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Hybrid numbers with balancing and Lucas balancing hybrid number coefficients

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Abstract. In this note, we made a connection between hybrid numbers and hybrid balancing, and hybrid Lucas balancing number. Initially, we obtained for these new number recurrence relation, some important relations among new numbers, and Binet's like formula. Afterwards, using Binet's like formula, we obtained some identities such as D'ocagne's identity, Honsberger identity, Catalan identity, and Cassini identity.

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1. Introduction

Triangular number sequences, Fibonacci, Lucas sequences, Pell sequences, related Pell sequences, and balancing number sequences are a few of the brilliant topics in the wonderful magnificence of integer sequences. Their fascination held both noviciates and professional mathematicians fascinated for a considerable amount of time.

Among the most renowned mathematicians, Diophantus of Alexandria (c. 200–300 B.C.) was well-known for his work in algebraic problem solving and number theory. Typically, he would use positive rational numbers to solve equations; however, these days, equations with integer solutions are known as Diophantine equations. Balancing number sequence which is denoted by $\{B_n\}_{n=0}^{\infty}$ was presented firstly by Behera and Panda in [2], in such a way that a balancing number B is the solution of the Diophantine equation

$$1 + 2 + 3 + ... + (B - 1) = (B + 1) + (B + 2) + ... + (B + R),$$

with R as a balancer corresponding to B. They gave functions which generate balancing numbers. The elements of the sequence $\{B_n\}_{n=0}^{\infty}$ were defined by the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \qquad n \geq 1,$$

with $B_0 = 0$ and $B_1 = 1$ initial elements, here nth balancing number is indicated as B_n. A few instances of numbers were mentioned as 6, 35 and 204 with balancers in order of 2, 14 and 84. Also they determined that n is a balancing number if and only if n^2 is a triangular number and n is a balancing number if and only if $8n^2 + 1$ is a perfect square.

Following the definition of balancing numbers, cobalancing numbers—which are determined by the Diophantine equation—were modified by Panda and Ray [16].

$$1 + 2 + 3 + ... + n = (n + 1) + (n + 2) + ... + (n + R).$$

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Here, $n \in Z^+$ is called the cobalancing number with cobalancer $R \in Z^+$ and the recurrence relation for cobalancing numbers is $b_{n+1} = 6b_n - b_{n-1} + 2, \quad n \geq 2$. All cobalancing numbers are even, all cobalancers are balancing numbers, and all balancers are cobalancing numbers, as they proved.

Numerous mathematicians worked on this problem after the foundation of balancing and cobalancing numbers emerged in [3, 5, 6, 8, 9, 11–14, 18–21]. The generalization of balancing numbers was made and studied in [3, 13, 14]. The relationships between balancing and cobalancing numbers were then formed with a different kind of numbers in [17]. Pell numbers as well as related Pell numbers. A remarkable result was made: the nth balancing number is produced by multiplying the nth Pell number by the nth associated Pell number.

Moreover, Liptai proved that neither the Lucas sequence term in [12] nor Fibonacci balancing numbers exist In [11]. But Panda associated nth balancing number B_n with C_n, as in the way of the link between Fibonacci number F_n and Lucas number L_n, in [19]. But Panda associated nth balancing number B_n with C_n, as in the way of the link between Fibonacci number F_n and Lucas number L_n, in [19]. This identity, which establishes the relationship between nth balancing number B_n and nth Lucas-balancing number C_n, is $C_n = \sqrt{8B_n^2 + 1}$. The recurrence relation also defined the Lucas-balancing sequence $\{C_n\}_{n=0}^{\infty}$.

$$C_{n+1} = 6C_n - C_{n-1}, \qquad n \geq 1, \tag{2}$$

where $C_0 = 1$ and $C_1 = 3$ are initial terms.

Additionally, balancing and Lucas-balancing numbers were studied in [22,24] using matrices. Balancing and Lucas-balancing numbers were generated by using powers of matrices and many new and known formulas for these numbers can be presented by the matrix representation. The descriptions and several properties of these matrices were given.

