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Qualitative Behavior of Solutions of a Two-Dimensional Rational System of Difference Equations

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Equilibrium point, Periodicity, Rate of convergence, Stability, System of difference equations

Abstract — In this study, the rational system

$$x_{n+1}=\frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 x_n},\quad n\in\mathbb{N}_0,$$

where α_i, β_i, a_i, b_i, $(i=1,2)$, and x_{-j}, y_{-j}, $(j=0,1)$, are positive real numbers, is defined and its qualitative behavior is discussed. The system in question is a two-dimensional extension of an old difference equation in the literature. The results obtained generalize the results in the literature on the equation in question.

Subject Classification (2020): 39A20, 39A23, 39A30.

1. Introduction

Difference equations have occurred in many scientific areas such as biology, physics, engineering, and economics. Particularly, rational difference equations and their systems have great importance in applications. See [4, 11, 23, 24]. As a natural consequence of this, it is very worthy to examine the qualitative analyses of such equations and their systems. Over the past two decades, many studies have been published on the qualitative behavior of difference equations and systems. For example, see [1–3, 5, 6, 8–10, 12–15, 21, 22, 25, 29, 30, 32, 34, 36, 38, 40–42, 44] and therein references. Below, we present a prototype, among others, that caught our attention, along with its two extensions. Gibbons et al. [16] analyzed the boundedness, the oscillatory and periodicity, and the global stability of the nonnegative solutions of the rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{\gamma+x_n},\quad n\in\mathbb{N}_0, \tag{1.1}$$

where the parameters α, β and γ are nonnegative and real. Din et al. [8] investigated the boundedness, the local and global stability, the periodicity, and the rate of convergence of positive solutions of the system of difference equations

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_{n-1}}{a_1+b_1 y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 y_{n-1}}{a_2+b_2 x_n},\quad n\in\mathbb{N}_0, \tag{1.2}$$

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where α_i, β_i, a_i, b_i, $(i = 1,2)$, and x_{-j}, y_{-j}, $(j = 0,1)$, are positive real numbers. Din [10] investigated the boundedness, the local and global stability behavior, the periodicity, and the rate of convergence of positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 y_n}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where α_i, β_i, a_i, b_i, $(i = 1,2)$, and x_{-j}, y_{-j}, $(j = 0,1)$, are positive real numbers.

Studies on the qualitative behavior of the difference equations and systems still continue actively. For recent studies, see, for example [7, 17–20, 26–28, 32, 33, 35, 37, 39, 43] and therein references.

The systems in (1.2) and (1.3) are two-dimensional symmetric extensions of (1.1). Apart from these, there is another two-dimensional symmetric extension of (1.1). In this paper, we define the aforementioned extension of (1.1). That is, we define the rational system

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 x_n}, \tag{1.4}$$

where α_i, β_i, a_i, b_i, $(i = 1,2)$ are positive real parameters, and x_{-j}, y_{-j}, $(j = 0,1)$ are positive real initial conditions, and discuss qualitative behavior of its solutions. More concretely, we investigate existence of a unique positive equilibrium, local and global stability of the equilibrium, rate of convergence of a solution converging to the equilibrium, existence of unbounded solutions and the periodicity of solutions.

2. Preliminaries

Assume that I, J are some intervals of real numbers and

$$f_1 : I^2 \times J^2 \to I, \quad f_2 : I^2 \times J^2 \to J$$

are continuously differentiable functions. Then, for every set of initial conditions $x_{-1}, x_0 \in I$ and $y_{-1}, y_0 \in J$, the system of difference equations

$$x_{n+1} = f_1\left(x_n, x_{n-1}, y_n, y_{n-1}\right), \quad y_{n+1} = f_2\left(x_n, x_{n-1}, y_n, y_{n-1}\right), \quad n \in \mathbb{N}_0, \tag{2.1}$$

has a unique solution denoted by $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$. An equilibrium point of system (2.1) is a point $\left(\overline{x}, \overline{y}\right) \in I \times J$ that satisfies

$$\overline{x} = f_1\left(\overline{x}, \overline{x}, \overline{y}, \overline{y}\right), \quad \overline{y} = f_2\left(\overline{x}, \overline{x}, \overline{y}, \overline{y}\right).$$

For stability analysis, we use some key results of the multivariable calculus. Hence we transform system (2.1) into the vector system

$$X_{n+1} = F\left(X_n\right), \quad n \in \mathbb{N}_0, \tag{2.2}$$

where $X_n = \left(x_n, y_n, x_{n-1}, y_{n-1}\right)^T$, F is a vector map such that $F : I^2 \times J^2 \to I^2 \times J^2$ and

$$F\begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} f_1\left(x_n, y_n, x_{n-1}, y_{n-1}\right) \\ f_2\left(x_n, y_n, x_{n-1}, y_{n-1}\right) \\ x_n \\ y_n \end{pmatrix}.$$

It is obvious that if an equilibrium point of system (2.1) is $(\overline{x}, \overline{y})$, then the corresponding equilibrium point of system (2.2) is the point $\overline{X} = (\overline{x}, \overline{y}, \overline{x}, \overline{y})^T$.

By $\|\cdot\|$, we denote any convenient vector norm and the corresponding matrix norm. Also, $X_0 \in I \times J \times I \times J$ is an initial condition of the vector system (2.2) corresponding to the initial conditions $x_{-1}, x_0 \in I$ and $y_{-1}, y_0 \in J$ of system (2.1).

Definition 2.1. [23] Let $\overline{X}$ be an equilibrium of system (2.2). Then,

i) The equilibrium $\overline{X}$ is called stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\left\|X_0 - \overline{X}\right\| < \delta$ implies $\left\|X_n - \overline{X}\right\| < \epsilon$, for all $n \geq 0$. Otherwise, the equilibrium point $\overline{X}$ is called unstable.

ii) The equilibrium $\overline{X}$ is called locally asymptotically stable if it is stable and there exists $\gamma > 0$ such that $\left\|X_0 - \overline{X}\right\| < \gamma$ and $X_n \to \overline{X}$ as $n \to \infty$.

iii) The equilibrium $\overline{X}$ is called a global attractor if $X_n \to \overline{X}$ as $n \to \infty$.

iv) The equilibrium $\overline{X}$ is called globally asymptotically stable if it is both locally asymptotically stable and global attractor.

