_1) be the least solution of $S(s_0)$. If (2.1) has a solution $(M,N,K)\in S(s_0)$ satisfying $M>0$ and $N=1$, then $N_1=1$. Additionally, if $(M,K)\neq(M_1,K_1)$, in that case, at least one of the following conditions is satisfied*

(i) $W_1M_1^2=\frac{1}{4}(q^{K_1}\pm1),\qquad W_1=\frac{1}{4}(3q^{K_1}\pm1)$

$(M,K)=(M_1|W_1M_1^2-3W_2|,3K_1)$

(ii) $W_1K_1^2=\frac{1}{4}F_{3a+3\varepsilon},\quad W_2=\frac{1}{4}L_{3a},\ \ q^{K_1}=F_{3a+\varepsilon}$

$(M,K)=(M_1|W_1^2M_1^4-10W_1W_2M_1^2+5W_2^2|,\ 5K_1)$

where a is a positive integer, $\varepsilon\in\{1,-1\}$, and F_n is the n-th Fibonacci number in which each number is the sum of the two preceding ones.

Let γ and θ be algebraic integers. A Lucas pair refers to a pair (γ,θ) such that $\gamma+\theta$ and $\gamma\theta$ are non-zero relatively prime integers, and $\frac{\gamma}{\theta}$ is not a root of unity. For any given pair (γ,θ) forming a Lucas pair, the resulting sequences of Lucas numbers are given by

$$L_n(\gamma,\theta)=\frac{\gamma^n-\theta^n}{\gamma-\theta},\quad n=0,1,2,\ldots$$

It's worth noting that primitive divisors of $L_n(\gamma,\theta)$ are prime numbers p for which $p|L_n(\gamma,\theta)$ and $p\nmid(\gamma,\theta)^2L_1(\gamma,\theta)\ldots L_{n-1}(\gamma,\theta)$. For any Lucas sequence $L_n(\gamma,\theta)$ determined by a finite set of parameters (n,γ,θ), if $n\geq5$ and $n\neq6$, it is guaranteed that the sequence has always a primitive divisor.

Lemma 2.6. *[25] If $n>30$, then $L_n(\gamma,\theta)$ is guaranteed to have a primitive divisor.*

Lemma 2.7. *[26] For $4<n\leq 30$ and $n\neq 6$, aside from equivalence, $L_n(\gamma,\theta)$ contains a primitive divisor, except for the following pairs of parameters (k,l):*

- $(1,-15),(1,-11),(1,-7),(1,5),(2,-40),(12,-76)$ *or* $(12,-1364)$ *if* $n=5$,
- $(1,-19)$ *or* $(1,-7)$ *if* $n=7$,
- $(1,-7)$ *or* $(2,-24)$ *if* $n=8$,
- $(2,-8),(5,-47)$ *or* $(5,-3)$ *if* $n=10$,
- $(1,-19),(1,-15),(1,-11),(1,-7),(1,-5)$ *or* $(2,-56)$ *if* $n=12$,
- $(1,-7)$ *if* $n=13,18$ *or* 30. *where* $(\gamma,\theta)=(\frac{k+\sqrt{l}}{2},\frac{k-\sqrt{l}}{2})$.

Lemma 2.8. *[9] If a,b,c and $r>1$ are positive integers satisfying $a+b=c^2$, and $(x,y,z)\geq 0$ is a solution to the exponential Diophantine equation*

$$(ar^2+1)^x+(br^2-1)^y=(cr)^z,$$

where x is the larger of the two values $\{x,y\}$, In this case, the following inequalities are satisfied

$$\left(2-\frac{\log\left(\frac{c^2}{a}\right)}{\log(cr)}\right)x<z\leq 2x.$$

On the other hand, if y is the larger value, then

$$\left(2-\frac{\log\left(\frac{c^2r^2}{br^2-1}\right)}{\log(cr)}\right)y<z\leq 2y.$$

In particular, when $M=\max\{x,y\}>1$, it follows that

$$\left(2-\frac{\log\left(\frac{c^2}{\min\{a,b-\frac{1}{r^2}\}}\right)}{\log(cr)}\right)M<z<2M.$$

This offers a more precise description of the range of z based on M and the specified parameters.

