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Complete Rewriting System of Schützenberger – Crossed Product of Monoids

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Abstract

In this paper, we study on complete rewriting system for the Schützenberger - crossed product of two cyclic monoids, which is defined in (Emin et al. 2013). Additionally, we obtain normal form structure of elements of this monoid construction and give solvability of the word problem. Finally, we present an example part.

Key Words: Schützenberger product, crossed product, rewriting system, normal form

Monoidlerin Schützenberger – Çapraz Çarpımının Tam Yeniden Yazma Sistemi

Öz

Bu makalede, (Emin ve ark. 2013) de tanımlanan iki devirli monoidin Schützenberger – çapraz çarpımının tam yeniden yazma sistemi çalışılmıştır. Ayrıca bu monoid yapısının elemanlarının normal form yapısı elde edilmiştir ve kelime probleminin çözülebilirliği verilmiştir. Son olarak, bir örnek bölüm sunulmuştur.

Anahtar Kelimeler: Schützenberger çarpım, çapraz çarpım, yeniden yazma sistemi, normal form

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1. INTRODUCTION AND PRELIMINARIES

The origin of Combinatorial Group Theory can be traced back to 1911 when Max Dehn posed three questions concerning groups defined by finite presentations: the word, conjugacy and isomorphism problems (Adyan and Durney 2000). These questions prompted the idea of using algorithms to solve problems related to Group Theory.

In this paper, we study on solvability of the word problem for a new monoid product, which is defined in (Emin et al. 2013) and called Schützenberger-crossed product of monoids. To have solvability word problem for this construction we study on complete rewriting system.

The aim of the rest of this section is just to give the standard definitions and information about complete rewriting system, crossed product and Schützenberger product of monoids.

Let X be a finite alphabet and let X^* be the free monoid consisting of all words obtained by the letters of X. A *string rewriting system*, or simply a rewriting system, on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(x, y) \in R,$ also written as $x \to y$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions. Furthermore an element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$; otherwise x is called reducible.

The rewriting system R is called

- *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in X^*$,
- *Confluent* if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in X^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- *Complete* if R is both Noetherian and confluent.

If R is a complete rewriting system, then for every word x there is a unique irreducible word y such that $x \to^* y$; this word is called the normal form of x. Each element of the monoid presented by $\langle X \mid R \rangle$ has a unique normal form representive. For $u, v \in X^*$, if $|u| > |v|$ or if $|u| = |v|$ and v precedes u in the lexicograpfic ordering induced by a linear ordering on X then we write $v < u$ and $<$ is called *length-lexicographic ordering*. A rewriting system is R is called a length-lexicographic rewriting system if $s < r$, for all $(r, s) \in R$. It is clear that length-lexicographic rewriting system is Noetherian. A critical pair of a rewriting system R is a pair of overlapping rules such that one of the forms

(i) $(r_1 r_2, s), (r_2 r_3, t) \in R$ with $r_2 \neq 1$ or

(ii) $(r_1 r_2 r_3, s), (r_2, t) \in R,$

is satisfied. Also a critical pair is resolved in R if there is a word z such that $s r_3 \to^* z$ and $r_1 t \to^* z$ in the first case or $s \to^* z$ and $r_1 t r_3 \to^* z$ in the second case. A Noetherian rewriting system is complete if and only if every critical pair is resolved. A rewriting system is complete then the word problem for given rewriting system is solvable.

We note that the reader is referred to (Book 1987; Book and Otto 1993; Sims 1994) for a detailed survey on (complete) rewriting systems and to (Çetinalp et al. 2016; Çetinalp and Karpuz 2018; Çetinalp et al. 2019; Dehn 1911) for complete rewriting systems of some group and semigroup constructions.

As known crossed product construction appears in different areas of algebra such as Lie algebras, C^*-algebras and group theory. This product has also many applications in other fields of mathematics like group representation theory and topology. This product is more important than known group constructions since it contains direct and semidirect products of groups/monoids. One can also study this new product in many applications of Hopf algebra and C^*-algebra.

