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GRÖBNER-SHIRSHOV BASES AND NORMAL FORMS FOR THE INFINITE COXETER GROUPS OF TYPES $\widetilde{B}_n$ AND $\widetilde{D}_n$

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Abstract. In this paper, we will investigate the infinite Coxeter groups $\widetilde{B}_n$ and $\widetilde{D}_n$. Their Gröbner-Shirshov bases and classifications of normal forms are achieved by leveraging results from the infinite Coxeter groups of types $\widetilde{C}_n$. Additionally, new algorithms are presented for obtaining normal forms of elements within these groups.

1. Introduction

To begin with, we revisit certain ideas related to the Gröbner-Shirshov basis theory. Let S represent a set, and S^* denote the free monoid of strings formed by S. We refer to the empty string as e. A well-ordering $<$ on S^* is referred to as a monomial order if $x < y$ implies $axb < ayb$ for all $a, b \in S^*$. Let $\langle S \rangle$ denote the free associative algebra generated by S over a field k. Given $0 \neq f \in \langle S \rangle$, we denote by $\overline{f}$ the leading word of f concerning a specified monomial order. For two monic polynomials f and g, $f\langle S \rangle$ if there exists a word w such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in S^*$. The intersection composition of f and g is defined by $\langle f, g \rangle_w = fb - ag$. If $\overline{f} = a\overline{g}b$ for some $a, b \in S^*$, the inclusion composition is defined as $\langle f, g \rangle = f - agb$. In this scenario, the transformation $f \to f - agb$ is known as the elimination of the leading word (ELW) of f in g. Let $R \subseteq \langle S \rangle$ be a collection of monic polynomials, and let f be another monic polynomial. We say that f is reduced to h modulo R if f is derived from a sequence of ELWs involving elements of R, and no further ELWs of r are possible. A set $R \subseteq \langle S \rangle$ is termed a Gröbner-Shirshov basis, denoted by GSB if every composition of polynomials from R is reduced to zero modulo R. A GSB R is considered minimal if there are no inclusion compositions within R. If $R \subseteq \langle S \rangle$ is not a GSB, take a composition of intersections of polynomials from R and reduce it modulo R. If this reduction results in a non-zero polynomial r, add r to the set R. Continue this process for each composition of polynomials from R until no further enlargements are required. The final set obtained will be a GSB.

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This procedure is referred to as the Shirshov algorithm. The Composition Diamond Lemma ([13]) is valuable for finding the normal form of a group through its GSB.

When a group G is defined by generators S and relations R, each relation $x = y$ in R can be associated with a polynomial $x - y$. Thus, the set of relations can be viewed as a subset of $\urcorner\langle S\rangle$. Consequently, a GSB of R, referred to as a GSB for the group G, can be found. It's worth noting that R comprises "biwords," essentially differences of words. The Shirshov algorithm maintains this property throughout the computation. Therefore, a GSB of a group can be considered a unique set of relations for that group. Furthermore, the set

$$Red(R) = \{w \in S^* | w \neq x\bar{s}y, x, y \in S^*, s \in R\}$$

constitutes the set of all normal forms of G, as established by the Composition Diamond lemma.

Coxeter groups, known as Weyl groups, represent one of the most significant examples of groups defined by generators and defining relations. Consequently, the pursuit of finding GSB for these groups has attracted considerable attention from researchers. GSB for finite Coxeter groups can be found in [1]. For the finite exceptional Coxeter group of type E_8, a GSB has been established in [3], while for the finite exceptional Coxeter groups of type E_6 and E_7, a GSB can be found in [4]. The method of GSB bases introduces a new algorithm for deriving normal forms of elements in groups, monoids, and semigroups, providing a fresh approach to solving the word problem in these algebraic structures. The word problem for a finitely generated group G involves the algorithmic challenge of determining whether two words formed by the generators represent the same element. A novel algorithm for obtaining normal forms and addressing the word problem for Extended Modular, Extended Hecke, and Picard groups through their GSB is explored in [5]. Comparable findings for the singular part of the Brauer semigroup and braid groups via the complex reflection group G_{12} are presented in [11] and [14], respectively. In [6], the authors establish a connection between graph theory and GSB of groups. This article aims to pave the way for further research in this area. GSB for infinite Coxeter groups of type $\widetilde{A}_n$, $\widetilde{C}_n$, as well as for finite Coxeter groups of type A_n, B_n, and D_n, have been obtained in [8], [15], and [12], respectively. Additionally, for the infinite exceptional Weyl group of type F_4, a GSB has been constructed in [10]. The author worked on GSB bases for infinite Coxeter group of type $\widetilde{A}_n$ in [7] and the results in this article were obtained from [13].

The primary objective of this article is to derive GSB and normal forms for infinite Coxeter groups of types $\widetilde{B}_n$ and $\widetilde{D}_n$.

2. Gröbner-Shirshov Bases

This section focuses on the discussion of GSB for the infinite Coxeter groups of Types $\widetilde{B}_n$ and $\widetilde{D}_n$.

2.1. GSB for $\widetilde{B}_n$.

Definition 2.1. The presentation of the infinite Coxeter group of type $\widetilde{B}_n$ includes generators $S = \{s_0, s_1, \dots, s_n\}$ for a positive integer $n \geq 2$ and the following defining relations:

(RB_1) $s_a s_a = e$ for $0 \leq a \leq n$,
(RB_2) $s_a s_b = s_b s_a$ for $0 \leq a < b - 1 < n$ but $(a, b) \neq (0, 2)$,

(RB_3) $s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}$ for $1 \leq a < n-1$,
(RB_4) $s_0 s_1 = s_1 s_0$,
(RB_5) $s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}$,
(RB_6) $s_0 s_2 s_0 = s_2 s_0 s_2$.

where e represents the identity element of the group.

For the sake of convenience, let us assume that

$$s_{ab} = \begin{cases} s_a s_{a+1} \cdots s_b, & \text{if } 1 \leq a \leq b < n \text{ ;} \\ s_a s_{a+1} \cdots s_n s_{n-1} \cdots s_c, & \text{if } 1 \leq a \leq c = 2n-j \leq n; \\ e, & \text{if } 0 \leq b = a-1 < n. \end{cases}$$

and

$$s_{ab}^{-1} = \begin{cases} s_b s_{b-1} \cdots s_a, & \text{if } 1 \leq a \leq b < n \text{ ;} \\ s_c s_{c+1} \cdots s_n s_{n-1} \cdots s_a, & \text{if } 1 \leq a \leq c = 2n-j \leq n; \\ e, & \text{if } 0 \leq b = a-1 < n. \end{cases}$$

It is important to note that s_{ab}^{-1} is, in fact, the inverse of s_{ab} since $s_a s_a = e$ for each a.