Proposition 2.9. *[27] Consider C and D be relatively prime integers with $C>D\geq 1$. Let $\{a_n\}_{n\geq 1}$ be the sequence defined as*

$$a_n=C^n+D^n.$$

If $n>1$, then a_n has a prime factor not dividing $a_1a_2a_3\cdots a_{n-1}$, whenever $(C,D,n)\neq(2,3,1)$.

3. Proof of Theorem 1.1

3.1 The case $2|r$

This section demonstrates that Theorem 1.1 is valid under the condition $2 \mid r$.

Lemma 3.1. *If $2|r$, then $(x,y,z)=(1,1,2)$ constitutes the sole positive integer solution of the equation (1.2).*

Proof. For $z \leq 2$, it is evident that $(x,y,z)=(1,1,2)$ is the unique solution to equation (1.2). Thus, the assumption $z \geq 3$ is made. Considering equation (1.2) modulo r^2, the relation $1+(-1)^y \equiv 0 \pmod{r^2}$ holds, implying that y must be odd, given that $r^2 > 2$. Further, reducing equation (1.2) modulo r^3, the following is obtained

$$1+8r^2x+(-1)+r^2y \equiv 0 \pmod{r^3},$$

$$8x+y \equiv 0 \pmod{r},$$

which results in a contradiction, since y is odd and r is even. Therefore, it is concluded that equation (1.2) has no positive integer solutions for $z \geq 3$. Consequently, the only positive integer solution to equation (1.2) when r is even is $(1,1,2)$. The case where r is odd will now be considered. □

3.2 The case $2 \nmid r$ where $r \equiv 0 \pmod{3}$

This section demonstrates that Theorem 1.1 is valid under the condition $2 \nmid r$ where $r \equiv 0 \pmod{3}$.

Proof. Let (x,y,z) be any solution to equation (1.2). It is clear that $(x,y,z)=(1,1,2)$ constitutes a solution of (1.2). For $r>1$, examining equation (1.2) modulo r^2, it can be concluded, similar to the earlier scenario, that y must be odd. The investigation then continues by splitting into two cases depending on the parity of x. First, let us assume x is odd. Next, the focus turns to the Diophantine equation

$$(8r^2+1)M^2+(r^2-1)N^2=(3r)^K, \quad K>0 \quad \text{and} \quad M,N,K \in \mathbb{Z}. \tag{3.1}$$

Since (x,y,z) represents any solution of equation (1.2), it follows from Lemma 2.3 that

$$(M,N,K)=\left((8r^2+1)^{\frac{x-1}{2}},(r^2-1)^{\frac{y-1}{2}},z\right) \tag{3.2}$$

is a solution of equation (3.1). Let $s=\langle(8r^2+1)^{\frac{x-1}{2}},(r^2-1)^{\frac{y-1}{2}},z\rangle$ be the characteristic number corresponding to the solution given in (3.2). From the congruence

$$(8r^2+1)^{\frac{x+1}{2}} \equiv -s(r^2-1)^{\frac{y-1}{2}} \pmod{3r},$$

it follows that $s \equiv \pm 1 \pmod{3r}$.