On the other hand, Ozdemir recently defined a new class of numbers in [15]. These numbers belong to the non-commutative number system and are known to as hybrid numbers. Hybrid numbers is as the generalization of dual, hyperbolic, and complex numbers. A hybrid number can be combined by these numbers and encounters the relation $ih = -hi = i + \varepsilon$ if the units for complex, hyperbolic, and dual numbers are i, h, and ε. The definition of the hybrid number set is

$$\mathbb{K} = \left\{ a + bi + c\varepsilon + dh \quad | \, a, b, c, d \in \mathbb{R}, \; i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = i + \varepsilon \right\}. \tag{3}$$

For hybrid numbers unit multiplication table are as following:

.	**1**	**i**	ϵ	**h**
1	1	i	ϵ	h
i	i	-1	1-h	ϵ+i
ϵ	ϵ	h+1	0	-ϵ
h	h	-ϵ-i	ϵ	1

The set $\mathbb{K}$ was determined to scalar operations for multiplication, subtraction, and equality. For addition operations, both association and commutation identities are provided; for multiplication operations, association identities are offered but commutation identities are not. The roots of a hybrid number were found according to its type, and the hybrid numbers were classified.

Researchers in the field of number theory are interested in hybrid numbers since they are defined as the generalization of three different number types. In [26, 28], Horadam hybrid numbers were investigated. For these numbers, the Binet formula, the Poisson and exponential generating functions, a few identities, the Honsberger formulas, and a few summation formulas were given. In more recent times, [27], [10], and [7], respectively, defined and characterized Jacobsthal and Jacobsthal-Lucas hybrid numbers, Fibonacci hybrid and Lucas hybrid numbers, and k-Fibonacci and k-Lucas hybrid numbers. For these numbers, the Binet formula, the Poisson and exponential generating functions, a few identities, the Honsberger formulas, and a few summation formulas were given. In addition, the hybrid numbers were investigated by Alp and Kocer [1] in terms of Leonardo numbers, which are connected to Fibonacci numbers and have the formula $Le_n = 2F_{n+1} - 1$, where nth is the Fibonacci number F_n and nth is the Leonardo number Le_n.

In [29] were studied combinating balancing numbers, Lucas-balancing numbers and hybrid numbers. They imputed the properties of hybrid balancing and hybrid Lucas-balancing numbers and establish some relations between balancing number types and hybrid numbers. Additionally, special identities which are

Cassini, Catalan identity and Binet's formula were obtained. Also generating function for these numbers are given. Finally, they illustrated the identities with examples. We motivated particularly by this paper.

Definition 1. *Let B_n and C_n denote the nth balancing and Lucas-balancing numbers, respectively. The nth hybrid balancing number is defined as*

$$HB_n = B_n + B_{n+1}i + B_{n+2}\varepsilon + B_{n+3}h, \tag{1}$$

and the nth hybrid Lucas-balancing number is defined as

$$HC_n = C_n + C_{n+1}i + C_{n+2}\varepsilon + C_{n+3}h, \tag{2}$$

where i, ε and h are the units of the hybrid number set $\mathbb{K}$.

Some examples for HB_n and HC_n are as following:

	Hybrid Balancing(HB_n)	**Hybrid Lucas Balancing(HC_n)**
n=0	i+6ϵ+35h	1+3i+17ϵ+99h
n=1	1+ 6i+35ϵ+204h	3+ 17i+99ϵ+577h
n=2	6+35i+204ϵ1189h	17+99i+577ϵ+3363h
n=3	35+204 i+1189ϵ+6930h	99+ 577i+3363ϵ+19601h
n=4	204+1189 i+6930ϵ+40391h	577+3363i+19601ϵ+114243h

Let n$\geq$ 2 be an integer. Then

$$(i)\ HB_n = 6HB_{n-1} - HB_{n-2}. \tag{3}$$

$$(ii)\ HC_n = 6HC_{n-1} - HC_{n-2}.$$

Let n$\geq$ 1 be an integer. Then

$$HC_n = 3HB_n - HB_{n-1}.$$

Let n$\geq$ 2 be an integer. Then

$$HB_{n+2} - HB_{n-2} = 12HC_n.$$

For n$\geq$ 2 we have following identities

$$(i)\ HB_n = 3HB_{n-1} + HC_{n-1}. \tag{4}$$

$$(ii)\ HC_n = 8HB_{n-1} + 3HC_{n-1.} \tag{5}$$

$$(iii)\ 2HC_n = HB_{n+1} - HB_{n-1}. \tag{6}$$

$$(iv)\ HC_n - HC_{n-1} = 2(HB_{n-1} + HB_n). \tag{7}$$

Let n$\geq$ 0 be an integer. Then

$$HB_n = \frac{\overline{\Phi}\Phi^n - \overline{\Psi}\Psi^n}{\Phi - \Psi}. \tag{8}$$