Definition 1.1 *Let A and B two monoids. A crossed system of these monoids is a quadruple (A, B, α, f), where $\alpha : B \to End(A)$ and $f : B \times B \to A$ are two maps such that the following compatibility conditions hold:*

$$\alpha_{b_1}(\alpha_{b_2}(a)) f(b_1, b_2) = f(b_1, b_2) \alpha_{b_1 b_2}(a), \quad (1)$$

$$f(b_1, b_2) f(b_1 b_2, b_3) = \alpha_{b_1}(f(b_2, b_3)) f(b_1, b_2 b_3) \quad (2)$$

for all $b_1, b_2, b_3 \in B$ and $a \in A$. The crossed system (A, B, α, f) is called normalized if $f(1_B, 1_B) = 1_A$. If (A, B, α, f) is normalized crossed system then $f(1_B, b) = f(b, 1_B) = 1_A$ and $\alpha_{1_B}(a) = a$ (see (Agore and Militaru 2008)).

Now let $A \#_\alpha^f B := A \times B$ be a set with a binary operation defined by the formula: $(a_1, b_1).(a_2, b_2) = (a_1(\alpha_{b_1}(a_2)) f(b_1, b_2), b_1 b_2),$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $(A \#_\alpha^f B, .)$ is a monoid with the unit $1_{A \#_\alpha^f B} = (1_A, 1_B)$ if and only if (A, B, α, f) is a normalized crossed system. In this case the monoid $A \#_\alpha^f B$ is called *the crossed product of A and B associated to the crossed system* (A, B, α, f) (Agore and Militaru 2008).

The reader is referred to (Agore and Militaru 2008; Agore and Fratila 2010; Ateş et al. 2021; Çetinalp 2022; Çetinalp 2023; Çevik et al. 2020; Karpuz and Çetinalp 2016) for more details on crossed product constuctions and related results.

The Schützenberger product operation on monoids is a well known construction introduced by M. P. Schützenberger (Schützenberger 1965). This product was originally defined for two monoids in view of applications to Language Theory. The Schützenberger product plays an important role in the study of several problems of Automata Theory, such as, the Dot Depth Hierarchy of regular languages and studying concatenation product. In (Howie and Ruškuc 1994), the authors obtained a presentation for Schützenberger product of two monoids and gave the normal form structure of the elements of this product. In (Ateş et al. 2011), the authors obtained normal form of elements of Schützenberger product of two monoids by using Gröbner-Shirshov bases theory. Many authors combined Schützenberger product with different products to obtain a new monoid construction. As an example of these works, in (Ateş 2009), Ateş obtained a new monoid construction under semidirect product and Schützenberger product.

The reader is also referred to (Ateş et al. 2009; Straubing 1981) for related results on Schützenberger product constuction.

Definition 1.2 *Let A and B be monoids. For $P \subseteq A \times B$ and $a \in A, b \in B$, we define $aP = \{(ac, d) \mid (c, d) \in P\}$ and $Pb = \{(c, db) \mid (c, d) \in P\}$. Then the Schützenberger product of monoids A and B, denoted by $A \Diamond B$, is the set $A \times \mathrm{P}(A \times B) \times B$ (where $\mathrm{P}(.)$ denotes the power set) with the multiplication given by*

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 a_2, P_1 b_2 \cup a_1 P_2, b_1 b_2).$$

It is known that $A \Diamond B$ is a monoid with identity $(1_A, \varnothing, 1_B)$ (see (Howie and Ruškuc 1994)).

The reader is referred to (Gracinda et al. 2006; Karpuz et al. 2010; Karpuz et al. 2016; Karpuz and Çetinalp 2024; Schützenberger 1965) for some algebraic results of Schützenberger product.

2. SCHUTZENBERGER – CROSSED PRODUCT OF MONOIDS

In this section, we obtain a complete rewriting system for the Schützenberger-crossed product of two cyclic monoids by using the presentation of this product given in (Emin et al. 2013).