It is noteworthy that $(M_1,N_1,K_1)=(1,1,2)$ also satisfies equation (3.1), and let $s_0=\langle 1,1,2\rangle$ denote the characteristic number of this solution. Hence, the following holds

$$8r^2+1 \equiv -s_0 \pmod{3r} \tag{3.3}$$

$$s_0 \equiv -1 \pmod{3r}$$

Thus, it is observed by the equation (3.3) $s \equiv \pm s_0 \pmod{3r}$, indicating that the solutions $(M_1,N_1,K_1)=(1,1,2)$ and the one given in (3.2) belong to the same solution class $S(s_0)$ of equation (3.1). Furthermore, $(M,N,K)=(1,1,2)$ is clearly the least solution within $S(s_0)$. Therefore, applying Lemma 2.4, it follows that

$$z=2t, \ 2 \nmid t, \ t \in \mathbb{N},$$

$$(8r^2+1)^{\frac{x-1}{2}}\sqrt{8r^2+1}+(r^2-1)^{\frac{y-1}{2}}\sqrt{1-r^2}=\lambda_1\left(\sqrt{8r^2+1}+\lambda_2\sqrt{1-r^2}\right)^t. \tag{3.4}$$

By expanding the right-hand side of equation (3.4) and equating the coefficients of $\sqrt{1-r^2}$, the following result is obtained

$$(r^2-1)^{\frac{y-1}{2}}=\lambda_1\lambda_2\sum_{i=0}^{\frac{t-1}{2}}\binom{t}{2i+1}(8r^2+1)^{\frac{t-1}{2}-i}(r^2-1)^i \tag{3.5}$$

At this point, it is asserted that $y=1$. Suppose $y>1$. From equation (3.5), it can be deduced that

$$0 \equiv \lambda_1\lambda_2 t\cdot(8r^2+1)^{\frac{t-1}{2}} \pmod{(r^2-1)}$$

$$0 \equiv \lambda_1\lambda_2 t\cdot 9^{\frac{t-1}{2}} \pmod{(r^2-1)}.$$

This leads to a contradiction, as $2\nmid t\cdot 9^{\frac{t-1}{2}}$ and $2\mid(r^2-1)$. Therefore, it is concluded that $y=1$, and consequently $N=(r^2-1)^{\frac{y-1}{2}}=1$. The two conditions in Lemma 2.5 will now be verified. Given that $(M_1,N_1,K_1)=(1,1,2)$ represents the smallest solution of $S(s_0)$, Lemma 2.5 implies that either

$$8r^2+1=\frac{1}{4}((3r)^2\pm 1)$$

or

$$F_{3a+\varepsilon}=(3r)^2$$

where $\varepsilon=\pm1$. The first equation leads to

$$4(8r^2+1)=(3^2r^2\pm1),$$

resulting in $4\equiv\pm1 \pmod{r^2}$, which is not possible. Moreover, since the only square Fibonacci number greater than 1 is $F_{12}=12^2$ [29], the second condition implies $3r=12$, which is also impossible due to the parity of r. Consequently, by Lemma 2.5, it follows that $(M,K)=((8r^2+1)^{\frac{t-1}{2}},z)=(M_1,K_1)=(1,2)$. Thus, equation (1.2) has no positive integer solutions other than $(x,y,z)=(1,1,2)$ when x is odd.

Next, the case when $2|x$ is considered. From equation (1.2), the Diophantine equation

$$M^2+(r^2-1)N^2=(3r)^K,\quad \gcd(M,N)=1,\quad K>0,$$

admits the solution

$$(M,N,K)=\left((8r^2+1)^{\frac{x}{2}},(r^2-1)^{\frac{y-1}{2}},z\right).$$

Hence, by Lemma 2.2, it is concluded that

$$z=K_1t,\quad t\in\mathbb{N}$$

$$(8r^2+1)^{\frac{x}{2}}+(r^2-1)^{\frac{y-1}{2}}\sqrt{1-r^2}=\lambda_1(M_1+\lambda_2N_1\sqrt{1-r^2})^t \tag{3.6}$$

where $\lambda_{1,2}\in\{-1,1\}$ and M_1,N_1,K_1 are positive integers satisfying

$$M_1^2+(r^2-1)N_1^2=(3r)^{K_1},\ \gcd(M_1,N_1)=1 \tag{3.7}$$

$$h(-4(r^2-1))\equiv 0 \pmod{K_1}. \tag{3.8}$$

Suppose that $2|t$ and let

$$M_2+N_2\sqrt{1-r^2}=(M_1+\lambda_2N_1\sqrt{1-r^2})^{\frac{t}{2}}. \tag{3.9}$$

By taking the norm of both sides of equation (3.8) in the field $\mathbb{Q}(\sqrt{1-r^2})$ and applying equation (3.7), the following result is obtained