The linearized system of (2.2) about the equilibrium $\overline{X}$ is of the form

$$Z_{n+1} = J_F Z_n, \quad n \in \mathbb{N}_0, \tag{2.3}$$

where J_F is the Jacobian of the map F at the equilibrium $\overline{X}$. The characteristic polynomial of (2.3) at the equilibrium $\overline{X}$ is

$$P(\lambda) = a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{2.4}$$

with real coefficients and $a_0 > 0$.

Theorem 2.2. [23] Let $\overline{X}$ be any equilibrium of (2.2). If all eigenvalues of J_F at $\overline{X}$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{X}$ is local asymptotically stable. If one of the eigenvalues has a modulus greater than one, then the equilibrium point $\overline{X}$ is unstable.

The next results deal with the rate of convergence for a solution converging to an equilibrium of a system of difference equations. See [11, 31] for more details.

Consider the system of difference equations

$$X_{n+1} = (A + B_n) X_n, \quad n \in \mathbb{N}_0, \tag{2.5}$$

where X_n is an $m-$dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : Z^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B_n\| \to 0 \tag{2.6}$$

as $n \to \infty$.

Theorem 2.3 (Perron's First Theorem)**.** Suppose that condition (2.6) holds. If X_n is a solution of (2.5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{2.7}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 2.4 (Perron's Second Theorem)**.** Suppose that condition (2.6) holds. If X_n is a solution of (2.5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \tag{2.8}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

The following lemma is the second part of Lemma 3.1 in [30].

Lemma 2.5. Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions and a_1, b_1, a_2, b_2 be positive numbers such that $a_1 < b_1$, $a_2 < b_2$. Suppose that

$$f : [a_2, b_2] \times [a_2, b_2] \to [a_1, b_1], \quad g : [a_1, b_1] \times [a_1, b_1] \to [a_2, b_2].$$

In addition, assume that $f(u, v)$ is a decreasing (resp. increasing) function with respect to u (resp. v) for every v (resp. u) and $g(z, w)$ is a decreasing (resp. increasing) function with respect to z (resp. w) for every w (resp. z). Finally suppose that if the real numbers m, M, r, R satisfy the system

$$M = f(r,R), \quad m = f(R,r), \quad R = g(m,M), \quad r = g(M,m)$$

then $m = M$ and $r = R$. Then the system of difference equations

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{2.9}$$

has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (2.9) which satisfies

$$x_{n_0} \in [a_1, b_1], \quad x_{n_0+1} \in [a_1, b_1], \quad y_{n_0} \in [a_2, b_2], \quad y_{n_0+1} \in [a_2, b_2], \quad n_0 \in \mathbb{N}$$

tends to the unique positive equilibrium of (2.9).

3. Main results

In this section, we express and prove our main results on the system of difference equations (1.4).

3.1. Boundedness and persistence of the system

In this subsection, the boundedness and the persistence of (1.4) are investigated. The following theorem states the result obtained.

Theorem 3.1. If $\beta_1\beta_2 < a_1a_2$, then every solution of the system of difference equations (1.4) is bounded and persist.

Proof.
From (1.4), we have the following system of difference inequalities

$$x_{n+1} \le \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1} y_{n-1}, \quad y_{n+1} \le \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2} x_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

We pay regard to the system of nonhomogeneous linear difference equations

$$u_{n+1}=\frac{\alpha_1}{a_1}+\frac{\beta_1}{a_1}v_{n-1},\quad v_{n+1}=\frac{\alpha_2}{a_2}+\frac{\beta_2}{a_2}u_{n-1},\quad n\in\mathbb{N}_0,\tag{3.2}$$

with $u_{-1}=x_{-1}$, $u_0=x_0$, $v_{-1}=y_{-1}$ and $v_0=y_0$. System (3.2) yields the following independent equatios

$$u_{n+1}=\frac{\alpha_1}{a_1}+\frac{\beta_1}{a_1}\frac{\alpha_2}{a_2}+\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}u_{n-3},\quad n\geq 2,\tag{3.3}$$

and

$$v_{n+1}=\frac{\alpha_2}{a_2}+\frac{\beta_2}{a_2}\frac{\alpha_1}{a_1}+\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}v_{n-3},\quad n\geq 2.\tag{3.4}$$

The general solutions of (3.3) and (3.4) are given by

$$\begin{aligned}u_n &= \frac{\alpha_1a_2+\alpha_2\beta_1}{a_1a_2-\beta_1\beta_2}+c_1\left(\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n\\ &\quad +c_2\left(-\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n+c_3\left(-i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n+c_4\left(i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}v_n &= \frac{\alpha_2a_1+\alpha_1\beta_2}{a_1a_2-\beta_1\beta_2}+c_5\left(\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n\\ &\quad +c_6\left(-\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n+c_7\left(-i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n+c_8\left(i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}}\right)^n,\end{aligned}\tag{3.6}$$

where c_s, $(s=1,2,\dots,8)$, are arbitrary constants and i is the imaginary unit. From (3.5) and (3.6), it follows that if $\beta_1\beta_2<a_1a_2$, then there exist the limits

$$\lim_{n\to\infty}u_n=\frac{\alpha_1a_2+\alpha_2\beta_1}{a_1a_2-\beta_1\beta_2}\tag{3.7}$$

and

$$\lim_{n\to\infty}v_n=\frac{\alpha_2a_1+\alpha_1\beta_2}{a_1a_2-\beta_1\beta_2},\tag{3.8}$$

and so the sequences $\{u_n\}$ and $\{v_n\}$ are bounded. Also, since $u_{-1}=x_{-1}$, $u_0=x_0$, $v_{-1}=y_{-1}$ and $v_0=y_0$, by comparison method, we find $x_n\leq u_n$ and $y_n\leq v_n$, and so

$$x_n\leq\frac{\alpha_1a_2+\alpha_2\beta_1}{a_1a_2-\beta_1\beta_2}=U_1\tag{3.9}$$

and

$$y_n\leq\frac{\alpha_2a_1+\alpha_1\beta_2}{a_1a_2-\beta_1\beta_2}=U_2.\tag{3.10}$$

Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are also bounded. On the other hand, from (1.4), (3.9) and (3.10), it follows that

$$x_{n+1}\geq\frac{\alpha_1}{a_1+b_1y_n}\geq\frac{\alpha_1}{a_1+b_1\frac{a_1\alpha_2+\alpha_1\beta_2}{a_1a_2-\beta_1\beta_2}}=\frac{\alpha_1\left(a_1a_2-\beta_1\beta_2\right)}{a_1(a_1a_2-\beta_1\beta_2)+b_1\left(a_1\alpha_2+\alpha_1\beta_2\right)}=L_1\tag{3.11}$$

and

$$y_{n+1} \geq \frac{\alpha_2}{a_2 + b_2 x_n} \geq \frac{\alpha_2}{a_2 + b_2 \frac{\alpha_1 a_2 + \beta_1 \alpha_2}{a_1 a_2 - \beta_1 \beta_2}} = \frac{\alpha_2 \left(a_1 a_2 - \beta_1 \beta_2\right)}{a_2 \left(a_1 a_2 - \beta_1 \beta_2\right) + b_2 \left(\alpha_1 a_2 + \beta_1 \alpha_2\right)} = L_2. \tag{3.12}$$

Consequently, from (3.9), (3.10), (3.11) and (3.12), for $n \geq 1$, we have

$$L_1 \leq x_n \leq U_1, \quad L_2 \leq y_n \leq U_2 \tag{3.13}$$

which means that $\{x_n\}$ and $\{y_n\}$ are bounded and persist. The proof is completed.

Theorem 3.2. If $\beta_1 \beta_2 < a_1 a_2$, then the set $[L_1, U_1] \times [L_2, U_2]$ is invariant set of (1.4).

Proof.
Let $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$ be an arbitrary positive solution of (1.4). If $\beta_1 \beta_2 < a_1 a_2$, then the bounds L_1, U_1, L_2 and U_2 exist. Also, let $x_{-1}, x_0 \in [L_1, U_1]$ and $y_{-1}, y_0 \in [L_2, U_2]$. Then, from (1.4), we have

$$\begin{aligned}
x_1 &= \frac{\alpha_1 + \beta_1 y_{-1}}{a_1 + b_1 y_0} \leq \frac{\alpha_1 + \beta_1 U_2}{a_1} = U_1, \quad y_1 = \frac{\alpha_2 + \beta_2 x_{-1}}{a_2 + b_2 x_0} \leq \frac{\alpha_2 + \beta_2 U_1}{a_2} = U_2, \\
x_2 &= \frac{\alpha_1 + \beta_1 y_0}{a_1 + b_1 y_1} \leq \frac{\alpha_1 + \beta_1 U_2}{a_1} = U_1, \quad y_2 = \frac{\alpha_2 + \beta_2 x_0}{a_2 + b_2 x_1} \leq \frac{\alpha_2 + \beta_2 U_1}{a_2} = U_2, \\
x_3 &= \frac{\alpha_1 + \beta_1 y_1}{a_1 + b_1 y_2} \leq \frac{\alpha_1 + \beta_1 U_2}{a_1} = U_1, \quad y_3 = \frac{\alpha_2 + \beta_2 x_1}{a_2 + b_2 x_2} \leq \frac{\alpha_2 + \beta_2 U_1}{a_2} = U_2, \\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
x_1 &= \frac{\alpha_1 + \beta_1 y_{-1}}{a_1 + b_1 y_0} \geq \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad y_1 = \frac{\alpha_2 + \beta_2 x_{-1}}{a_2 + b_2 x_0} \geq \frac{\alpha_2}{a_2 + b_2 U_1} = L_2, \\
x_2 &= \frac{\alpha_1 + \beta_1 y_0}{a_1 + b_1 y_1} \geq \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad y_2 = \frac{\alpha_2 + \beta_2 x_0}{a_2 + b_2 x_1} \geq \frac{\alpha_2}{a_2 + b_2 U_1} = L_2, \\
x_3 &= \frac{\alpha_1 + \beta_1 y_1}{a_1 + b_1 y_2} \geq \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad y_3 = \frac{\alpha_2 + \beta_2 x_1}{a_2 + b_2 x_2} \geq \frac{\alpha_2}{a_2 + b_2 U_1} = L_2, \\
&\vdots
\end{aligned}$$

Considering inductively, it can be easily shown that $x_n \in [L_1, U_1]$ and $y_n \in [L_2, U_2]$ for $n \geq -1$. So the proof is completed.

3.2. Stability analysis

In this subsection, the existence of the unique positive equilibrium of (1.4) and local asymptotic stability and global asymptotic stability of the equilibrium are investigated.

Lemma 3.3. System (1.4) possesses a unique positive equilibrium point. If $\beta_1 \beta_2 < a_1 a_2$, then the equilibrium point is in the set $[L_1, U_1] \times [L_2, U_2]$.

Proof.

For the equilibrium points of (1.4) we consider the system

$$\overline{x} = \frac{\alpha_1 + \beta_1 \overline{y}}{a_1 + b_1 \overline{y}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 \overline{x}}{a_2 + b_2 \overline{x}}. \tag{3.14}$$

From (3.14) we have the independent quadratic equations

$$D_1 \overline{x}^2 + (C_1 - B_1)\overline{x} - A_1 = 0, \quad D_2 \overline{y}^2 + (C_2 - B_2)\overline{y} - A_2 = 0, \tag{3.15}$$

where

$$\begin{aligned}
A_1 &= \alpha_1 a_2 + \beta_1 \alpha_2, \\
B_1 &= \alpha_1 b_2 + \beta_1 \beta_2, \\
C_1 &= a_1 a_2 + b_1 \alpha_2, \\
D_1 &= a_1 b_2 + b_1 \beta_2, \\
A_2 &= a_1 \alpha_2 + \alpha_1 \beta_2, \\
B_2 &= \alpha_2 b_1 + \beta_1 \beta_2, \\
C_2 &= a_1 a_2 + b_2 \alpha_1, \\
D_2 &= a_2 b_1 + b_2 \beta_1.
\end{aligned}$$

Hence, from (3.15), we have

$$\Delta_{\overline{x}} = (C_1 - B_1)^2 + 4A_1 D_1 > 0, \quad \Delta_{\overline{y}} = (C_2 - B_2)^2 + 4A_2 D_2 > 0$$

which implies that they have two real simple roots. Also, since $-A_1/D_1 < 0$ and $-A_2/D_2 < 0$, both equations in (3.15) have one negative and one positive root. Therefore there exists the unique positive equilibrium point of (1.4).