Lemma 2.2. *Assume that $<$ denotes the degree lexicographic order on S^*. A GSB for the infinite Coxeter group of type $\widetilde{B}_n$ with respect to $<$ includes the following polynomials:*

- $f_1^{(a)} = s_a s_a - 1$ if $0 \leq a \leq n$,
- $f_2^{(a,b)} = s_a s_b - s_b s_a$ if $0 \leq a < b-1 < n$ but $(a,b) \neq (0,2)$,
- $f_3^{(a,b)} = s_{ab} s_a - s_{a+1} s_{ab}$ if $1 \leq a \leq n-2$ and $a < b < 2n-a-1$,
- $f_4^{(a)} = s_{a,2n-a} s_{a+1} - s_{a+1} s_{a,2n-a}$ if $1 \leq a \leq n-1$,
- $f_5^{(a)} = s_0 s_{2a} s_{1a} - s_1 s_0 s_{2a} s_{1,a-1}$ if $1 \leq a \leq n-1$,
- $f_6^{(a)} = s_0 s_{2,2n-a} s_{1,2n-a+1} - s_1 s_0 s_{2,2n-a} s_{1,2n-a}$ if $2 \leq a \leq n$,
- $f_7^{(a,b)} = s_0 s_{2a} s_{1b} s_0 - s_2 s_0 s_{2a} s_{1b}$ if $2 \leq a \leq 2n-3$ and $0 \leq b \leq 1$,
- $f_8^{(a,b)} = s_0 s_{2a} s_{1b} s_0 s_{2b} - s_2 s_0 s_{2a} s_{1b} s_0 s_{2,a-1}$ if $2 \leq b < a \leq n$,
- $f_9^{(a,b)} = s_0 s_{2,2n-a} s_{1b} s_0 s_{2b} - s_2 s_0 s_{2,2n-a} s_{1a} s_0 s_{2,b-1}$ if $3 \leq a \leq n-1$ and $2 \leq b \leq n-1$,
- $f_{10}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_1 - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2b}$ if $1 \leq a \leq 2$ and $2 \leq b \leq 2n-3$,
- $f_{11}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_{1,a-1} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_{1,a-2}$ if $3 \leq a \leq n-1$, $3 \leq b \leq n$ and $a \leq b$,
- $f_{12} = s_0 s_{2,2n-2} s_0 s_2 - s_2 s_0 s_{2,2n-2} s_0$,
- $f_{13}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-b} s_{1a} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-b} s_{1,a-1}$ if $2 \leq b \leq a \leq n-1$,
- $f_{14}^{(a,b)} = s_0 s_{2,2n-2} s_{1,2n-a-1} s_0 s_{2,2n-b} s_{1,2n-a}$ $- s_2 s_0 s_{2,2n-2} s_{1,2n-a-1} s_0 s_{2,2n-b} s_{1,2n-a-1}$ if $2 \leq b \leq a \leq n-1$,
- $f_{15}^{(a,b)} = s_0 s_{2,2n-a} s_{1,2n-b-1} s_0 s_{2,2n-b}$ $- s_2 s_0 s_{2,2n-a} s_{1,2n-b-1} s_0 s_{2,2n-b-1}$ if $2 \leq a-1 \leq b \leq n-1$,
- $f_{16} = s_0 s_{2,2n-2} s_1 s_0 s_{2,2n-2} s_1 s_2 - s_2 s_0 s_{2,2n-2} s_1 s_0 s_0 s_{2,2n-2} s_1$,
- $f_{17}^{(a,b)} = s_0 s_{2,2n-2} s_{1b} s_0 s_{2,2n-a} s_{1,b-1} - s_2 s_0 s_{2,2n-2} s_{1b} s_0 s_{2,2n-a} s_{1,b-2}$ if $3 \leq b < a \leq n-1$.

Proof. The proof is conducted using the Shirshov algorithm.
$< f_{12}, f_5^{(2)} >= f_{10}^{(1,2)} - s_2s_0s_{2,2n-2}f_5^{(1)}s_2$,
$< f_{10}^{(1,b)}, f_7^{(b,1)} >= f_{10}^{(2,b)} - s_2s_0s_{2,2n-2}s_1f_7^{(b,0)}$ if $2 \le b \le n$,

Similarly, other elements can also be found.For a detailed proof, you can refer to the thesis [13] □

Let R^B denote the set of polynomials as outlined in Lemma 2.2. Currently, we are unable to demonstrate that the provided polynomials in the lemma form a GSB for the infinite Coxeter group of type $\widetilde{B}_n$. Verifying this would involve intricate computations to confirm that the remaining compositions in R^B reduce to zero modulo R^B. Instead, we will utilize the Composition Diamond lemma to establish that R serves as a GSB for the infinite Coxeter group of type $\widetilde{B}_n$.

2.2. GSB for $\widetilde{D}_n$.

Definition 2.3. The presentation of the infinite Coxeter group of type $\widetilde{D}_n$ includes generators $S = \{s_0, s_1, \ldots, s_n\}$ for a positive integer $n \ge 4$ and the following defining relations:
(RD_1) $s_as_a = 1$ for $0 \le a \le n$,
(RD_2) $s_as_b = s_bs_a$ for $0 < a < b-1 < n$ but $(a,b) \ne (0,2)$ and $(a,b) \ne (n-2,n)$,
(RD_3) $s_as_{a+1}s_a = s_{a+1}s_as_{a+1}$ where $1 \le a < n-1$,
(RD_4) $s_{n-2}s_ns_{n-2} = s_ns_{n-2}s_n$,
(RD_5) $s_0s_2s_0 = s_2s_0s_2$.

For the sake of convenience, let us assume that

$$s_{ij} = \begin{cases} s_as_{a+1}\cdots s_b, & \text{if } 1 \le a < b < n \text{ ;} \\ s_as_{a+1}\cdots s_{n-2}s_ns_{n-1}\cdots s_{2n-b}, & \text{if } 1 \le a \le n-1 < b \le 2n-a; \\ s_a, & \text{if } b = a; \\ 1, & \text{if } b = a-1. \end{cases}$$

From this point forward, we will refrain from using superscripts unless it becomes necessary to distinguish between the groups $\widetilde{B}_n$ and $\widetilde{D}_n$.

Lemma 2.4. *Assume that $<$ denotes the degree lexicographic order on S^*. A GSB for the infinite Coxeter group of type $\widetilde{D}_n$ with respect to $<$ includes the following polynomials:*

- $g_1^{(a)} = s_as_a - 1$ *if* $0 \le a \le n$,
- $g_2^{(a,b)} = s_as_b - s_bs_a$ *if* $1 < b-a$ *but* $(a,b) \ne (0,2)$ *and* $(a,b) \ne (n-2,n)$,
- $g_3^{(a)} = s_{a,a+1} - s_{a+1}s_a$ *if* $a = 0, n-1$,
- $g_4 = s_{n-2,n}s_{n-2} - s_ns_{n-2,n}$,
- $g_5^{(a,b)} = s_{ab}s_a - s_{a+1}s_{ab}$ *if* $(1 \le a < b \le n-1)$ *or* $(1 \le a < n-2$ *and* $n \le b \le 2n-3$ *and* $2n-b-1 > 1)$,
- $g_6^{(a)} = s_{a,2n-a}s_{a+1} - s_{a+1}s_{a,2n-a}$ *if* $1 \le a \le n-3$,
- $g_7 = s_{n-2,n+2}s_n - s_{n-1}s_{n-2,n+2}$,
- $g_8 = s_{n-2,n+2}s_{n-1} - s_ns_{n-2,n+2}$,
- $g_9^{(a,b)} = s_0s_{2a}s_{1b}s_0 - s_2s_0s_{2a}s_{1b}$ *if* $0 \le b \le 1$ *and* $2 \le a \le 2n-3$,
- $g_{10}^{(a)} = s_0s_{2a}s_{1a} - s_1s_0s_{2a}s_{1,a-1}$ *if* $2 \le a \le n-1$,