$$M_2^2+(r^2-1)N_2^2=(3r)^{\frac{K_1t}{2}}=(3r)^{\frac{z}{2}}. \tag{3.10}$$

By substituting equation (3.9) into equation (3.6), the result is obtained as follows

$$(8r^2+1)^{\frac{x}{2}}+(r^2-1)^{\frac{y-1}{2}}\sqrt{1-r^2}=\lambda_1(M_2+N_2\sqrt{1-r^2})^2$$

and therefore it follows that

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1(M_2^2-N_2^2(r^2-1)), \tag{3.11}$$

$$(r^2-1)^{\frac{y-1}{2}}=2\lambda_1 M_2N_2. \tag{3.12}$$

Since $\gcd(8r^2+1,r^2-1)=1$, it follows from equations (3.11) and (3.12) that $|M_2|=1$. Thus, $|N_2|=\frac{1}{2}(r^2-1)^{\frac{y-1}{2}}$. Substituting $|M_2|$ and $|N_2|$ into equation (3.10), the result is

$$1+\frac{1}{4}(r^2-1)^y=(3r)^{\frac{z}{2}}$$

which leads to

$$3\equiv 0 \pmod{r^2}.$$

This presents a contradiction, leading to the conclusion that $2\nmid t$. Define

$$\gamma=M_1+N_1\sqrt{1-r^2},\quad \theta=M_1-N_1\sqrt{1-r^2}.$$

By taking the complex conjugate of equation (3.6), the following relation is obtained

$$(r^2-1)^{\frac{y-1}{2}}=N_1\left|\frac{\gamma^t-\theta^t}{\gamma-\theta}\right|=N_1|L_t(\gamma,\theta)|. \tag{3.13}$$

By equation (3.7), it holds that $\gamma+\theta=2M_1$, $\gamma-\theta=2N_1\sqrt{1-r^2}$, and $\gamma\theta=(3r)^{K_1}$. Since $\gcd(M_1,N_1)=1$, the integers $\gamma+\theta=2M_1$ and $\gamma\theta=(3r)^{K_1}$ are also relatively prime, as implied by equation (3.7), and $\frac{\gamma}{\theta}\neq\pm1$, with γ and θ being units in the ring of algebraic integers of $\mathbb{Q}(\sqrt{1-r^2})$. Consequently, $L_t(\gamma,\theta)$ forms a Lucas sequence.

From equation (3.13), it is evident that the Lucas numbers $L_t(\gamma,\theta)$ lack primitive divisors. By applying Lemma 2.6 and Lemma 2.7, it is concluded that $t\leq 30$. Furthermore, if $4<t\leq 30$ and $t\neq 6$, the parameters $(k,l)=(2M_1,4N_1^2(1-r^2))$ must match one of the parameter sets listed in Lemma 2.7. However, none of these sets align with the given parameters. Therefore, it follows that $t\leq 3$.

The case $t=3$ will be shown to be impossible. Assuming $t=3$, the right-hand side of equation (3.6) is expanded, and by equating the coefficients on both sides, it is determined that

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1 M_1(M_1^2-3(r^2-1)N_1^2) \tag{3.14}$$

$$(r^2-1)^{\frac{y-1}{2}}=\lambda_1\lambda_2 N_1(3M_1^2-(r^2-1)N_1^2). \tag{3.15}$$

From equation (3.7), it is evident that $\gcd(3M_1,r^2-1)=1$. Thus, from equation (3.15), the relation $3M_1^2-(r^2-1)N_1^2=\pm1$ holds. In fact, upon considering this equation modulo 3, it can be observed that only the positive sign is feasible, and the following equation is obtained

$$3M_1^2-(r^2-1)N_1^2=1. \tag{3.16}$$

Thus, it follows that

$$|N_1|=(r^2-1)^{\frac{y-1}{2}}. \tag{3.17}$$

By substituting equation (3.17) into equation (3.14), the following result is obtained