Definition 2.1 (Emin et al. 2013) Let A and B be monoids. For $P \subseteq A \times B$ and $b \in B$, we define $Pb = \{(a, db) : (a, d) \in P\}$. Then the Schützenberger - crossed product of monoids A and B, denoted by $A_{cp}\#_{\alpha}^{f} B$, is the set $A \times \mathrm{P}(A \times B) \times B$ with the multiplication
$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1\alpha_{b_1}(a_2) f(b_1, b_2), P_1 b_2 \cup P_2, b_1 b_2)$.

This product defines a monoid with unit element $(1_A, \varnothing, 1_B)$.

Theorem 2.2 (Emin et al. 2013) Let us suppose that the monoids $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined by presentations $\langle\, \boldsymbol{X} ; \boldsymbol{R} \,\rangle$ and $\langle\, \boldsymbol{Y} ; \boldsymbol{S} \,\rangle$, respectively. Then the Schützenberger-crossed product of monoids **A** and **B**, $\boldsymbol{A_{cp}\#_{\alpha}^{f} B}$, is defined by generators $\boldsymbol{Z} = \boldsymbol{X} \cup \boldsymbol{Y} \cup \{\, \boldsymbol{z_{a,b}};\ \boldsymbol{a} \in \boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{B} \,\}$, and relations

(1) R, (2) $S = W_S$, (3) $yx = \alpha_y(x)y$,

(4) $z_{a,b}^2 = z_{a,b}$, (5) $z_{a,b} z_{c,d} = z_{c,d} z_{a,b}$,

(6) $z_{a,b} y = y z_{a,by}$, (7) $x z_{a,b} = z_{a,b} x$,

$(x \in X, y \in Y, a, c \in A, b, d \in B)$, where W_S is the word on X.

In Theorem 2.2, by considering two finite cyclic monoids we get the following presentation.

Corollary 2.3 Let C_n and C_m be finite cyclic monoids presented by $C_n = \langle\, x\,;\ x^n = 1 \,\rangle$ and $C_m = \langle\, y\,;\ y^m = 1 \,\rangle$, respectively. Then the Schützenberger-crossed product of these monoids, $C_{n\ cp}\#_{\alpha}^{f} C_m$, has the following generators,

$$x, y, z_{1,1}, z_{1,y}, \dots, z_{1,y^{m-2}}, z_{1,y^{m-1}}, z_{x,1}, z_{x,y}, \dots, z_{x,y^{m-1}},$$
$$\dots, z_{x^{n-2},1}, z_{x^{n-2},y}, \dots, z_{x^{n-2},y^{m-2}}, z_{x^{n-2},y^{m-1}}, z_{x^{n-1},1},$$
$$z_{x^{n-1},y}, \dots, z_{x^{n-1},y^{m-2}}, z_{x^{n-1},y^{m-1}}$$

and the following relations

(1) $x^n = 1$, (2) $y^m = x^k$ $(m > k,\ 0 \le k \le n-1)$,

(3) $x^t y = yx$ $(0 < t \le n-1)$,

(4) $z_{x^p,y^q}^2 = z_{x^p,y^q}$,

(5) $z_{x^{p_1},y^{q_1}} z_{x^{p_2},y^{q_2}} = z_{x^{p_2},y^{q_2}} z_{x^{p_1},y^{q_1}}$,

(6) $z_{x^p,y^q}\, y = y\, z_{x^p,y^q y}$,

(7) $x z_{x^p,y^q} = z_{x^p,y^q} x$.

Here $0 \le p, p_1, p_2 \le n-1$ and $0 \le q, q_1, q_2 \le m-1$.

Regarding the numbers of relations given in the forms (1)-(7) in Corollary 2.3, the following numerical values and formulas are obtained.