Consider the inequalities

$$\overline{x} \le \frac{\alpha_1 + \beta_1 \overline{y}}{a_1}, \quad \overline{y} \le \frac{\alpha_2 + \beta_2 \overline{x}}{a_2},$$

which is obtained from (3.14). Using these two inequalities within each other we get the following inequalities

$$\begin{aligned}
\overline{x} &\le \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1}\overline{y} \le \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1}\frac{\alpha_2}{a_2} + \frac{\beta_1}{a_1}\frac{\beta_2}{a_2}\overline{x}, \\
\overline{y} &\le \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2}\overline{x} \le \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2}\frac{\alpha_1}{a_1} + \frac{\beta_2}{a_2}\frac{\beta_1}{a_1}\overline{y}.
\end{aligned}$$

If $\beta_1 \beta_2 < a_1 a_2$, from the last inequalities, it follows that

$$\overline{x} \le \frac{\alpha_1 a_2 + \alpha_2 \beta_1}{a_1 a_2 - \beta_1 \beta_2} = U_1, \quad \overline{y} \le \frac{\alpha_2 a_1 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2} = U_2.$$

Moreover, from (3.14) and the inequalities $\overline{x} \le U_1$, $\overline{y} \le U_2$, we obtain the inequalities

$$\overline{x} \ge \frac{\alpha_1}{a_1 + b_1\overline{y}} \ge \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad \overline{y} \ge \frac{\alpha_2}{a_2 + b_2\overline{x}} \ge \frac{\alpha_2}{a_2 + b_2 U_1} = L_2.$$

Thus, for the aforementioned equilibrium point, we have $\left(\overline{x}, \overline{y}\right) \in [L_1, U_1] \times [L_2, U_2]$. So the proof is completed.

Theorem 3.4. If $\beta_1\beta_2 < a_1 a_2$, then the unique positive equilibrium of system (1.4) is locally asymptotically stable.

Proof.
We know from Lemma 3.3 that (1.4) has the unique positive equilibrium $\left(\overline{x}, \overline{y}\right)$. In this case, the vector system corresponding to (1.4) also has the equilibrium point $\overline{X} = \left(\overline{x}, \overline{y}, \overline{x}, \overline{y}\right)^T$. The aforementioned vector system is given by the vector map

$$F\begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 y_n} \\ \frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 x_n} \\ x_n \\ y_n \end{pmatrix}$$

The linearized system of the vector system about $\overline{X} = \left(\overline{x}, \overline{y}, \overline{x}, \overline{y}\right)^T$ is the system

$$Z_{n+1} = J_F(\overline{X}) Z_n, \tag{3.16}$$

where the vector Z_n is

$$Z_n = \begin{pmatrix} z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \end{pmatrix}$$

and J_F at $\overline{X}$ is

$$J_F(\overline{X}) = \begin{pmatrix} 0 & -\frac{b_1\overline{x}}{a_1+b_1\overline{y}} & 0 & \frac{\beta_1}{a_1+b_1\overline{y}} \\ -\frac{b_2\overline{y}}{a_2+b_2\overline{x}} & 0 & \frac{\beta_2}{a_2+b_2\overline{x}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

The characteristic polynomial of (3.16) at $\overline{X}$ is

$$P(\lambda) = \lambda^4 - \frac{b_1 b_2 \overline{x}\overline{y}}{\left(a_2 + b_2\overline{x}\right)\left(a_1 + b_1\overline{y}\right)}\lambda^2 + \frac{b_1\beta_2\overline{x} + \beta_1 b_2\overline{y}}{\left(a_2 + b_2\overline{x}\right)\left(a_1 + b_1\overline{y}\right)}\lambda - \frac{\beta_1\beta_2}{\left(a_2 + b_2\overline{x}\right)\left(a_1 + b_1\overline{y}\right)}$$

or

$$P(\lambda) = \lambda^4 - \frac{\left(\beta_1 - b_1\overline{x}\lambda\right)\left(\beta_2 - b_2\overline{y}\lambda\right)}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)}. \tag{3.18}$$

Let us consider the polynomial equation $P(\lambda) = 0$. Obviously, since $\beta_1\beta_2 \neq 0$, $\lambda \neq 0$. In this case, it can be seen from (3.18) that there are two cases to consider.

(i) If $\beta_1 < b_1\overline{x}\lambda$ and $\beta_2 < b_2\overline{y}\lambda$, then we have

$$\lambda^4 = \frac{\left(\beta_1 - b_1\overline{x}\lambda\right)\left(\beta_2 - b_2\overline{y}\lambda\right)}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{b_1\overline{x}\lambda b_2\overline{y}\lambda}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{b_1\overline{x}\lambda b_2\overline{y}\lambda}{b_1\overline{y}b_2\overline{x}} = \lambda^2$$

from which it follows that $|\lambda| < 1$.

(ii) If $\beta_1 > b_1\overline{x}\lambda$ and $\beta_2 > b_2\overline{y}\lambda$, then we have

$$\lambda^4 = \frac{\left(\beta_1 - b_1\overline{x}\lambda\right)\left(\beta_2 - b_2\overline{y}\lambda\right)}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{\beta_1\beta_2}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{\beta_1\beta_2}{a_1a_2}.$$

Hence if $\beta_1\beta_2 < a_1a_2$, then we obtain that $|\lambda| < 1$. Therefore the proof is completed.

Theorem 3.5. If $\beta_1\beta_2 < a_1a_2$, then the unique positive equilibrium point of (1.4) is a global attractor.

Proof.