- $g_{11} = s_0 s_{2n} s_{1n} - s_1 s_0 s_{2n} s_{1,n-2}$,
- $g_{12} = s_0 s_{2,n-1} s_{1,n+1} - s_1 s_0 s_{2,n-1} s_{1n}$,
- $g_{13}^{(a)} = s_0 s_{2,2n-a} s_{1,2n-a+1} - s_1 s_0 s_{2,2n-a} s_{1,2n-a}$ if $2 \leq a < n$,
- $g_{14}^{(a,b)} = s_0 s_{2a} s_{1b} s_0 s_{2b} - s_2 s_0 s_{2a} s_{1b} s_0 s_{2,b-1}$ if $(2 \leq b \leq n-1$ and $n \leq a \leq 2n-3)$ or $(2 \leq b < n-1$ and $3 \leq a \leq n-1$ and $b < a)$,
- $g_{15} = s_0 s_{2,n-1} s_{1n} s_0 s_{2n} - s_2 s_0 s_{2,n-1} s_{1n} s_0 s_{2,n-2}$,
- $g_{16}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_{1,a-1} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_{1,a-2}$ if $2 \leq a \leq b \leq n-1$,
- $g_{17} = s_0 s_{2,2n-2} s_0 s_2 - s_2 s_0 s_{2,2n-2} s_0$,
- $g_{18}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_1 - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2b}$ if $(a = 1$ and $2 \leq b \leq n-1)$ or $(1 \leq a \leq 2$ and $n \leq b \leq 2n-3)$,
- $g_{19}^{(a)} = s_0 s_{2,2n-a} s_{1,n-1} s_0 s_{2,n+1} - s_2 s_0 s_{2,2n-a} s_{1,n-1} s_0 s_{2n}$ if $3 \leq a \leq n$,
- $g_{20}^{(a)} = s_0 s_{2,2n-a} s_{1n} s_0 s_{2n} - s_2 s_0 s_{2,2n-a} s_{1n} s_0 s_{2,n-2}$ if $3 \leq a \leq n-1$,
- $g_{21}^{(a)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-2} s_{12} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-2} s_1$ if $1 \leq a \leq 2$,
- $g_{22} = s_0 s_{2,2n-2} s_{1n} s_0 s_{2n} s_{1,n-2} - s_2 s_0 s_{2,2n-2} s_{1n} s_0 s_{2n} s_{1,n-3}$,
- $g_{23}^{(a)} = s_0 s_{2,2n-2} s_{1,n-1} s_0 s_{2,2n-a} s_{1n} - s_2 s_0 s_{2,2n-2} s_{1,n-1} s_0 s_{2,2n-a} s_{1,n-2}$ if $2 \leq a \leq n-1$,
- $g_{24}^{(a)} = s_0 s_{2,2n-2} s_{1n} s_0 s_{2,2n-a} s_{1,n-1} - s_2 s_0 s_{2,2n-2} s_{1n} s_0 s_{2,2n-a} s_{1,n-2}$ if $2 \leq a \leq n-1$,
- $g_{25}^{(a,b)} = s_0 s_{2,2n-2} s_{1,2n-a} s_0 s_{2,2n-b} s_{1,2n-a+1}$
  $- s_2 s_0 s_{2,2n-2} s_{1,2n-a} s_0 s_{2,2n-b} s_{1,2n-a}$ if $2 \leq b < a \leq n$,
- $g_{26}^{(a,b)} = s_0 s_{2,2n-a} s_{1,2n-b} s_0 s_{2,2n-b+1} - s_2 s_0 s_{2,2n-a} s_{1,2n-b} s_0 s_{2,2n-b}$ if $3 \leq a \leq b \leq n-1$,
- $g_{27}^{(a,b,c)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-b} s_{1c} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-b} s_{1,c-1}$ if $(c = a$ and $2 \leq b \leq n-2$ and $3 \leq a \leq n-2)$ or $(c = a-1$ and $3 \leq a \leq n-2$ and $a < b \leq n)$,

Proof. As in the case of $\widetilde{B}_n$, the proof is established using the Shirshov algorithm. For a detailed proof, you can refer to the thesis [13].

□

At this stage, we are unable to demonstrate that the polynomials provided in the lemma form GSB for the infinite Coxeter group of type $\widetilde{D}_n$. We will demonstrate that the set of polynomials found for $\widetilde{B}_n$ and $\widetilde{D}_n$ indeed forms GSB for the infinite Coxeter group of types $\widetilde{B}_n$ and $\widetilde{D}_n$ by examining their normal forms, respectively.

3. Normal Forms

The necessary definitions and properties for the normal forms of $\widetilde{C}_n$ are provided in [13] and [15].

3.1. **Normal Forms for $\widetilde{B}_n$.** For $v \in \widetilde{S_n}^C$, let us define $v[a,b] = |\{t \in \mathbb{Z} : t \leq a, v(t) \geq b\}|$ for all $a, b \in \mathbb{Z}$. Now, consider $\widetilde{S}_n^B = \{u \in \widetilde{S}_n^C : u[n, n+1] \equiv 0 \bmod 2\}$ which is a subgroup of $\widetilde{S}_n^C$ consisting of elements in the form $\{u \in S_n^C : u[n, n+1] \equiv 0 \bmod 2\}$. It is clear that $\widetilde{S}_n^B$ is a subgroup of $\widetilde{S}_n^C$ with an index of 2. Moreover, for any $u \in \widetilde{S}_n^B$, we can represent it as $u = (s_{nb_n}^C s_{n-1,b_{n-1}}^C \cdots s_{1b_1}^C)(s_0^C s_{1,2n-1}^C)^{\alpha_{2n-1}} \cdots (s_0^C s_1^C)^{\alpha_1} (s_0^C)^{\alpha_0}$ where $\sum_{t=0}^{2n-1} \alpha_t$ is an even number. The following proposition affirms that $\widetilde{S}_n^B$ is indeed the infinite Coxeter group of type $\widetilde{B}_n$.

Proposition 3.1. ([2], Proposition 8.5.3)
The group $\widetilde{S}_n^B$ with generating set $\{s_0^B, s_1^B, \dots, s_n^B\}$ is the infinite Coxeter group of type $\widetilde{B}_n$ where $s_a^B = s_i^C$ for $a = 1, 2, \dots, n$ and $s_0^B = [2n-1, 2n, 3, \dots, n]$.

First of all, we give some relations between words in $\widetilde{B}_n$ and words in $\widetilde{C}_n$.

Lemma 3.2. *The following statements are equivalent.*
(i) $s_0^C s_1^C s_0^C = s_0^B$,
(ii) $(s_0^C s_{1a}^C)(s_0^C s_{1b}^C) = s_0^B s_{2a}^B s_{1b}^B$ *for* $0 \le a \leqslant b \le 2n-2$.

Proof. (i) $s_0^C s_1^C s_0^C = [2n, 2, \dots, n][2, 1, 3, \dots, n][2n, 2, \dots, n] = s_0^B$.
(ii) $s_0^B s_{2a}^B s_{1b}^B = s_0^C s_1^C s_0^C s_{2a}^C s_{1b}^C = s_0^C s_{1a}^C s_0^C s_{1b}^C$ by a series of ELW in $f_2^{(0,c)}$. □

It's worth mentioning that the length of a word in $\widetilde{C}_n$ is two greater than the length of the corresponding word in $\widetilde{B}_n$.

Lemma 3.3. *In the context of the infinite Coxeter group of type $\widetilde{C}_n$, the following relation is valid:*

$$(s_0^C s_{1,2n-2}^C)(s_0^C s_{1b}^C)(s_0^C s_{1a}^C) = \begin{cases} (s_0^C s_{1,2n-1}^C)(s_0^C s_{1a}^C)(s_0^C s_{1,b-1}^C), & \text{if } a+b < 2n, \\ (s_0^C s_{1,2n-1}^C)(s_0^C s_{1,a-1}^C)(s_0^C s_{1b}^C), & \text{if } a+b \ge 2n. \end{cases}$$

This equation is applicable for $1 \le a, b \le 2n-1$ with the condition that $b \le a$ when $a < n$ or $a < b$ when $a \ge n$.