Number of relations of the form (1): 1

Number of relations of the form (2): 1

Number of relations of the form (3): 1

Number of relations of the form (4): mn

Number of relations of the form (5): $n\left(\frac{(m-1)m}{2}\right) + m^2\left(\frac{(n-1)n}{2}\right)$

Number of relations of the form (6): mn

Number of relations of the form (7): mn

With an easy calculation, the following result is obtained.

Corollary 2.4 The total numbers of generators and relations in the presentation given in Corollary 2.3 are formulated as $mn + 2$ and $\frac{mn(5+mn)+6}{2}$, respectively.

Now we can give the main result of this section. To do that, we consider the generators and relations given in Corollary 2.3 and order the generators as follows.

$$x > z_{x^{n-1},y^{m-1}} > z_{x^{n-1},y^{m-2}} > \cdots > z_{x^{n-1},y} > z_{x^{n-1},1} > \cdots > z_{x,y^{m-1}} > z_{x,y^{m-2}} > \cdots > z_{x,y} > z_{x,1} > \cdots > z_{1,y^{m-1}} > z_{1,y^{m-2}} > \cdots > z_{1,y^2} > z_{1,y} > z_{1,1} > y \quad (3)$$

We note that we consider the reductions steps on words by taking into account length-lexicographical order on words. We also note that the notation $(r) \cap (p)$ denotes the overlapping word of left-hand sides of relations (r) and (p).

Theorem 2.5 Let $C_n = \langle\, x\,;\ x^n = 1 \,\rangle$ and $C_m = \langle\, y\,;\ y^m = 1 \,\rangle$. A complete rewriting system for Schützenberger-crossed product of monoids C_n and C_m, $C_{n\ cp}\#_{\alpha}^{f} C_m$, consists of the following rules by considering the order on generators given by (3):

(1) $x^n \to 1$,

(2) $y^m \to x^k$ $(m > k,\ 0 \le k \le n-1)$,

(3) $x^t y \to yx$ $(0 < t \le n-1)$,

(4) $z_{x^p,y^q}^2 \to z_{x^p,y^q}$,

(5) $z_{x^{p_1},y^{q_1}} z_{x^{p_2},y^{q_2}} \to z_{x^{p_2},y^{q_2}} z_{x^{p_1},y^{q_1}}$,

(6) $z_{x^p,y^q}\, y = y\, z_{x^p,y^q y}$,

(7) $x z_{x^p,y^q} \to z_{x^p,y^q} x$.

Proof: This rewriting system is Noetherian since there is no infinite chain of rewritings of overlapping words for the given length-lexicographical order. In order to show the second condition, the confluent property (diamond rule), the words obtained by appropriate overlappings of the words on the left-hand side of all rewriting rules and the critical pairs formed by the first reductions of these words are given below.

$(1)\cap(1):\ x^{n+1}\rightarrow\begin{cases}x\\ x\end{cases}$

$(1)\cap(3):\ x^{n}y\ \rightarrow\begin{cases}y\\ x^{n-1}yx\rightarrow\ldots\rightarrow yx^{n}\rightarrow y\end{cases}$

$(1)\cap(7):\ x^{n}z_{x^p,y^q}$
$\rightarrow\begin{cases}z_{x^p,y^q}\\ x^{n-1}z_{x^p,y^q}x\rightarrow\ldots\rightarrow z_{x^p,y^q}x^{n}\rightarrow z_{x^p,y^q}\end{cases}$

$(2)\cap(2):\ y^{m+1}\rightarrow\begin{cases}x^{k}y\rightarrow x^{k-1}yx\rightarrow\ldots\rightarrow yx^{k}\\ yx^{k}\end{cases}$

$(3)\cap(2):\ x^{t}y^{m}\rightarrow\begin{cases}yxy^{m-1}\rightarrow\ldots\rightarrow y^{m}x\rightarrow x^{k+1}\\ x^{k+1}\end{cases}\quad(t=1)$

$(4)\cap(4):\ z^{3}_{x^p,y^q}\rightarrow\begin{cases}z^{2}_{x^p,y^q}\rightarrow z_{x^p,y^q}\\ z^{2}_{x^p,y^q}\rightarrow z_{x^p,y^q}\end{cases}$