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1 M_1(M_1^2-3(r^2-1)^y) \tag{3.18}$$

By considering equations (3.16) and (3.17) modulo $3r$, it follows that $3M_1^2-(r^2-1)^y\equiv 0 \pmod{3r}$, which implies $M_1\equiv 1 \pmod{r}$. Substituting this result into equation (3.18) yields

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1 M_1(M_1^2-3(r^2-1)^y)$$

leading to

$$1\equiv 0 \pmod{r}$$

which is evidently a contradiction. Therefore, the only possibility remaining is $t=1$. Consequently, $z=W_1t=K_1$, and according to equation (3.8), it is established that $K_1\leq -4(r^2-1)$. Utilizing the upper bound provided by Lemma 2.1, the following result is obtained

$$z<\frac{4}{\pi}\sqrt{r^2-1}\,\log\,(2e\sqrt{r^2-1}). \tag{3.19}$$

Assume $z=3$. In this case, at least one of x or y must be greater than 1. If $x\geq 2$, it follows that $(3r)^3>(8r^2+1)^x\geq(8r^2+1)^2>8^2r^4$, leading to $3^3>8^2r$, which implies $64>27$, resulting in a contradiction. Similarly, if $(3r)^3>(r^2-1)^2+(8r^2+1)$, this also results in a contradiction. Thus, it can be concluded that $z\geq 4$. Examining equation (1.2) modulo r^4 leads to

$$(8r^2+1)^x+(r^2-1)^y\equiv 0 \pmod{r^4}$$

and hence

$$8x+y\equiv 0 \pmod{r^2}$$

$$r^2\leq 8x+y. \tag{3.20}$$

The application of the logarithm function facilitates the straightforward derivation of the inequalities $x<z$ and $y<1.06z$. Consequently, from inequality (3.20), it follows that $r^2<9.06z$. Therefore, from the derived inequality

$$r^2<9.06z<9.06\cdot\frac{4}{\pi}\sqrt{r^2-1}\log(2e\sqrt{r^2-1}),$$

it can be concluded that $r\leq 63$. Furthermore, by consulting Lemma 2.8, the following upper bounds for x and y can be established

$$1.94x<\left(2-\frac{\log\left(\frac{9}{8}\right)}{\log(9)}\right)x<\left(2-\frac{\log\left(\frac{9}{8}\right)}{\log(3r)}\right)x<z\leq 2x \tag{3.21}$$

$$0.95y<\left(2-\frac{\log(10)}{\log(9)}\right)y<\left(2-\frac{\log\left(\frac{10r^2-10}{r^2-1}\right)}{\log(9)}\right)y<\left(2-\frac{\log\left(\frac{9r^2}{r^2-1}\right)}{\log(3r)}\right)y<z\leq 2y. \tag{3.22}$$

Based on equations (3.21) and (3.22), it can be concluded that equation (1.2) has no solutions in positive integers for $z\leq 6$. Assuming $z>6$, the analysis of equation (1.2) proceeds by considering it modulo r^4, r^6, and r^8.

1. Modulo r^4: By considering equation (1.2) modulo r^4, the following congruence is obtained

$$8r^2x+r^2y\equiv 0 \pmod{r^4}.$$

In other words,

$$8x+y\equiv 0 \pmod{r^2}. \tag{3.23}$$

2. Modulo r^6: Taking equation (1.2) modulo r^6, the following congruence is obtained

$$8r^2x+8^2r^4\frac{x(x-1)}{2}+r^2y-r^4\frac{y(y-1)}{2}\equiv 0 \pmod{r^6}.$$

Simplifying,

$$8x+8^2r^2\frac{x(x-1)}{2}+y-r^2\frac{y(y-1)}{2}\equiv 0 \pmod{r^4}. \tag{3.24}$$