We will use Lemma 2.5 to prove the theorem. Let $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$ be any solution of system (1.4). We know that if the inequality $\beta_1\beta_2 < a_1a_2$ is satisfied, then $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$ is bounded and persist. Suppose that

$$f(u,v) = \frac{\alpha_1 + \beta_1 v}{a_1 + b_1 u}, \quad g\left(x,y\right) = \frac{\alpha_2 + \beta_2 y}{a_2 + b_2 x}.$$

Then we have

$$f_u(u,v) = -\frac{\left(\alpha_1 + \beta_1 v\right)b_1}{\left(a_1 + ub_1\right)^2} < 0, \quad f_v(u,v) = \frac{\beta_1}{a_1 + ub_1} > 0$$

for $(u,v) \in (L_2, U_2) \times (L_2, U_2)$ and

$$g_x\left(x,y\right) = -\frac{\left(\alpha_2 + \beta_2 y\right)b_2}{\left(a_2 + xb_2\right)^2} < 0, \quad g_y\left(x,y\right) = \frac{\beta_2}{a_2 + xb_2} > 0$$

for $\left(x,y\right) \in (L_1, U_1) \times (L_1, U_1)$. Therefore, the function $f(u,v)$ is decreasing with respect to u for every $v \in (L_2, U_2)$ and it is increasing with respect to v for every $u \in (L_2, U_2)$, and also the function $g(x,y)$ is decreasing with respect to x for every $y \in (L_1, U_1)$ and it is increasing with respect to y for every $x \in (L_1, U_1)$.

Let

$$\limsup_{n\to\infty} x_n = M_1, \quad \liminf_{n\to\infty} x_n = m_1, \quad \limsup_{n\to\infty} y_n = M_2, \quad \liminf_{n\to\infty} y_n = m_2.$$

In this case we can define the system

$$M_1 = \frac{\alpha_1 + \beta_1 M_2}{a_1 + b_1 m_2}, \quad m_1 = \frac{\alpha_1 + \beta_1 m_2}{a_1 + b_1 M_2}, \quad M_2 = \frac{\alpha_2 + \beta_2 M_1}{a_2 + b_2 m_1}, \quad m_2 = \frac{\alpha_2 + \beta_2 m_1}{a_2 + b_2 M_1}. \tag{3.19}$$

From (3.19), we have

$$a_1M_1 + b_1M_1m_2 = \alpha_1 + \beta_1M_2, \qquad a_1m_1 + b_1m_1M_2 = \alpha_1 + \beta_1m_2, \tag{3.20}$$

$$a_2M_2 + b_2M_2m_1 = \alpha_2 + \beta_2M_1, \qquad a_2m_2 + b_2m_2M_1 = \alpha_2 + \beta_2m_1. \tag{3.21}$$

Furthermore, from (3.20) and (3.21), we have

$$a_1 (M_1 - m_1) + b_1 (M_1 m_2 - m_1 M_2) = \beta_1 (M_2 - m_2) \tag{3.22}$$

and

$$a_2 (M_2 - m_2) + b_2 (m_1 M_2 - M_1 m_2) = \beta_2 (M_1 - m_1), \tag{3.23}$$

respectively. If $M_1 = m_1$, then it is seen from (3.22) that $m_2 = M_2$. On the other hand, if $m_2 = M_2$, then it is seen from (3.23) that $M_1 = m_1$. Therefore, we will just show that $M_2 = m_2$. After some operations, the equalities (3.22) and (3.23) yield the equality

$$\left(\frac{a_1}{b_1} - \frac{\beta_2}{b_2}\right)(M_1 - m_1) + \left(\frac{a_2}{b_2} - \frac{\beta_1}{b_1}\right)(M_2 - m_2) = 0. \tag{3.24}$$

We rewrite (3.24) as

$$M_1 - m_1 = \frac{\frac{a_2}{b_2} - \frac{\beta_1}{b_1}}{\frac{a_1}{b_1} - \frac{\beta_2}{b_2}} (m_2 - M_2). \tag{3.25}$$

If $\beta_1 \beta_2 < a_1 a_2$, then (3.2) becomes

$$M_1 - m_1 = \frac{a_2}{\beta_2} (M_2 - m_2).$$

Using this result in (3.23), we obtain

$$m_1 M_2 - M_1 m_2 = 0.$$

Using the last two results in (3.22), we obtain

$$\left(a_1 a_2 - \beta_1 \beta_2\right)(M_2 - m_2) = 0$$

which implies that $M_2 = m_2$. So the proof is completed. In order to verify the theoretical result we obtained in Theorem 3.5, a special case obtained by giving some values to the parameters and initial conditions of system (1.4) is given in the example below.

Example 3.6. If $\alpha_1 = 1$, $\beta_1 = 13.1$, $a_1 = 7$, $b_1 = 3$, $\alpha_2 = 12$, $\beta_2 = 3.5$, $a_2 = 8.2$, $b_2 = 1$, then (1.4) becomes

$$x_{n+1} = \frac{1 + 12.1 y_{n-1}}{7 + 3 y_n}, \quad y_{n+1} = \frac{12 + 3.5 x_{n-1}}{6 + x_n}. \tag{3.26}$$

The unique positive equilibrium of (3.26) is $(2.364109242, 1.919175757)$. Plot of the corresponding solution to $x_{-1} = 5.4$, $x_0 = 9.5$, $y_{-1} = 7$ and $y_0 = 1.7$ is given by Figure 1 and Figure 2.

According to the item iv) of Definition 2.1, we give the next result from Theorem 3.4 and Theorem 3.5.

Theorem 3.7. If $\beta_1 \beta_2 < a_1 a_2$, then the unique positive equilibrium point of (1.4) is globally asymptotically stable.

3.3. Rate of convergence of solutions

In this subsection, the rate of convergence of a solution converging to the unique positive equilibrium of (1.4) is studied.