$(4)\cap(5):\ z^{2}_{x^{p_1},y^{q_1}}z_{x^{p_2},y^{q_2}}$
$\rightarrow\begin{cases}z_{x^{p_1},y^{q_1}}\,z_{x^{p_2},y^{q_2}}\rightarrow z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\\ z_{x^{p_1},y^{q_1}}\,z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\rightarrow z^{2}_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\rightarrow z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\end{cases}$

$(4)\cap(6):\ z^{2}_{x^p,y^q}y$
$\rightarrow\begin{cases}z_{x^p,y^q}y\rightarrow yz_{x^p,y^{q+1}}\\ z_{x^p,y^q}yz_{x^p,y^{q+1}}\rightarrow yz^{2}_{x^p,y^{q+1}}\rightarrow yz_{x^p,y^{q+1}}\end{cases}$

$(5)\cap(4):\ z^{2}_{x^{p_1},y^{q_1}}z_{x^{p_2},y^{q_2}}$
$\rightarrow\begin{cases}z_{x^{p_1},y^{q_1}}\,z_{x^{p_2},y^{q_2}}\rightarrow z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\\ z_{x^{p_1},y^{q_1}}\,z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\rightarrow z_{x^{p_2},y^{q_2}}z^{2}_{x^{p_1},y^{q_1}}\rightarrow z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\end{cases}$

$(5)\cap(6):\ z_{x^{p_1},y^{q_1}}z_{x^{p_2},y^{q_2}}y$
$\rightarrow\begin{cases}z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}y\rightarrow z_{x^{p_2},y^{q_2}}yz_{x^{p_1},y^{q_1+1}}\\ \quad\rightarrow yz_{x^{p_2},y^{q_2+1}}z_{x^{p_1},y^{q_1+1}}\\ z_{x^{p_1},y^{q_1}}\,yz_{x^{p_2},y^{q_2+1}}\rightarrow yz_{x^{p_1},y^{q_1+1}}z_{x^{p_2},y^{q_2+1}}\\ \quad\rightarrow yz_{x^{p_2},y^{q_2+1}}z_{x^{p_1},y^{q_1+1}}\end{cases}$

$(6)\cap(2)$: $z_{x^p,y^q}y^{m}$

$\rightarrow\begin{cases}yz_{x^p,y^{q+1}}y^{m-1}\rightarrow\cdots\rightarrow y^{m}z_{x^p,y^{q+m}}\rightarrow x^{k}z_{x^p,y^{q+m}}\\ \quad\rightarrow z_{x^p,y^{q+m}}x^{k}\rightarrow\cdots\rightarrow z_{x^p,y^q}x^{k}\\ z_{x^p,y^q}x^{k}\end{cases}$

$(7)\cap(4):\ xz^{2}_{x^p,y^q}\rightarrow\begin{cases}z_{x^p,y^q}xz_{x^p,y^q}\rightarrow z^{2}_{x^p,y^q}x\rightarrow z_{x^p,y^q}x\\ xz_{x^p,y^q}\rightarrow z_{x^p,y^q}x\end{cases}$

$(7)\cap(5):\ xz_{x^{p_1},y^{q_1}}z_{x^{p_2},y^{q_2}}$
$\rightarrow\begin{cases}z_{x^{p_1},y^{q_1}}xz_{x^{p_2},y^{q_2}}\rightarrow z_{x^{p_1},y^{q_1}}z_{x^{p_2},y^{q_2}}x\rightarrow z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}x\\ xz_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}\rightarrow z_{x^{p_2},y^{q_2}}xz_{x^{p_1},y^{q_1}}\rightarrow z_{x^{p_2},y^{q_2}}z_{x^{p_1},y^{q_1}}x\end{cases}$

$(7)\cap(6)$: $xz_{x^p,y^q}y$

$\rightarrow\begin{cases}z_{x^p,y^q}xy\rightarrow z_{x^p,y^q}yx\rightarrow yz_{x^p,y^{q+1}}x\\ xyz_{x^p,y^{q+1}}\rightarrow yxz_{x^p,y^{q+1}}\rightarrow yz_{x^p,y^{q+1}}x\end{cases}$

It is seen that all overlapping words are reduced to the same words after appropriate steps. Therefore, the confluent property for the given rewriting system is also satisfied. Consequently, since the presentation of $C_{n\,cp}\#^{f}_{\alpha}C_m$ is Noetherian and confluent, it is complete. Hence the result.