3. Modulo r^8: Finally, taking equation (1.2) modulo r^8, the following congruence is obtained

$$8r^2x+8^2r^4\frac{x(x-1)}{2}+8^3r^6\frac{x(x-1)(x-2)}{6}$$
$$+r^2y-r^4\frac{y(y-1)}{2}+r^6\frac{y(y-1)(y-2)}{6}\equiv 0 \pmod{r^8}.$$

Simplifying,

$$\begin{aligned}8x+8^2r^2\frac{x(x-1)}{2}+8^3r^4\frac{x(x-1)(x-2)}{6}\\+y-r^4\frac{y(y-1)}{2}+r^4\frac{y(y-1)(y-2)}{6}\equiv 0 \pmod{r^6}.\end{aligned} \tag{3.25}$$

In summary, equations (3.23), (3.24), and (3.25) represent the congruence conditions derived from equation (1.2) modulo r^2, r^4, and r^6, respectively. Utilizing equation (3.19) alongside the conditions $x,y<z$, and the congruences (3.23), (3.24), and (3.25), a brief computer program was developed using Maple to investigate all potential solutions of equation (1.2) within the range $3\le r\le 63$. The results show that there are no positive integer solutions (r,x,y,z) to equation (1.2) when $z\ge 3$. This concludes the proof. □

3.3 The case $r\nmid 2$ where $r\equiv\pm1 \pmod 3$

This section demonstrates that Theorem 1.1 is valid under the condition $r\nmid 2$ where $r\equiv\pm1 \pmod 3$.

Lemma 3.2. *If r is a positive odd integer such that $r\equiv\pm1 \pmod 3$, then equation (1.2) admits sole the positive integer solution $(x,y,z)=(1,1,2)$.*

Proof. Let k_1 and k_2 be positive integers, and consider the case where $r\equiv\pm1 \pmod 3$. In this context, equation (1.2) can be reformulated as follows

$$8r^2+1=3^{k_1}A, \qquad (8r^2+1)^x=3^{k_1x}A^x \tag{3.26}$$

$$r^2-1=3^{k_2}B, \qquad (r^2-1)^y=3^{k_2y}B^y \tag{3.27}$$

where $A,B\not\equiv 0 \pmod 3$. Then the equation (1.2) becomes

$$3^{k_1x}A^x+3^{k_2y}B^y=(3r)^z. \tag{3.28}$$

Firstly, let's consider $k_1x>k_2y$, then equation (3.28) can be written as

$$3^{k_2y}(3^{k_1x-k_2y}A^x+B^y)=3^zr^z$$

this implies that

$$k_2y=z \tag{3.29}$$

then equation (1.2) becomes

$$(8r^2+1)^x=((3r)^{k_2})^y-(r^2-1)^y.$$

Apply Proposition 2.9 , $y=1$ is found. When $y=1$ equation (3.27) turns into,

$$(r^2-1)^y=3^{k_2y}B^y=3^{k_2}B. \tag{3.30}$$

And substituting (3.29) into (3.30) with $y=1$

$$r^2=3^zB+1. \tag{3.31}$$

If $z \leq 2$, then $(x,y,z)=(1,1,2)$ is evidently the sole solution of equation (1.2). Therefore, let's assume $z=3$. Equation (1.2) becomes $(8r^2+1)^x+r^2-1=(3r)^3$. $x\geq 2$ gives $(3r)^3>(8r^2+1)^x\geq(8r^2+1)^2>8^2r^4$, and hence $3^3>8^2r>64$, a contradiction. Also it seen that $y=1$ and $x=1$, the equation (1.2) turns into $8r^2+1+r^2-1=(3r)^3$ also leads us a contradiction under the condition $r\equiv\pm1 \pmod 3$. Now, consider the scenario in which $z\geq 4$. Upon taking equation (1.2) modulo r^4, it becomes evident that $y=1$ as a result of Proposition 2.9 [27]. Consequently, the following congruence is established.