$$HC_n = \frac{\overline{\Phi}\Phi^n + \overline{\Psi}\Psi^n}{2}. \tag{9}$$

Where $\overline{\Phi}$ and $\overline{\Psi}$ are defined by

$$\overline{\Phi} = 1 + \Phi i + \Phi^2\varepsilon + \Phi^3 h. \tag{10}$$

$$\overline{\Psi} = 1 + \Psi i + \Psi^2\varepsilon + \Psi^3 h. \tag{11}$$

2. Hybrid Numbers with Balancing and Lucas balancing Hybrid Number Coefficients

Definition 2. *Let $\mathbb{B}_n$ and $\mathbb{C}_n$ denote the n-th Hybrid Numbers with Balancing and Lucas balancing Hybrid Number Coefficients, respectively. These numbers are defined as:*

$$\mathbb{B}_n = HB_n + HB_{n+1}i + HB_{n+2}\varepsilon + HB_{n+3}h, \tag{12}$$

$$\mathbb{C}_n = HC_n + HC_{n+1}i + HC_{n+2}\varepsilon + HC_{n+3}h, \tag{13}$$

where i, ε and h are the units of the hybrid number set $\mathbb{K}$.

Remark 1. *If we expand the definition of $\mathbb{B}_n$ and $\mathbb{C}_n$, we obtain*

$$\mathbb{B}_n = B_n + 2B_{n+3} + B_{n+6} - B_{n+2} + 2(B_{n+1})i + 2(B_{n+2})\varepsilon + 2B_{n+3}h.$$

$$\mathbb{C}_n = C_n + 2C_{n+3} + C_{n+6} - C_{n+2} + 2(C_{n+1})i + 2(C_{n+2})\varepsilon + 2C_{n+3}h.$$

Theorem 1. *Let $n \geq 2$ be an integer. Then*

$$(i)\mathbb{B}_n = 6\mathbb{B}_{n-1} - \mathbb{B}_{n-2}.$$
$$(ii)\mathbb{C}_n = 6\mathbb{C}_{n-1} - \mathbb{C}_{n-2}.$$

Proof. (i) Using (12) and (3), we have

$$\begin{aligned} 6\mathbb{B}_{n-1} - \mathbb{B}_{n-2} &= 6\left(HB_{n-1} + HB_n i + HB_{n+1}\varepsilon + HB_{n+2}h\right) \\ &- \left(HB_{n-2} + HB_{n-1}i + HB_n\varepsilon + HB_{n+1}h\right), \\ &= (6HB_{n-1} - HB_{n-2}) + i(6HB_n - HB_{n-1}) \\ &\quad +\varepsilon(6HB_{n+1} - HB_n) + h(6HB_{n+2} - HB_{n+1}), \\ &= \mathbb{B}_n. \end{aligned}$$

We similarly can show proof of (ii). □

Theorem 2. *Let $n \geq 1$ be an integer. Then*

$$\mathbb{C}_n = 3\mathbb{B}_n - \mathbb{B}_{n-1}.$$

Proof. By identity (12), (1) and (13), we get

$$\begin{aligned} 3\mathbb{B}_n - \mathbb{B}_{n-1} &= 3(HB_n + iHB_{n+1} + \varepsilon HB_{n+2} + hBH_{n+3}) \\ &\quad -HB_{n-1} - iHB_n - \varepsilon HB_{n+1} - hHB_{n+2}, \\ &= 3HB_n - HB_{n-1} + i(3HB_{n+1} - HB_n) + \varepsilon(3HB_{n+2} - HB_{n+1}) + h(3HB_{n+3} - HB_{n+2}), \\ &= HC_n + iHC_{n+1} + \varepsilon HC_{n+2} + hHC_{n+3}, \\ &= \mathbb{C}_n. \end{aligned}$$