By considering Theorem 2.5, we have the following other result of this section.

Corollary 2.6 The normal form of a word w, representing an element of $C_{n\,cp}\#^{f}_{\alpha}C_m$, is

$$y^{k}W_{z_{a',b'}}x^{l}\quad(0\le k\le m-1,\,0\le l\le n-1),\qquad(4)$$

where $W_{z_{a',b'}}$ $(a'\in C_n,\ b'\in C_m)$ is a reduced word obtained by generators $z_{a,b}$ $(a\in C_n, b\in C_m)$.

By considering Theorem 2.5 and Corollary 2.6, we can give the following result.

Corollary 2.7 Let C_n and C_m be finite cyclic monoids. Then the word problem for Schützenberger-crossed product of these monoids, $C_{n\,cp}\#^{f}_{\alpha}C_m$, is decidable.

Proof: By Theorem 2.5, since the rewriting system of Schützenberger-crossed product of $C_{n\,cp}\#^{f}_{\alpha}C_m$ is Noetherian and confluent, this system is complete. By this complete rewriting system, each element of the monoid of $C_{n\,cp}\#^{f}_{\alpha}C_m$ has a unique structure containing normal form given in (4). Thus, the word problem for of $C_{n\,cp}\#^{f}_{\alpha}C_m$ is decidable.

3. EXAMPLE PART

In this section, by considering two cyclic monoids with ranks 2 and 3, we give applications of Theorem 2.5 and Corollary 2.6.

Let $C_2=\langle\, x\,;\ x^2=1\,\rangle$ and $C_3=\langle\, y\,;\ y^3=1\,\rangle$ be two finite cyclic monoids. The generator set of the Schützenberger-crossed product of these monoids, $C_{2\,cp}\#^{f}_{\alpha}C_3$, is $\{\,x, y, z_{1,1}, z_{1,y}, z_{1,y^2}, z_{x,1}, z_{x,y}, z_{x,y^2}\,\}$.

Now we order these generators as follows.

$$x > z_{x,y^2} > z_{x,y} > z_{x,1} > z_{1,y^2} > z_{1,y} > z_{1,1} > y.$$

By the ordering given above, the monoid $C_{2\,cp}\#^{f}_{\alpha}C_3$ has the following rewriting system.

(1) $x^2\rightarrow 1$, (2) $y^3\rightarrow x$, (3) $xy\rightarrow yx$,

(4) $z_{1,1}^2 \to z_{1,1}$, $z_{1,y}^2 \to z_{1,y}$, $z_{1,y^2}^2 \to z_{1,y^2}$, $z_{x,1}^2 \to z_{x,1}$,
$z_{x,y}^2 \to z_{x,y}$, $z_{x,y^2}^2 \to z_{x,y^2}$,

(5) $z_{x,y^2} z_{x,y} \to z_{x,y} z_{x,y^2}$, $z_{x,y^2} z_{x,1} \to z_{x,1} z_{x,y^2}$, $z_{x,y^2} z_{1,y^2} \to z_{1,y^2} z_{x,y^2}$
$z_{x,y^2} z_{1,y} \to z_{1,y} z_{x,y^2}$, $z_{x,y^2} z_{1,1} \to z_{1,1} z_{x,y^2}$, $z_{x,y} z_{x,1} \to z_{x,1} z_{x,y}$
$z_{x,y} z_{1,y^2} \to z_{1,y^2} z_{x,y}$, $z_{x,y} z_{1,y} \to z_{1,y} z_{x,y}$, $z_{x,y} z_{1,1} \to z_{1,1} z_{x,y}$
$z_{x,1} z_{1,y^2} \to z_{1,y^2} z_{x,1}$, $z_{x,1} z_{1,y} \to z_{1,y} z_{x,1}$, $z_{x,1} z_{1,1} \to z_{1,1} z_{x,1}$
$z_{1,y^2} z_{1,y} \to z_{1,y} z_{1,y^2}$, $z_{1,y^2} z_{1,1} \to z_{1,1} z_{1,y^2}$, $z_{1,y} z_{1,1} \to z_{1,1} z_{1,y}$,