Proof. In the scenario where $a + b < 2n$, there are two distinct cases to consider:
(i) $1 \le b \le a < n$,
(ii) $1 \le b < n \le a < 2n - b$.

In both of these cases, the following relationships hold:
$(s_0^C s_{1,2n-2}^C)(s_0^C s_{1b}^C)(s_0^C s_{1a}^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1b}^C)(s_0^C s_{1b-1}^C)(s_0^C s_{b+1,a}^C)$ applying by an ELW in $f_5^{(b)}$.
$(s_0^C s_{1,2n-1}^C)(s_0^C s_{1b}^C)(s_0^C s_{1b-1}^C)(s_0^C s_{b+1,a}^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1a}^C)(s_0^C s_{1,b-1}^C)$ applying by a series of ELW in f_2.
In the case where $2n \le a + b$, we have $n \le b < a \le 2n - 2$. Let $a = 2n - c$ and $j = 2n - d$. Therefore;

$$(s_0^C s_{1,2n-2}^C)(s_0^C s_{1b}^C)(s_0^C s_{1a}^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1b}^C)(s_0^C s_{1b}^C) s_{d-2} s_{d-3} \cdots s_c$$

due to an ELW in $f_6^{(b)}$. Furthermore; $(s_0^C s_{1a}^C) s_t = (s_0^c s_{1,t-1}^C) s_{t+1}^C s_{tb}^C$ by an ELW in $f_3^{(t,b)}$. $(s_0^c s_{1,t-1}^C) s_{t+1}^C s_{tb}^C = s_{t+1}^C s_{1b}^C$ by a series of ELW in f_2. This results in the desired equality. □

Corollary 3.4.

$$(s_0^C s_{1,2n-1}^C)(s_0^C s_{1a}^C)(s_0^C s_{1b}^C) = \begin{cases} (s_0^B s_{2,2n-2}^B s_{1,b+1}^B)(s_0^C s_{1a}^C), & a+b < 2n-1, \\ (s_0^B s_{2,2n-2}^B s_{1b}^B)(s_0^C s_{1,a+1}^C), & a+b \ge 2n-1. \end{cases}$$

This equation holds for $1 \le b \leqslant a \le 2n-2$.

Lemma 3.5. *Let $m \geq 1$.*

(i) $(s_0^C s_{1,2n-1}^C)^{2m} = (s_0^B s_{2,2n-2}^B s_1^B)^{2m}$,

(ii) $(s_0^C s_{1,2n-1}^C)^{2m-1}(s_0^C s_{1b}^C) = (s_0^B s_{2,2n-2}^B s_1^B)^{2m-1}(s_0^B s_{2b}^B)$ *for* $2 \leq b \leq 2n-2$,

(iii) $(s_0^C s_{1,2n-1}^C)^{2m-1} s_0^C = (s_0^B s_{2,2n-2}^B s_1^B)^{2(m-1)}(s_0^B s_{2,2n-2}^B)(s_0^B)$,

Proof.

(i) We will utilize induction with respect to m. $(s_0^B s_{2,2n-2}^B s_1^B)(s_0^B s_{2,2n-2}^B s_1^B) = (s_0^C s_{1,2n-2}^C s_0^C s_1^C)(s_0^C s_{1,2n-2}^C s_0^C s_1^C) = (s_0^C s_{1,2n-2}^C)(s_1^C s_0^C s_1^C s_0^C)(s_{2,2n-2}^C s_0^C s_1^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1,2n-2}^C)(s_0^C s_0^C s_1^C) = (s_0^C s_{1,2n-1}^C)^2$. The first equality is derived from Lemma 3.2, and the second and the third equalities stem from ELW in $f_5^{(1)}$ and $f_2^{(0,c)}$, respectively. Assume that $(s_0^B s_{2,2n-2}^B s_1^B)^{2c} = (s_0^C s_{1,2n-1}^C)^{2c}$ for a positive integer c. Consequently, $(s_0^B s_{2,2n-2}^B s_1^B)^{2(c+1)} = (s_0^C s_{1,2n-1}^C)^{2c}(s_0^B s_{2,2n-2}^B s_1^B)^2 = (s_0^C s_{1,2n-1}^C)^{2(c+1)}$.

(ii) $(s_0^B s_{2,2n-2}^B s_1^B)^{2m+1}(s_0^B s_{2b}^B) = (s_0^C s_{1,2n-1}^C)^{2m}(s_0^C s_{1,2n-2}^C s_0^C s_1^C)(s_0^C s_{1b}^C s_0^C)$ by Lemma 3.2.
$(s_0^C s_{1,2n-1}^C)^{2m}(s_0^C s_{1,2n-2}^C s_0^C s_1^C)(s_0^C s_{1b}^C s_0^C) = (s_0^C s_{1,2n-1}^C)^{2m} s_0^C s_{1,2n-2}^C s_1^C s_0^C s_1^C s_0^C s_{2b}^C s_0^C$ by ELW in $f_5^{(1)}$.
$(s_0^C s_{1,2n-1}^C)^{2m} s_0^C s_{1,2n-2}^C s_1^C s_0^C s_1^C s_0^C s_{2b}^C s_0^C = (s_0^C s_{1,2n-1}^C)^{2m+1} s_0^C s_{1b}^C s_0^C s_0^C$ by a series of ELW in $f_2^{(0,c)}$.
$(s_0^C s_{1,2n-1}^C)^{2m+1} s_0^C s_{1b}^C s_0^C s_0^C = (s_0^C s_{1,2n-1}^C)^{2m+1} s_0^C s_{1b}^C$ by ELW in $f_1^{(0)}$.

(iii) $(s_0^B s_{2,2n-1}^B)(s_0^B) = (s_0^C s_{1,2n-2}^C s_0^C)(s_0^C s_1^C s_0^C)$ by Lemma 3.2.
$(s_0^C s_{1,2n-2}^C s_0^C)(s_0^C s_1^C s_0^C) = s_0^C s_{1,2n-1}^C s_0^C$.

The remaining part follows as a straightforward consequence of part (i). □

It should be noted that the length of word in $\widetilde{C}_n$ is $2m$ greater than the length of the corresponding word in $\widetilde{B}_n$.

Definition 3.6. The following words are defined in $\widetilde{B}_n$:

(i) $w_0 = s_{nd_n}^B \cdots s_{ad_a}^B \cdots s_{1d_1}^B$ for $a-1 \leq d_a \leq 2n-a$ and $a = 1, \ldots, n$.

(ii) $w_1 = \prod_{i=1}^t (s_0^B s_{2,2n-2}^B s_{1a_i}^B)$ for $t \geq 0$ and $1 \leq a_i \lessdot a_{i-1} \leq 2n-2$.

(iii) $w_2 = \prod_{i=1}^s (s_0^B s_{2,b_{2i-1}}^B s_{1b_{2i}}^B)$ for $s \geq 0$ and $0 \leq b_i \lessdot b_{i-1} \leq 2n-3$.

(iv) $$w_3 = \begin{cases} (s_0^B s_{2n-2}^B s_1^B)^{2m}, & \\ (s_0^B s_{2n-2}^B s_1^B)^{2m-1}(s_0^B s_{2b}^B), & \text{for } m \geq 0 \text{ and } 1 \leq b \leq 2n-2, \\ (s_0^B s_{2n-2}^B s_1^B)^{2(m-1)}(s_0^B s_{2,2n-2}^B) s_0^B. & \end{cases}$$

(v) $w_4 = w_0 w_1 w_2$ where $a_t \geq 2$ and either $b_1 \lessdot a_t$ or $b_1 \not\lessdot a_t$ but
$$\begin{cases} b_2 \lessdot a_t, & a_t + b_1 \geq 2n; \\ b_2 + 1 < a_t, & a_t + b_1 < 2n. \end{cases},$$

(vi) $w_5 = w_0 w_1 w_3$.