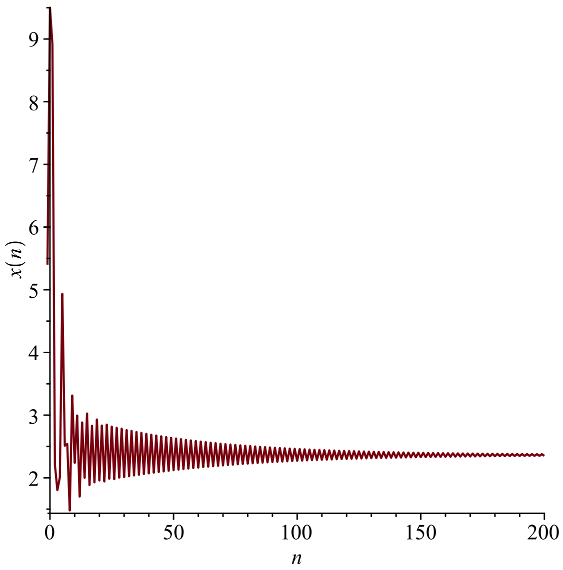

Figure 1. Plot of (x_n) converging to $\overline{x}$

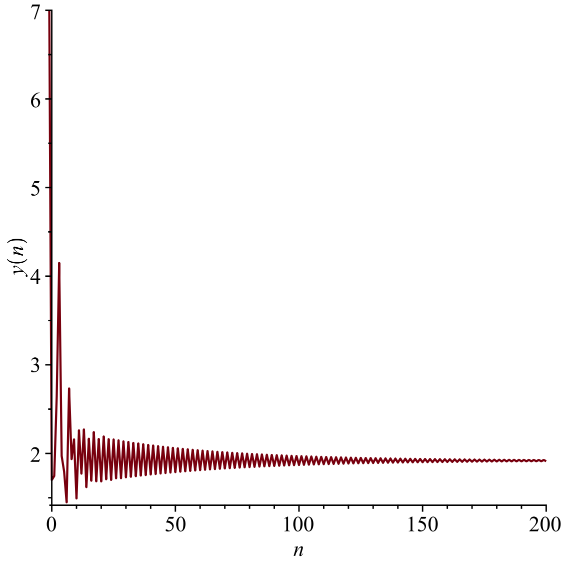

Figure 2. Plot of (y_n) converging to $\overline{y}$

Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be any solution of (1.4) such that

$$\lim_{n\to\infty} x_n = \overline{x} \quad and \quad \lim_{n\to\infty} y_n = \overline{y}, \tag{3.27}$$

where $\overline{x} \in [L_1, U_1]$ and $\overline{y} \in [L_2, U_2]$. From (1.4), we have

$$\begin{aligned} x_{n+1} - \overline{x} &= \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 y_n} - \frac{\alpha_1 + \beta_1 \overline{y}}{a_1 + b_1 \overline{y}} \\ &= \frac{-b_1\left(\alpha_1 + \beta_1 \overline{y}\right)}{(a_1 + b_1 y_n)\left(a_1 + b_1 \overline{y}\right)}\left(y_n - \overline{y}\right) + \frac{\beta_1\left(a_1 + b_1 \overline{y}\right)}{(a_1 + b_1 y_n)\left(a_1 + b_1 \overline{y}\right)}\left(y_{n-1} - \overline{y}\right) \end{aligned}$$

or after some operations and by using (3.14)

$$x_{n+1} - \overline{x} = \frac{-b_1 \overline{x}}{\left(a_1 + b_1 y_n\right)}\left(y_n - \overline{y}\right) + \frac{\beta_1}{\left(a_1 + b_1 y_n\right)}\left(y_{n-1} - \overline{y}\right). \tag{3.28}$$

Similarly, from (1.4), we have

$$\begin{aligned} y_{n+1} - \overline{y} &= \frac{\alpha_{2+}\beta_2 x_{n-1}}{a_2 + b_2 x_n} - \frac{\alpha_2 + \beta_2 \overline{x}}{a_2 + b_2 \overline{x}} \\ &= \frac{-b_2(\alpha_2 + \beta_2 \overline{x})}{(a_2 + b_2 x_n)\left(a_2 + b_2 \overline{x}\right)}\left(x_n - \overline{x}\right) + \frac{\beta_2\left(a_2 + b_2 \overline{x}\right)}{(a_2 + b_2 x_n)\left(a_2 + b_2 \overline{x}\right)}\left(x_{n-1} - \overline{x}\right) \end{aligned}$$

and so, by (3.14),

$$y_{n+1} - \overline{y} = \frac{-b_2 \overline{y}}{(a_2 + b_2 x_n)}\left(x_n - \overline{x}\right) + \frac{\beta_2}{(a_2 + b_2 x_n)}\left(x_{n-1} - \overline{x}\right). \tag{3.29}$$

If the error terms $e_n^1 = x_n - \bar{x}$, $e_n^2 = y_n - \bar{y}$, then we can write the system of the error terms as follows

$$\begin{aligned} e_{n+1}^1 &= a_n e_n^2 + b_n e_{n-1}^2, \\ e_{n+1}^2 &= c_n e_n^1 + d_n e_{n-1}^1, \end{aligned}$$

where

$$a_n = \frac{-b_1\overline{x}}{a_1 + b_1 y_n}, \quad b_n = \frac{\beta_1}{a_1 + b_1 y_n}, \quad c_n = \frac{-b_2\overline{y}}{a_2 + b_2 x_n}, \quad d_n = \frac{\beta_2}{a_2 + b_2 x_n}. \tag{3.30}$$

From (3.30), we obtain the limits

$$\lim_{n\to\infty} a_n = \frac{-b_1\overline{x}}{a_1 + b_1\overline{y}}, \tag{3.31}$$

$$\lim_{n\to\infty} b_n = \frac{\beta_1}{a_1 + b_1\overline{y}}, \tag{3.32}$$

$$\lim_{n\to\infty} c_n = \frac{-b_2\overline{y}}{a_2 + b_2\overline{x}}, \tag{3.33}$$

$$\lim_{n\to\infty} d_n = \frac{\beta_2}{a_2 + b_2\overline{x}}. \tag{3.34}$$

Consequently, from (3.31)-(3.34), we have the following system

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^1_n \\ e^2_n \end{pmatrix} = \begin{pmatrix} 0 & \frac{-b_1\overline{x}}{a_1 + b_1\overline{y}} & 0 & \frac{\beta_1}{a_1 + b_1\overline{y}} \\ \frac{-b_2\overline{y}}{a_2 + b_2\overline{x}} & 0 & \frac{\beta_2}{a_2 + b_2\overline{x}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^2_n \\ e^1_{n-1} \\ e^2_{n-1} \end{pmatrix}, \tag{3.35}$$

which resembles the linearized system of (1.4) about the equilibrium $\overline{X}$. In this case, one can obtain from Theorem 2.3 and Theorem 2.4 the following results.

Theorem 3.8. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be any positive solution of (1.4) satisfying (3.27). Then, the error vector $\left(e^1_n, e^2_n, e^1_{n-1}, e^2_{n-1}\right)^T$ of the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of (1.4) satisfies the asymptotic relations

$$\lim_{n\to\infty} (||e_n||)^{\frac{1}{n}} = \left|\lambda_{1,2,3,4} J_F\left(\overline{x}, \overline{y}\right)\right|$$

and

$$\lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = \left|\lambda_{1,2,3,4} J_F\left(\overline{x}, \overline{y}\right)\right|,$$

where the values $\lambda_{1,2,3,4}$ are the eigenvalues of the Jacobian $J_F(\overline{x}, \overline{y})$.