□

Theorem 3. *Let $n \geq 2$ be an integer. Then*

$$\mathbb{B}_{n+2} - \mathbb{B}_{n-2} = 12\mathbb{C}_n.$$

Proof. Using (12), (1) and (13), we obtain

$$\begin{aligned} \mathbb{B}_{n+2} - \mathbb{B}_{n-2} &= (HB_{n+2} + iHB_{n+3} + \varepsilon HB_{n+4} + hHB_{n+5}) \\ &\quad - \left(HB_{n-2} - iHB_{n-1} - \varepsilon HB_n - hHB_{n+1}\right), \\ &= HB_{n+2} - HB_{n-2} + i(HB_{n+3} - HB_{n-1}) \\ &\quad +\varepsilon(HB_{n+4} - HB_n) + h(HB_{n+5} - HB_{n+1}), \\ &= 12HC_n + i12HC_{n+1} + \varepsilon 12HC_{n+2} + h12HC_{n+3}, \\ &= 12(HC_n + iHC_{n+1} + \varepsilon HC_{n+2} + hHC_{n+3}), \\ &= 12\mathbb{C}_n. \end{aligned}$$

□

Theorem 4. *For n≥ 2 we have following identities*

$$(i)\mathbb{B}_n = 3\mathbb{B}_{n-1} + \mathbb{C}_{n-1}. \tag{14}$$

$$(ii)\mathbb{C}_n = 8\mathbb{B}_{n-1} + 3\mathbb{C}_{n-1}. \tag{15}$$

$$(iii)2\mathbb{C}_n = \mathbb{B}_{n+1} - \mathbb{B}_{n-1}. \tag{16}$$

$$(iv)\mathbb{C}_n - \mathbb{C}_{n-1} = 2(\mathbb{B}_{n-1} + B_n). \tag{17}$$

Proof. Using (12), (13) and (4) in proof in (i), we have

$$\begin{aligned} 3\mathbb{B}_{n-1} + \mathbb{C}_{n-1} &= 3\left(HB_{n-1} + iHB_n + \varepsilon HB_{n+1} + hHB_{n+2}\right) \\ &+ \left(HC_{n-1} + iHC_n + \varepsilon HC_{n+1} + hHC_{n+2}\right), \\ &= 3HB_{n-1} + HC_{n-1} + (3HB_n + HC_n)i + (3HB_{n+1} + HC_{n+1})\varepsilon + (3HB_{n+1} + HC_{n+1})h, \\ &= HB_n + HB_{n+1}i + HB_{n+2}\varepsilon + HB_{n+3}h, \\ &= \mathbb{B}_n. \end{aligned}$$

Similarly, we can show (ii), (iii), and (iv).

□

The next theorem gives Binet's formulas for Hybrid Numbers with Balancing and Lucas Balancing Hybrid Number Coefficients, respectively.

Theorem 5. *Let $n \geq 0$ be an integer. $\mathbb{B}_n$ and $\mathbb{C}_n$ are Hybrid Numbers with Balancing and Lucas Balancing Hybrid Number Coefficients. For n≥ 0 Binet's formula for these numbers, respectively, is as follows:*

$$\mathbb{B}_n = \frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}. \tag{18}$$

$$\mathbb{C}_n = \frac{\overline{\Phi}^2\Phi^n + \overline{\Psi}^2\Psi^n}{2}. \tag{19}$$

Where $\overline{\Phi}$ and $\overline{\Psi}$ are defined by

$$\overline{\Phi} = 1 + \Phi i + \Phi^2\varepsilon + \Phi^3 h \tag{20}$$

$$\overline{\Psi} = 1 + \Psi i + \Psi^2\varepsilon + \Psi^3 h \tag{21}$$

respectively.

Proof. Using (12) and (8). We find

$$\begin{aligned} \mathbb{B}_n &= HB_n + HB_{n+1}i + HB_{n+2}\varepsilon + HB_{n+3}h, \\ &= \left(\frac{\overline{\Phi}\Phi^n - \overline{\Psi}\Psi^n}{\Phi - \Psi}\right) + \left(\frac{\overline{\Phi}\Phi^{n+1} - \overline{\Psi}\Psi^{n+1}}{\Phi - \Psi}\right)i + \left(\frac{\overline{\Phi}\Phi^{n+2} - \overline{\Psi}\Psi^{n+2}}{\Phi - \Psi}\right)\varepsilon + \left(\frac{\overline{\Phi}\Phi^{n+3} - \overline{\Psi}\Psi^{n+3}}{\Phi - \Psi}\right)h \\ &= \frac{\left(1 + \Phi i + \Phi^2\varepsilon + \Phi^3 h\right)\overline{\Phi}\Phi^n - \left(1 + \Psi i + \Psi^2\varepsilon + \Psi^3 h\right)\overline{\Psi}\Psi^n}{\Phi - \Psi}, \\ &= \frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}. \end{aligned}$$