Let $W_B = \{w_4, w_5\}$.

Theorem 3.7. *Any word $w \in W_C$ in which number of appearance of s_0 is even can be converted into a word in W_B.*

Proof. Since $s_a^B = s_a^C$ for $a = 1, \ldots, n$, we focus on words of the form $w = (s_0^C s_{1,2n-1}^C)^m \prod_{i=1}^t (s_0^C s_{1b_i}^C)$ where $m + t$ is even $0 \le b_i \leqslant b_{i-1} \le 2n - 2$.

If $m = 0$, then according to Lemma 3.2,we can write $w = \prod_{i=1}^{\frac{t}{2}} (s_0^B s_{2,b_{2i-1}}^B s_{b_{2i}}^B)$. As a result, w belongs to W_B.

Suppose that $m \ge 1$ and $2n - 2 = b_1 = b_2 = \cdots = b_d > b_{d+1}$. Then $w = \big(\prod_{i=1}^{\lfloor \frac{d+1}{2} \rfloor} (s_0^B s_{2,b_{2i-1}}^B s_{1,b_{2i}}^B)\big) w^{'}$ where $w^{'} = (s_0^C s_{1,2n-1}^C)^m \prod_{i=2\lfloor \frac{d+1}{2} \rfloor + 1}^t (s_0^C s_{1b_i}^C)$ by repeated applications of Corollary 3.4 and Lemma 3.2. Let us rewrite $w^{'}$ as follows, $w^{'} = (s_0^C s_{1,2n-1}^C)^m (s_0^C s_{1a}^C) \prod_{i=0}^p (s_0^C s_{1a_i}^C)$. Assume that $a + i + a_i \ge 2n - 1$ for $0 \le i \le q \le p$ and $a + q + 1 + a_i < 2n - 1$ for $q + 1 \le i \le p$. Let $x = (2n-2) - a$. Now we investigate each case separately. There are 6 cases.

Case (i): $q \ge x - 1$ and $m > x$. Corollary 3.4 and Lemma 3.2 imply that $w^{'} = \prod_{i=0}^x (s_0^B s_{2,2n-2}^B s_{1a_i}^B) w^{''}$ where $w^{''} = (s_0^C s_{1,2n-1}^C)^{m-x} \prod_{i=x+1}^p (s_0^C s_{1,x_i}^C)$. Now same process can be applied to $w^{''}$. This should be repeated until one of the conditions is not met. Therefore we can assume that $w^{'}$ does not satisfy one of the conditions without loss of generality.

Case (ii): $q \ge x - 1$ and $m = x$. Corollary 3.4 and Lemma 3.2 suggest that $w^{'} = \prod_{i=0}^m (s_0^B s_{2,2n-2}^B s_{1a_i}^B) \prod_{i=\frac{m+2}{2}}^{\frac{p}{2}} (s_0^B s_{2,a_{2i-1}}^B s_{1,a_{2i}})$ because of $a_x \leqslant a_{x+1}$, $w^{'} \in W_B$ and so is w.

Case (iii): $q \ge x - 1$ and $m < x$. Corollary 3.4 and Lemma 3.2 suggest that $w^{'} = \big(\prod_{i=0}^{m-1} (s_0^B s_{2,2n-2}^B s_{1a_i}^B)\big)(s_0^B s_{2,a+m}^B s_{1a_m}^B) \prod_{i=\frac{m+2}{2}}^{\frac{p}{2}} (s_0^B s_{2,a_{2i-1}}^B s_{1,a_{2i}})$. If $a + m \leqslant a_{m-1}$, then clearly $w^{'} \in W_B$ which implies $w \in W_B$. Suppose $a + m \nleqslant a_{m-1}$. Since $a_{m-1} + m + a \ge 2n$ and $a_m \leqslant a_{m-1}$, $w^{'} \in W_B$ and so is w.

Case (iv): $q < x - 1$ and $m \le q$. Similar to the scenario in case (iii).

Case (v): $q < x - 1$ and $q < m \le p$. $w^{'}$ equals

$$\Big(\prod_{i=0}^{q} (s_0^B s_{2,2n-2} s_{1a_i}^B)\Big)\Big(\prod_{i=q+1}^{m-1} (s_0^B s_{2,2n-2}^B s_{1a_i+1}^B)\Big)(s_0^B s_{2,a+q+1}^B s_{1a_m}^B) \prod_{i=\frac{m+2}{2}}^{\frac{p}{2}} (s_0^B s_{2,a_{2i-1}}^B s_{1,a_{2i}})$$

by Corollary 3.4 and Lemma 3.2. We can observe that $a_q > a_{q+1}$. If $a + q + 1 \leqslant a_{m-1} + 1$, then it is evident that $w^{'} \in W_B$ which consequently implies that $w \in W_B$. Now consider the scenario where $a + q + 1 \nleqslant a_{m-1} + 1$. In this case, $a_{m-1} + 1 \le a + q + 1$ and $a_m + a + q + 1 < 2n - 1$. It follows that $a_m < n$ and consequently $a_m + 1 < a_{m-1} + 1$. Therefore, we can conclude that $w^{'} \in W_B$ and hence w is also an element of W_B.

Case (vi): Applying Corollary 3.4 and Lemma 3.2 repeatedly provides the following, $w^{'} = \big(\prod_{i=0}^q (s_0^B s_{2,2n-2} s_{1a_i}^B)\big)\big(\prod_{i=q+1}^p (s_0^B s_{2,2n-2}^B s_{1a_i+1}^B)\big) w^{''}$ where

$$w^{''} = \begin{cases} (s_0^B s_{2,2n-2}^B s_1^B)^{m-p}, & a + q + 1 = 2n - 2 \\ (s_0^B s_{2,2n-2}^B s_1^B)^{m-p-1} (s_0^B s_{2,a+q+1}^B), & 1 \le a + q + 1 \le 2n - 3 \\ (s_0^B s_{2,2n-2}^B s_1^B)^{m-p-2} (s_0^B s_{2,2n-2}^B)(s_0^B), & a + q + 1 = 0 \end{cases}$$

by Lemma 3.5. Thus, it is evident that $w^{'} \in W_B$ and consequently, w is also an element of W_B.

□

Lemma 3.8. *The generating function for words in W_B is given by the expression:*

$$\prod_{a=1}^{n}(1+y+\cdots+y^{2a-1})\frac{1+y^a}{1-y^{n+a}}.$$

Proof. We have established a one to one correspondence between words in W_B and words in W_C with the even number of occurrence of s_0. Consider a word in W_C of the form $w = (s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1d_1})\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ where t is even and $0 \le b_i \leqslant b_{i-1} \le 2n-1$. Since $s^C_a = s^B_a$ for $a = 1,\dots,n$, $s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1d_1} = s^B_{nd_n}s^B_{n-1,d_{n-1}}\cdots s^B_{1d_1}$, we can express this word in W_B as $s^B_{nd_n}s^B_{n-1,d_{n-1}}\cdots s^B_{1d_1}$. The generating function for this form of word in W_B is $\prod_{a=1}^{n}(1+y+\cdots+y^{2a-1})$. When converting the $\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ part into a word in W_B, the corresponding word losses length by the number of occurrences of s_0. The generating function for the words in the form $\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ where $t \ge 0$ in W_C is $\prod_{a=1}^{n}\frac{1+y^a}{1-y^{n+a}}$. It is important to note we consider all words of the form $\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ where $t \ge 0$, and we can add or remove s^C_0 from the end of the word if the number of occurrences of s^C_0 is odd, without affecting the result. □

Consider the generating function for the infinite Coxeter group of type $\widetilde{B}_n$

$$\prod_{a=1}^{n}\frac{1+y+\cdots+y^{2a-1}}{1-y^{2a-1}}.$$

Using Section 7.1 in [2] we can express this as:

$$\prod_{a=1}^{n}(1+y+\cdots+y^{2a-1})(\frac{1+y^a}{1-y^{n+a}}) = \prod_{a=1}^{n}\frac{1+y+\cdots+y^{2a-1}}{1-y^{2a-1}}$$

which corresponds to the generating function for words in W_B.