3.4. Existence of unbounded solutions

In this subsection, the existence of unbounded solutions of (1.4) is proven.

Theorem 3.9. If $\beta_1\beta_2 > a_1 a_2$, then every positive solution of (1.4) is unbounded.

Proof.
From (1.4) we have the system of difference inequalities

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 y_n} \geq \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 U_2}, \tag{3.36}$$

and

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 x_n} \geq \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 U_1}, \tag{3.37}$$

where U_1 and U_2 are given by (3.9) and (3.10), respectively. Now we can consider the system of nonhomogeneous linear equations

$$w_{n+1} = c_2 + d_2 z_{n-1}, \quad z_{n+1} = c_1 + d_1 w_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.38}$$

where

$$c_1 = \frac{\alpha_1}{a_1 + b_1 U_2}, \quad d_1 = \frac{\beta_1}{a_1 + b_1 U_2}, \quad c_2 = \frac{\alpha_2}{a_2 + b_2 U_1}, \quad d_2 = \frac{\beta_2}{a_2 + b_2 U_1}$$

and $w_{-1} = x_{-1}$, $w_0 = x_0$, $z_{-1} = y_{-1}$, $z_0 = y_0$. The general solution of (3.38) is given by the formulas

$$w_n = \frac{c_2 + c_1 d_2}{1 - d_1 d_2} + k_1 \left(\sqrt[4]{d_1 d_2}\right)^n + k_2 \left(-\sqrt{d_1 d_2}\right)^n + k_3 \left(-i \sqrt[4]{d_1 d_2}\right)^n + k_4 \left(i \sqrt[4]{d_1 d_2}\right)^n \tag{3.39}$$

and

$$z_n = \frac{c_1 + c_2 d_1}{1 - d_1 d_2} + k_5 \left(\sqrt[4]{d_1 d_2}\right)^n + k_6 \left(-\sqrt{d_1 d_2}\right)^n + k_7 \left(-i \sqrt[4]{d_1 d_2}\right)^n + k_8 \left(i \sqrt[4]{d_1 d_2}\right)^n, \tag{3.40}$$

where k_s, $(s = 1, 2, ..., 8)$ are arbitrary constants and i is the imaginary unit. It is easy to see from (3.39) and (3.40) that if $d_1 d_2 > 1$, that is,

$$\beta_1 \beta_2 > (a_1 + b_1 U_2)(a_2 + b_2 U_1) > a_1 a_2$$

then the sequences (w_n) and (z_n) are unbounded. Therefore, since $w_{-1} = x_{-1}$, $w_0 = x_0$, $z_{-1} = y_{-1}$ and $z_0 = y_0$, by comparison method, we have the inequalities $x_n \geq w_n$, $y_n \geq z_n$. Hence the sequences $\{x_n\}$ and $\{y_n\}$ are unbounded. The proof is completed.

Example 3.10. If $\alpha_1 = 1$, $\beta_1 = 12.1$, $a_1 = 3.6$, $b_1 = 3$, $\alpha_2 = 12$, $\beta_2 = 3.5$, $a_2 = 6$, $b_2 = 1$, then (1.4) becomes

$$x_{n+1} = \frac{1 + 12.1 y_{n-1}}{3.6 + 3 y_n}, \quad y_{n+1} = \frac{12 + 3.5 x_{n-1}}{6 + x_n}. \tag{3.41}$$

The unique positive equilibrium of (3.41) is $(2.808100791, 2.478213327)$ and unstable. Plot of the corresponding solution to $x_{-1} = 5.4$, $x_0 = 9.5$, $y_{-1} = 7$ and $y_0 = 1.7$ is given by Figure 3 and Figure 4.

3.5. Period two solutions

In this subsection, the existence of two-periodic solutions of (1.4) is investigated. The next result states the existence of such solutions.

Theorem 3.11. If $a_1 a_2 = \beta_1 \beta_2$, then the system of difference equations (1.4) has two-periodic solutions.

Proof.
Let a two-periodic solution of (1.4) be

$$..., (p_1, q_1), (p_2, q_2), (p_1, q_1), (p_2, q_2), ..., \tag{3.42}$$

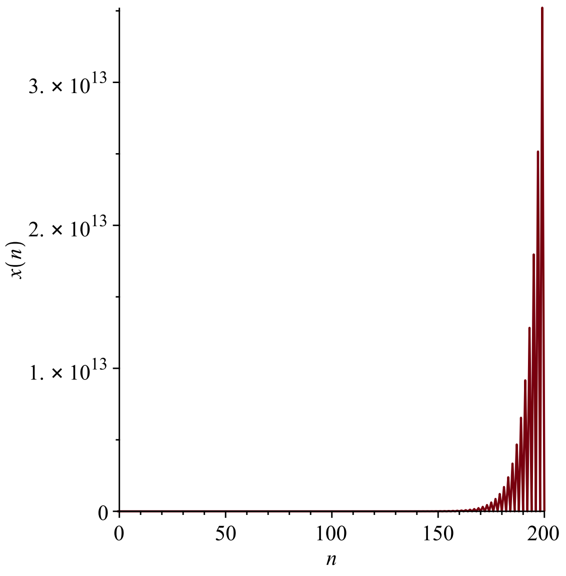

Figure 3. Plot of unbounded (x_n)