Similarly, If we use (13) and (9) we find Binet formula for this number

$$\begin{aligned} \mathbb{C}_n &= HC_n + HC_{n+1}i + HC_{n+2}\varepsilon + HC_{n+3}h, \\ &= \left(\frac{\overline{\Phi}\Phi^n + \overline{\Psi}\Psi^n}{2}\right) + \left(\frac{\overline{\Phi}\Phi^{n+1} + \overline{\Psi}\Psi^{n+1}}{2}\right)i + \left(\frac{\overline{\Phi}\Phi^{n+2} + \overline{\Psi}\Psi^{n+2}}{2}\right)\varepsilon + \left(\frac{\overline{\Phi}\Phi^{n+3} + \overline{\Psi}\Psi^{n+3}}{2}\right)h \\ &= \frac{\left(1 + \Phi i + \Phi^2\varepsilon + \Phi^3 h\right)\overline{\Phi}\Phi^n + \left(1 + \Psi i + \Psi^2\varepsilon + \Psi^3 h\right)\overline{\Psi}\Psi^n}{2}, \\ &= \frac{\overline{\Phi}^2\Phi^n + \overline{\Psi}^2\Psi^n}{2}. \end{aligned}$$

□

Theorem 6. *Let $n \geq 0$ be an integer. $\mathbb{B}_n$ and $\mathbb{C}_n$ are Hybrid Numbers with Balancing and Lucas Balancing Hybrid Number Coefficients. Then*

$$\mathbb{C}_n^2 - 8\mathbb{B}_n^2 = \frac{\overline{\Phi}^2\overline{\Psi}^2 + \overline{\Psi}^2\overline{\Phi}^2}{2}.$$

Proof. Using (18) and (19)

$$\begin{aligned}
\mathbb{C}_n^2 - 8\mathbb{B}_n^2 &= \left(\frac{\overline{\Phi}^2\Phi^n + \overline{\Psi}^2\Psi^n}{2}\right)\left(\frac{\overline{\Phi}^2\Phi^n + \overline{\Psi}^2\Psi^n}{2}\right), \\
&- 8\left(\frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}\right)\left(\frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}\right), \\
&= \frac{\overline{\Phi}^2\overline{\Psi}^2 + \overline{\Psi}^2\overline{\Phi}^2}{2}.
\end{aligned}$$

□

Theorem 7. *For $n, m \geq 0$, $\mathbb{B}_n$ is n-th Hybrid Numbers with Balancing Hybrid Number Coefficients, d'Ocagne's identity is as follows:*

$$\mathbb{B}_n\mathbb{B}_{m+1} - \mathbb{B}_{n+1}\mathbb{B}_m = \frac{\overline{\Phi}^2\overline{\Psi}^2\Phi^n\Psi^m(\Phi - \Psi) - \overline{\Psi}^2\overline{\Phi}^2\Phi^m\Psi^n(\Phi - \Psi)}{(\Phi - \Psi)}. \tag{22}$$

Proof. Using (18), we get

$$\begin{aligned}
\mathbb{B}_n\mathbb{B}_{m+1} - \mathbb{B}_{n+1}\mathbb{B}_m &= \left(\frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}\right)\left(\frac{\overline{\Phi}^2\Phi^{m+1} - \overline{\Psi}^2\Psi^{m+1}}{\Phi - \Psi}\right) \\
&- \left(\frac{\overline{\Phi}^2\Phi^{n+1} - \overline{\Psi}^2\Psi^{n+1}}{\Phi - \Psi}\right)\left(\frac{\overline{\Phi}^2\Phi^m - \overline{\Psi}^2\Psi^m}{\Phi - \Psi}\right), \\
&= \frac{\overline{\Phi}^2\overline{\Psi}^2\Phi^n\Psi^m(\Phi - \Psi) - \overline{\Psi}^2\overline{\Phi}^2\Phi^m\Psi^n(\Phi - \Psi)}{(\Phi - \Psi)}.
\end{aligned}$$