With t is understanding in place, we can now proceed to unveil the main result about a GSB for the infinite Coxeter group of type $\widetilde{B}_n$..

Theorem 3.9. *Let R^B represent the set of all polynomials as described in Lemma 2.2. Then,*

(i) *$W_B = Red(R^B)$.*
(ii) *R^B serves as a GSB for the infinite Coxeter group of type $\widetilde{B}_n$.*

Proof. (i) It is evident that any word in W_B is R^B-reduced. Thus, we have $W_B \subseteq \mathrm{Red}(R^B)$. Conversely, if $w \in \mathrm{Red}(R^B)$, then w can be expressed as a permutation in $\widetilde{S}^B_n$. According to Theorem 3.7, this permutation corresponds to a word in W_B. Consequently, we obtain $\mathrm{Red}(R^B) \subseteq W_B$.

(ii) We know that any polynomial in R^B is part of a GSB for the infinite Coxeter group of type $\widetilde{B}_n$. If R^B were not a GSB, then, by the Composition Diamond lemma, $\mathrm{Red}(R^B)$ should be a proper subset of the set of normal forms in the infinite Coxeter group of type $\widetilde{B}_n$. This would contradict the fact that W_B and the normal forms of the infinite Coxeter group of type $\widetilde{B}_n$ share the same generating functions.

□

3.2. **Normal Forms for $\widetilde{D}_n$.** Define $\widetilde{S}_n^D$ as a subgroup of $\widetilde{S}_n^B$ consisting of those elements in $\widetilde{S}_n^B$ which, in their complete notation, exhibit an even number of negative entries to the right of 0. $\widetilde{S}_n^D = \{u \in \widetilde{S}_n^B \ : \ u[0,1] \equiv 0 \ (mod 2)\}$. Hence, it follows that $\widetilde{S}_n^D$ is a subgroup of $\widetilde{S}_n^B$ with an index of 2.

Proposition 3.10. ([2], Proposition 8.6.3)
The group $\widetilde{S}_n^D$ generated by $\{s_0^D, s_1^D, \ldots, s_n^D\}$, constitutes the infinite Coxeter group of type $\widetilde{D}_n$. In this group, $s_a^D = s_a^B$ for $a = 0, 1, 2, \ldots, n-1$ and $s_n^D = [(n-1 \ -n)]$.

Now, let's attempt to find normal form representations of elements in $\widetilde{D}_n$ with respect to these generators. First and foremost, we'll present some relations between words in $\widetilde{D}_n$ and words in $\widetilde{B}_n$.

Lemma 3.11. (i) $s_n^B s_{n-1}^B = s_n^D s_n^B$,
(ii) $s_n^B s_{n-1}^B s_n^B = s_n^D$,
(iii) $s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D s_n^B$,
(iv) $s_n^B s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D$.

Proof. (i) $s_n^B s_{n-1}^B = [(n \ -n)][(n-1 \ n)] = [(n-1 \ -n)][(n \ -n)] = s_n^D s_n^B$.
(ii) $s_n^B s_{n-1}^B s_n^B = s_n^D s_n^B s_n^B = s_n^D$ by part (i).
(iii) $s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D s_n^B$ by applying part (i) and ELW in $g_3^{(n-1)}$, respectively.
(iv) $s_n^B s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D$ by part (ii).

□

Lemma 3.12. *For* $1 \le a \le n-2$

$$s_{ab_a}^B = \begin{cases} s_{ab_a}^D, & b_a < n; \\ s_{a,n-1}^D s_n^B, & b_a = n; \\ s_{ab_i}^D s_n^B, & b_a > n. \end{cases}$$

Proof. Since $s_a^B = s_a^D$ for $1 \le a \le n-1$, we also have $s_{ab_a}^B = s_{ab_a}^D$ for $b_a < n$. Similarly $s_{an}^B = s_{a,n-1}^D s_n^B$. Now, let us consider the case where $b_a > n$ and $a \le n-2$. Then, if part (ii) of Lemma 3.11, ELW's in $f_2^{(i,n)}$ where $i = 2n - b_a, \ldots, n-2$ and ELW's in $f_4^{(n-1)}$ are applied, respectively, then $s_{ab_a}^D s_n^B = s_{ab_a}^B$ will be obtained.

□

Lemma 3.13. *For* $1 \le a \le n-2$

$$s_n^B s_{ab_a}^B = \begin{cases} s_{ab_a}^D s_n^B, & b_a \le n-2; \\ s_{an}^D s_n^B, & b_a = n-1; \\ s_{an}^D, & b_a = n; \\ s_{ab_a}^D, & b_a > n. \end{cases}$$

Proof. (i) $s_n^B s_{ab_a}^B = [(n \ -n)][(a \ a+1 \ \cdots \ b_a+1)] = [(a \ a+1 \ \cdots \ b_a+1)][(n \ -n)] = s_{ab_a}^B s_n^B$ because $b_a + 1 < n$. $s_{ab_a}^B s_n^B = s_{ab_a}^D s_n^B$ because $s_{ab_a}^B = s_{ab_a}^D$.
(ii) $s_{an}^D s_n^B = s_{a,n-2}^B s_n^B s_{n-1}^B s_n^B s_n^B$. If ELW's in $f_1^{(n)}$ and ELW's in $f_2^{(i,n)}$ for $i = n-2, \ldots, a$ are applied, respectively, then $s_{a,n-2}^B s_n^B s_{n-1}^B s_n^B s_n^B = s_n^B s_{a,n-1}^B$.
(iii) $s_n^B s_{an}^B = s_n^B s_{a,n-1}^B s_n^B$. Using part (ii), then $s_n^B s_{a,n-1}^B s_n^B = s_{a,n}^D s_n^B s_n^B = s_{a,n}^D$.

(iv) $s_n^B s_{ab_a}^B = s_n^B s_{a,n}^B s_{n-1}^B \cdots s_{2n-b_a}^B$. Using part (ii), then $s_n^B s_{a,n}^B s_{n-1}^B \cdots s_{2n-b_a}^B = s_{an}^D s_{n-1}^B \cdots s_{2n-b_a}^B = s_{ab_a}^D$ since $s_i^B = s_i^D$ for $i \neq n$.

$\square$

Definition 3.14. Let us consider a word w of the form $s_{nj_n}^B s_{n-1,b_{n-1}}^B \cdots s_{ab_a}^B \cdots s_{1b_1}^B$ where each b_a satisfies $a-1 \leq b_a \leq 2n-a$ for $1 \leq a \leq n$. We will define a function $n(w)$, which counts the number of occurrences of s_n in the word w.

The following corollary is a result of the equalities $s_n^B s_0^B = s_0^B s_n^B$, $s_0^B = s_0^D$ and the lemmas discussed above.