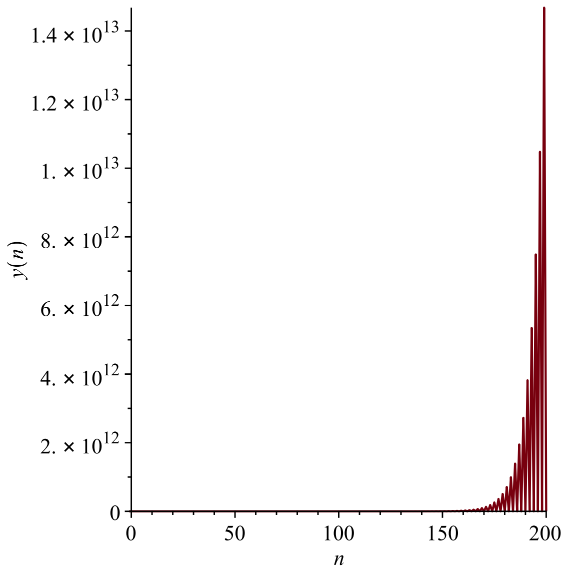

Figure 4. Plot of unbounded (y_n)

where p_1, p_2, q_1, q_2 are positive real numbers such that $p_1 \neq p_2$ ve $q_1 \neq q_2$. Then, from (1.4) and (3.42), we have the system

$$p_1 = \frac{\alpha_1 + \beta_1 q_1}{a_1 + b_1 q_2}, \quad p_2 = \frac{\alpha_1 + \beta_1 q_2}{a_1 + b_1 q_1}, \quad q_1 = \frac{\alpha_2 + \beta_2 p_1}{a_2 + b_2 p_2}, \quad q_2 = \frac{\alpha_2 + \beta_2 p_2}{a_2 + b_2 p_1},$$

from which it follows that

$$a_1 p_1 + b_1 p_1 q_2 = \alpha_1 + \beta_1 q_1, \quad a_1 p_2 + b_1 p_2 q_1 = \alpha_1 + \beta_1 q_2 \tag{3.43}$$

and

$$a_2 q_1 + b_2 q_1 p_2 = \alpha_2 + \beta_2 p_1, \quad a_2 q_2 + b_2 q_2 p_1 = \alpha_2 + \beta_2 p_2. \tag{3.44}$$

After some basic operations, from (3.43) and (3.44), we get the equalities

$$a_1 \left(p_1 - p_2\right) + b_1 \left(p_1 q_2 - p_2 q_1\right) = \beta_1 \left(q_1 - q_2\right)$$

and

$$a_2 \left(q_1 - q_2\right) + b_2 \left(q_1 p_2 - q_2 p_1\right) = \beta_2 \left(p_1 - p_2\right).$$

The last equalities yield

$$(a_1 b_2 - b_1 \beta_2)(p_1 - p_2) + (a_2 b_1 - b_2 \beta_1)(q_1 - q_2) = 0. \tag{3.45}$$

It is obvious from (3.45) and the assumptions $p_1 \neq p_2$ and $q_1 \neq q_2$ that if

$$a_1 b_2 - b_1 \beta_2 = 0 \quad \text{and} \quad a_2 b_1 - b_2 \beta_1 = 0, \tag{3.46}$$

then system (1.4) has two-periodic solutions. Note that (3.46) is equivalent to the desired equality $a_1 a_2 = \beta_1 \beta_2$. So the proof is completed.

Example 3.12. If $\alpha_1 = 3$, $\beta_1 = 6$, $a_1 = 12$, $b_1 = 9$, $\alpha_2 = 2$, $\beta_2 = 4$, $a_2 = 2$, $b_2 = 3$, then system (1.4) becomes

$$x_{n+1} = \frac{3+6y_{n-1}}{12+9y_n}, \quad y_{n+1} = \frac{2+4x_{n-1}}{2+3x_n}. \tag{3.47}$$

The unique positive equilibrium point of (3.47) is $(0.4413911092, 1.132782218)$ and it is unstable. Also, the solution converges a two-periodic solution of the system. Plot of the corresponding solution with $x_{-1} = 3$, $x_0 = 2$, $y_{-1} = 1.3$ and $y_0 = 7$ is given by Figure 5 and Figure 6.

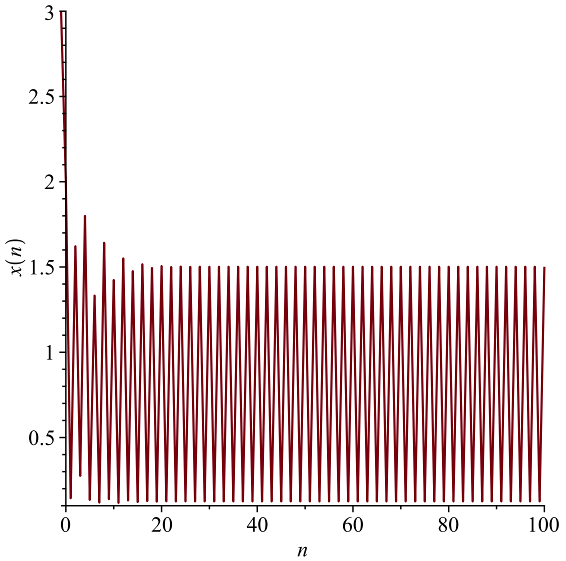

Figure 5. (x_n) converging to a two-periodic solution

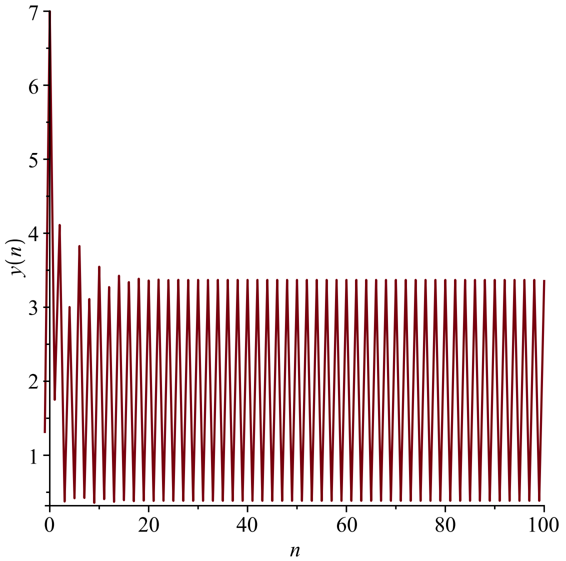

Figure 6. (y_n) converging to a two-periodic solution

4. Conclusion

In this study, the qualitative behavior of the positive solutions of (1.4) was investigated. The results obtained are summarized below.

1. If $\beta_1\beta_2 < a_1a_2$, then the solutions of the system are bounded and persist. In addition, the unique positive equilibrium of the system is globally asymptotically stable.
2. If $\beta_1\beta_2 = a_1a_2$, then the system has two-periodic solutions.
3. If $\beta_1\beta_2 > a_1a_2$, then the system has unbounded solutions.

Availability of data and materials Not applicable.

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