□

Theorem 8. *For $n, m \geq 0$, $\mathbb{C}_n$ is n-th Hybrid Numbers with Lucas Balancing Hybrid Number Coefficients, d'Ocagne's identity is as follows:*

$$\mathbb{C}_n\mathbb{C}_{m+1} - \mathbb{C}_{n+1}\mathbb{C}_m = \frac{(\Psi - \Phi)\left(\overline{\Phi}^2\overline{\Psi}^2\Psi^{m-n} - \overline{\Psi}^2\overline{\Phi}^2\Phi^{m-n}\right)}{4}. \tag{23}$$

We can obtain proof of (23) similary as (22).

Theorem 9. *For $n \geq r$, $\mathbb{B}_n$ is n-th Hybrid Numbers with Balancing Hybrid Number Coefficients, Catalan identity is as follows:*

$$\mathbb{B}_n^2 - \mathbb{B}_{n+r}\mathbb{B}_{n-r} = \frac{\overline{\Phi}^2\overline{\Psi}^2\left(\Phi^r\Psi^{-r} - 1\right) + \overline{\Psi}^2\overline{\Phi}^2\left(\Phi^{-r}\Psi^r - 1\right)}{(\Phi - \Psi)^2}. \tag{24}$$

Proof. Using (18)

$$\begin{aligned}
\mathbb{B}_n^2 - \mathbb{B}_{n+r}\mathbb{B}_{n-r} &= \left(\frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}\right)\left(\frac{\overline{\Phi}^2\Phi^n - \overline{\Psi}^2\Psi^n}{\Phi - \Psi}\right) \\
&- \left(\frac{\overline{\Phi}^2\Phi^{n+r} - \overline{\Psi}^2\Psi^{n+r}}{\Phi - \Psi}\right)\left(\frac{\overline{\Phi}^2\Phi^{n-r} - \overline{\Psi}^2\Psi^{n-r}}{\Phi - \Psi}\right), \\
&= \frac{\overline{\Phi}^2\overline{\Psi}^2\left(\Phi^r\Psi^{-r} - 1\right) + \overline{\Psi}^2\overline{\Phi}^2\left(\Phi^{-r}\Psi^r - 1\right)}{(\Phi - \Psi)^2}.
\end{aligned}$$

□

Theorem 10. *For $n \geq r$, $\mathbb{B}_n$ is n-th Hybrid Numbers with Lucas Balancing Hybrid Number Coefficients, Catalan identity is as follows:*

$$\mathbb{C}_n^2 - \mathbb{C}_{n+r}\mathbb{C}_{n-r} = \frac{\overline{\Phi}^2\overline{\Psi}^2\left(1-\Phi^r\Psi^{-r}\right)+\overline{\Psi}^2\overline{\Phi}^2\left(1-\Phi^{-r}\Psi^r\right)}{4}. \tag{25}$$

We can obtain proof of (25) similary as (24).

If we take r = 1 on Catalan identity, we will obtain Cassini's identity as follows.

Theorem 11. *For $r = 1$, $\mathbb{B}_n$ is n-th Hybrid Numbers with Balancing Hybrid Number Coefficients, Cassini identity is as follows:*

$$\mathbb{B}_n^2 - \mathbb{B}_{n+1}\mathbb{B}_{n-1} = \frac{\overline{\Phi}^2\overline{\Psi}^2\left(\Phi^1\Psi^{-1}-1\right)+\overline{\Psi}^2\overline{\Phi}^2\left(\Phi^{-1}\Psi^1-1\right)}{\left(\Phi-\Psi\right)^2}. \tag{26}$$

Theorem 12. *For $r = 1$, $\mathbb{B}_n$ is n-th Hybrid Numbers with Lucas Balancing Hybrid Number Coefficients, Cassini identity is as follows:*

$$\mathbb{C}_n^2 - \mathbb{C}_{n+1}\mathbb{C}_{n-1} = \frac{\overline{\Phi}^2\overline{\Psi}^2\left(1-\Phi^1\Psi^{-1}\right)+\overline{\Psi}^2\overline{\Phi}^2\left(1-\Phi^{-1}\Psi^r\right)}{4}. \tag{27}$$

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