Corollary 3.15. Let $1 \leq b \leqslant a \leq 2n-2$.

$$s_0^B s_{2a}^B s_{1b}^B = \begin{cases} s_0^D s_{2a}^D s_{1b}^D, & a \leq n-1 \text{ or } b > n \\ s_0^D s_{2,n-1}^D s_{1b}^D s_n^B, & a = n \text{ and } b < n-1 \\ s_0^D s_{2,n-1}^D s_{1n}^D s_n^B, & a = n \text{ and } b = n-1 \\ s_0^D s_{2,n-1}^D s_{1n}^D, & a = n \text{ and } b = n \\ s_0^D s_{2a}^D s_{1b}^D s_n^B, & a > n \text{ and } b < n-1 \\ s_0^D s_{2a}^D s_{1n}^D s_n^B, & a > n \text{ and } b = n-1 \\ s_0^D s_{2a}^D s_{1n}^D, & a > n \text{ and } a = n \end{cases}$$

Corollary 3.16. Let $1 \leq b \leqslant a \leq 2n-2$.

$$s_n^B s_0^B s_{2a}^B s_{1b}^B = \begin{cases} s_0^D s_{2a}^D s_{1b}^D s_n^B, & a \leq n-1 \text{ or } b > n \\ s_0^D s_{2n}^D s_{1b}^D s_n^B, & a = n-1 \\ s_0^D s_{2a}^D s_{1b}^D, & a \geq n \text{ and } b < n \\ s_0^D s_{2a}^D s_{1,n-1}^D s_n^B, & a \geq n \text{ and } b = n \end{cases}$$

Definition 3.17.

$$a \lessdot b = \begin{cases} a \leq b, & \text{if } a \geq n+1; \\ b = n-1 \text{ or } b \geq n+1, & \text{if } a = n; \\ a < b, & \text{if } a \leq n-1. \end{cases}$$

It is clear that n and $n-1$ are not directly comparable. However, we can say that $n \lessdot n-1$ and $n-1 \lessdot n$.

Definition 3.18.

$$a \lesssim b = \begin{cases} a \leq b, & \text{if } a \geq n; \\ b = n-1 \text{ or } b \geq n+1, & \text{if } a = n-1; \\ a < b, & \text{if } a < n-1. \end{cases}$$

Indeed, it is important to note that n and $n-1$ are not directly comparable to each other.

Definition 3.19. We define the following words in $\widetilde{D}_n$,

(i) $w_0 = s_{nd_n}^D \cdots s_{ad_a}^D \cdots s_{1d_1}^D$ where $a-1 \leq d_a \leq 2n-a$ for $a = 1, \ldots, n$ except $n-2 \leq d_{n-1} \leq n-1$.

(ii) $w_1 = \prod_{i=1}^{t} (s_0^D s_{2,2n-2}^D s_{1,a_i}^D)$ for $t \geq 0$, $1 \leq a_i \lesssim a_{i-1} \leq 2n-2$.

(iii) $w_2 = \prod_{i=1}^{s} (s_0^D s_{2,b_{2i-1}}^D s_{1,b_{2i}}^D)$ for $s \geq 0$, $1 \leq b_i \lessdot b_{i-1} \leq 2n-3$.

(iv) $w_3 = \begin{cases} (s_0^D s_{2,2n-2}^D s_{1,b_{2i}}^D)^{2m}, \\ (s_0^D s_{2,2n-2}^D s_{1,b_{2i}}^D)^{2m-1} (s_0^D s_{2b}^D), \\ (s_0^D s_{2,2n-2}^D s_{1,b_{2i}}^D)^{2(m-1)} (s_0^D s_{2,2n-2}^D) s_0^D, \end{cases}$ for $m \geq 0$ and $1 \leq b \leq 2n-2$.

(v) $w_4 = w_0w_1w_2$ where $a_t \geq 2$ and either $b_1 \lessdot a_t$ or $b_1 \not\lessdot a_t$
but $\begin{cases} b_2 \lesssim a_t, & a_t + b_1 \geq 2n; \\ b_2 + 1 < a_t, & a_t + b_1 < 2n. \end{cases}$

(vi) $w_5 = w_0w_1w_3$

Let $W_D = \{w_4, w_5\}$.

Theorem 3.20. *Any word $w \in W_B$ where $n(w)$ is even can be transformed into a word in W_D.*

Proof. Let $w_0 = s^B_{nb_n}s^B_{n-1,b_{n-1}}\cdots s^B_{ab_a}\cdots s^B_{1b_1}$ where $a-1 \leq b_a \leq 2n-a$ for $1 \leq a \leq n$. Let $t_a = n(s^B_{nb_n}\cdots s^B_{a+1,b_{a+1}})$. Then

$$w_0 = \begin{cases} (s^D_{n,d_n}\cdots s^D_{ad_a}\cdots s^D_{1,d_1}), & n(w) \text{ is even;} \\ (s^D_{n,d_n}\cdots s^D_{ad_a}\cdots s^D_{1,d_1})s^B_n, & n(w) \text{ is odd.} \end{cases}$$

where

$$d_n = \begin{cases} n, & b_n = n \text{ or } b_{n-1} = n+1; \\ n-1, & \text{otherwise.} \end{cases},$$

$$d_{n-1} = \begin{cases} n-1, & b_{n-1} = n-1 \text{ or } b_{n-1} = n; \text{ and } b_n = n-1; \\ n-1, & b_{n-1} = n+1; \\ n-2, & \text{otherwise.} \end{cases}$$

and

$$d_i = \begin{cases} b_a, & b_a \neq n-1, n; \\ n-1, & b_a = n-1 \text{ or } b_a = n; \text{ and } t_a \text{ is even;} \\ n, & b_a = n-1 \text{ or } b_a = n; \text{ and } t_a \text{ is odd.} \end{cases}$$

for $a = n-2, n-3, \ldots, 1$.

The values of d_n and d_{n-1} can be easily determined using Lemma 3.11. To find the values of other d_a, apply recursively either Lemma 3.12 or Lemma 3.13 for $a = n-2, n-3, \ldots, 1$ while using the fact that $s^B_n s^B_n = 1$.

Consider $w_1 = \prod_{i=1}^{t}(s^B_0 s^B_{2,2n-2}s^B_{a_i})$ for $t \geq 0$ and $1 \leq a_i \lessdot a_{i-1} \leq 2n-2$ and let ζ be the count of a_i's that are less than or equal to $n-1$ in w_1. Through multiple applications of Corollary 3.15 and Corollary 3.16 imply that

$$w_1 = \begin{cases} \prod_{i=1}^{t}(s^D_0 s^D_{2,2n-2}s^D_{a_i}), & \zeta \text{ is even;} \\ (\prod_{i=1}^{t}(s^D_0 s^D_{2,2n-2}s^D_{a_i}))s^B_0, & \zeta \text{ is odd.} \end{cases}$$

where $b_i = a_i$ if $a_i \neq n-1$ and $b_i = n$ if $a_i = n-1$.

Now consider $\bar{w}_1 = s^B_n w_1$. Similarly

$$\bar{w}_1 = \begin{cases} \prod_{i=1}^{t}(s^D_0 s^D_{2,2n-2}s^D_{a_i}), & \zeta \text{ is odd;} \\ (\prod_{i=1}^{t}(s^D_0 s^D_{2,2n-2}s^D_{a_i}))s^B_0, & \zeta \text{ is even.} \end{cases}$$

where $b_i = a_i$ if $a_i \neq n$ and $b_i = n-1$ if $a_i = n$.