$$8r^2x+r^2\equiv 0 \pmod{r^4}.$$

This implies that

$$8x+1\equiv 0 \pmod{r^2}$$

$$r^2\leq 8x+1. \tag{3.32}$$

Substituting (3.31) into inequality (3.32), the following inequality is obtained.

$$3^zB\leq 8x. \tag{3.33}$$

Also x is bounded as $x<z$. So (3.33) turns into (3.34)

$$3^zB\leq 8x<8z$$

$$3^zB\leq 8z. \tag{3.34}$$

Consequently, it is evident that no positive integer z can satisfy the condition $z\geq 4$. Similarly, upon conducting a comparable analysis in the context where $k_2y>k_1x$, it becomes clear that no positive integer z can satisfy $z\geq 3$.

Finally, consider the scenario where $k_1x=k_2y$. By summing equations (3.26) and (3.27), the following relation is established.

$$9r^2=3^{k_1}A+3^{k_2}B. \tag{3.35}$$

An examination of this equation will proceed based on the various cases concerning the positive integers k_1 and k_2.

3.3.1 $k_1=2$ and $k_2\geq 3$

In the scenario where $k_1=2$, it is evident that k_2 must be even, given that y is odd. From equation (3.35), the following relationship can be established

$$2x=k_2y.$$

This implies the existence of a positive integer k_3 such that $2k_3=k_2$. Substituting this into the aforementioned equation yields $x=k_3y$. Consequently, equation (1.2) can be expressed as

$$((8r^2+1)^{k_3})^y+(r^2-1)^y=(3r)^z.$$

Applying Proposition 2.9, it follows that $y=1$. Therefore, it is concluded that no solutions exist for $x>2$.

3.3.2 $k_1 \geq 3$ and $k_2 = 2$

It can be expressed that

$$\frac{k_1}{k_2} = \frac{y}{x}$$

where $k_1 x = k_2 y$.Notably, since $\gcd(x,y) = 1$, if there exists an odd prime $p \geq 1$ such that $p \mid x$ and $p \mid y$, then, by Zsigmondy's Theorem, no solutions for x and y would exist. As a result, it follows that $x = 2$ and $k_2 = 2$, with y being an odd integer. Consequently, one can derive

$$y = k_1 \geq 3 \quad \text{and} \quad x = k_2 = 2.$$

Thus, equation (3.28) transforms into

$$3^{k_1 x} A^x + 3^{k_2 y} B^y = (3r)^z.$$

This further simplifies to:

$$3^{2y}(A^2 + B^y) = (3r)^z.$$

If $3 \nmid (A^2 + B^y)$, it follows that $2y = z$. Hence, equation (1.2) can be rewritten as

$$(8r^2+1)^x = ((3r)^2)^y - (r^2-1)^y.$$

Applying Zsigmondy's Proposition, it is concluded that $y = 1$, which leads to a contradiction. Thus, it can be stated that no positive integer solutions exist for x and y, and therefore, $z \leq 2$.
Assuming $3 \mid (A^2 + B^y)$, equations (3.26) and (3.27) can be expressed as

$$r^2 - 1 = 3^{k_2} B = 9B,$$

$$8r^2 + 1 = 3^{k_1} A.$$

Adding these two equations results in

$$9r^2 = 3^{k_1} + 9B.$$

Taking the equation modulo 3, it follows that

$$1 \equiv B \pmod{3}.$$

Consequently, it becomes evident that no positive integer A can satisfy the condition
$3 \mid (A^2 + B^y)$. This concludes the proof. □

4. Conclusion

This study investigates equation (1.1) with the parameters $(a,b,c) = (8,1,3)$, identifying the unique solution $(x,y,z) = (1,1,2)$ for $r > 1$. The findings provide additional evidence supporting Terai's Conjecture. The objective is to advance the understanding of such equations and contribute to the development of a generalized form.

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