Hence both w_1 and $\bar{w}_1$ can be transformed one of the following

$$\begin{cases} \prod_{i=1}^{t}(s^D_0 s^D_{2,2n-2}s^D_{a_i}), \\ (\prod_{i=1}^{t}(s^D_0 s^D_{2,2n-2}s^D_{a_i}))s^B_0, \end{cases}$$

where for $t \geq 0$, $1 \leq a_i \lesssim a_{i-1} \leq 2n-2$.

∎

Lemma 3.21.

$$\prod_{a=1}^{n-1}[(1+y+y^2+\ldots+y^a)(1+y^a)] = (1+y+y^2+\ldots+y^{n-1})\prod_{a=1}^{n-1}(1+y+y^2+\ldots+y^{2a-1})$$

Proof. If n is odd, then

$$\begin{aligned}
\prod_{i=1}^{\frac{n-3}{2}}(1+y+y^2+\cdots+y^{2k})\prod_{t=1}^{n-1}(1+y^t) &= \prod_{i=1}^{\frac{n-3}{2}}(\frac{1-y^{2i+1}}{1-y})\prod_{t=1}^{n-1}(\frac{1-y^{2t}}{1-y^t}) \\
&= \frac{(1-y^{n+1})(1-y^{n+3})\cdots(1-y^{2n-2})}{(1-y)^{\frac{n-1}{2}}} \\
&= \frac{1-y^{n+1}}{1-y}\frac{1-y^{n+3}}{1-y}\cdots\frac{1-y^{2n-2}}{1-y} \\
&= \prod_{m=0}^{\frac{n-3}{2}}(1+y+y^2+\cdots+y^{n+2m}).
\end{aligned}$$

If n is even, then

$$\begin{aligned}
\prod_{i=1}^{\frac{n-2}{2}}(1+y+y^2+\cdots+y^{2k})\prod_{t=1}^{n-1}(1+y^t) &= \prod_{i=1}^{\frac{n-2}{2}}(\frac{1-y^{2i+1}}{1-x^i})\prod_{t=1}^{n-1}(\frac{1-y^{2t}}{1-y^t}) \\
&= \frac{(1-y^{n})(1-y^{n+2})\cdots(1-y^{2n-2})}{(1-y)^{\frac{n}{2}}} \\
&= \frac{1-y^{n}}{1-y}\frac{1-y^{n+2}}{1-y}\cdots\frac{1-y^{2n-2}}{1-y} \\
&= \prod_{m=0}^{\frac{n-1}{2}}(1+y+y^2+\cdots+y^{n+2m-2}).
\end{aligned}$$

□

Lemma 3.22. *The generating function for word in W_D is given by:*

$$\frac{1+y+\cdots+y^{n-1}}{1-y^{n-1}}\prod_{a=1}^{n-1}\frac{1+y^a}{1-y^{n-1+a}}.$$

Proof. We have established one to one correspondence between words in W_D and the words in W_C where the numbers of occurrences of both s_0 and s_n are even. Let us consider a word w of the form:

$$w = (s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1,d_1})\prod_{i=1}^{t}(s^C_0 s^C_{1b_i}).$$

Here, t is even, $n(w)$ is even and $0 \le b_i \leqslant b_{i-1} \le 2n-1$. First, we examiner the part of the word $s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1,d_1}$, which corresponds to $s^B_{nd_n}s^B_{n-1,d_{n-1}}\cdots s^B_{1,d_1}$ in W_B. According to Theorem 3.20, the corresponding word in W_D has

$$s^D_n s^D_{n-1}s^D_{n-2,b_{n-2}}\cdots s^D_{1b_1}$$

where $a-1 \le b_a \le 2n-a$. The generating function for these words is $(1+y)^2\prod_{a=2}^{n-1}(1+y+y^2+\cdots+y^{a-1}+2y^a+y^{a+1}+\cdots+y^{2a}) = \prod_{a=1}^{n-1}(1+y+\cdots+y^i)(1+y^i) = (1+y+y^2+\cdots+y^{n-1})\prod_{a=1}^{n-1}(1+y+y^2+\cdots+y^{2a-1})$ as given by

Lemma 3.21. Now, let us analyze the word $\overline{w} = \prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$, where t is even, and $n(\overline{w})$ is even. We are assuming that $n(s_{nb_n}^B s_{n-1,b_{n-1}}^B \cdots s_{1,b_1}^B)$ is even; otherwise, we would consider the word $s_n^B\overline{w}$. When converting the word $\prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$ into a word in W_D, the resulting word loses its length due to the number of occurrences of both s_0 and s_n. The generating function for words in the form $\prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$ in W_C is given by $\prod_{a=1}^{n}\frac{1+y^a}{1-y^{n+a}}$. Hence, the generating function for the corresponding words in W_D is $\frac{(1+y)(1+y^2)\cdots(1+y^{n-1})}{(1-y^{n-1})(1-y^n)\cdots(1-y^{2n-2})} = \frac{1}{y^{n-1}}\prod_{a=1}^{n-1}\frac{1+y^a}{1-y^{n-1+a}}$. □

We've established that the generating function for the infinite Coxeter group of type $\widetilde{D}_n$ can be expressed as:

$$\frac{1+y+\cdots+y^{n-1}}{1-y^{n-1}}\prod_{a=1}^{n-1}\frac{1+y+\ldots+y^{2a-1}}{1-y^{2a-1}}.$$

Using Section 7.1 in [2], we can simplify this expression to

$$\frac{1+y+\cdots+y^{n-1}}{1-y^{n-1}}\prod_{a=1}^{n-1}\frac{1+y+\ldots+y^{2a-1}}{1-y^{2a-1}} = (\prod_{a=1}^{n-1}(1+y+\ldots+y^{2a-1})(\frac{1+y^a}{1-y^{n-1+a}})).$$

This result matches the generating function of words in W_D.

Now, we are ready to present the main result about a GSB for the infinite Coxeter group of type $\widetilde{D}_n$.

Theorem 3.23. *Let R^D be the set of all polynomials as provided in Lemma 2.4. Then*

(i) $W_D = Red(R^D)$.
(ii) *R^D is a GSB for the infinite Coxeter group of type $\widetilde{D}_n$.*

Proof. (i) It is evident that any word in W_D is R^D-reduced. Therefore, we have $W_D \subseteq \mathrm{Red}(R^D)$. Conversely, if $w \in \mathrm{Red}(R^D)$, then w can be expressed as a permutation in $\widetilde{S}_n^D$, and this permutation corresponds to a word in W_D according to Theorem 3.20. Hence, we have $\mathrm{Red}(R^D) \subseteq W_D$.

(ii) We understand that any polynomial in R^D forms part of a GSB of the infinite Coxeter group of type $\widetilde{D}_n$. If, hypothetically, R^D were not a GSB, then according to Composition Diamond lemma, $\mathrm{Red}(R^D) = W_B$ would be a proper subset of the set of normal forms of the infinite Coxeter group of type $\widetilde{D}_n$. This would contradict to the fact that W_D and normal forms of the infinite Coxeter group of type $\widetilde{D}_n$ share same generating functions.

□

4. Conclusion

The main purpose of this article is to derive the GSB and normal forms for infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$. Similar to many previously mentioned papers, we use the Shirshov algorithm to obtain a set of R relations. We used it partially. We then asserted that $\mathrm{Red}(R)$ is equal to the set of normal forms of infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$. Then, by applying the Composition Diamond lemma, we find that R forms a GSB. At this stage, we took advantage of the combinatorial properties of infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$ as presented in [2]. Using this information, we determined a set of normal forms for this group and designed a method to determine the normal form of each element of the group when provided in permutation form. As a result, we have determined the normal form of the product of two normal forms. As a result, the group is completely characterized in terms of these normal forms.

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