(6) $z_{1,1} y \to y z_{1,y}$, $z_{1,y} y \to y z_{1,y^2}$, $z_{1,y^2} y \to y z_{1,1}$,
$z_{x,1} y \to y z_{x,y}$, $z_{x,y} y \to y z_{x,y^2}$, $z_{x,y^2} y \to y z_{x,1}$,

(7) $x z_{1,1} \to z_{1,1} x$, $x z_{1,y} \to z_{1,y} x$, $x z_{1,y^2} \to z_{1,y^2} x$
$x z_{x,1} \to z_{x,1} x$, $x z_{x,y} \to z_{x,y} x$, $x z_{x,y^2} \to z_{x,y^2} x$.

To show that this system is confluent, we check all overlapping words and corresponding critical pairs as follows.

Overlapping of the relation (1) with itself:
$(1) \cap (1) : x^3, \quad (x, x),$

Overlapping of the relation (1) with form (3):
$(1) \cap (3) : x^2 y, \quad (y, xyx),$

Overlappings of the relation (1) with form (7):
$(1) \cap (7) : x^2 z_{1,1}, \quad (y, xyx),$ $(1) \cap (7) : x^2 z_{1,y}, \quad (z_{1,y}, x z_{1,y} x),$
$(1) \cap (7) : x^2 z_{1,y^2}, (z_{1,y^2}, x z_{1,y^2} x),$ $(1) \cap (7) : x^2 z_{x,1}, (z_{x,1}, x z_{x,1} x),$
$(1) \cap (7) : x^2 z_{x,y}, (z_{x,y}, x z_{x,y} x),$ $(1) \cap (7) : x^2 z_{x,y^2}, (z_{x,y^2}, x z_{x,y^2} x),$

Overlapping of the relation (2) with itself:
$(2) \cap (2) : y^4, \quad (xy, yx),$

Overlapping of the relation (3) with form (2):
$(3) \cap (2) : xy^3, \quad (yxy^2, x^2),$

Overlappings of the relation (4) with itself:
$(4) \cap (4) : z_{1,1}^3, \quad (z_{1,1}^2, z_{1,1}^2),$ $(4) \cap (4) : z_{1,y}^3, \quad (z_{1,y}^2, z_{1,y}^2),$
$(4) \cap (4) : z_{1,y^2}^3, \ (z_{1,y^2}^2, z_{1,y^2}^2),$ $(4) \cap (4) : z_{x,1}^3, \ (z_{x,1}^2, z_{x,1}^2),$
$(4) \cap (4) : z_{x,y}^3, \ (z_{x,y}^2, z_{x,y}^2),$ $(4) \cap (4) : z_{x,y^2}^3, \ (z_{x,y^2}^2, z_{x,y^2}^2),$

Overlappings of the relation (4) with form (5):

$(4) \cap (5) : z_{1,y}^2 z_{1,1}, \quad (z_{1,y} z_{1,1}, z_{1,y} z_{1,1} z_{1,y}),$
$(4) \cap (5) : z_{1,y^2}^2 z_{1,y}, \quad (z_{1,y^2} z_{1,y}, z_{1,y^2} z_{1,y} z_{1,y^2}),$
$(4) \cap (5) : z_{1,y^2}^2 z_{1,1}, \quad (z_{1,y^2} z_{1,1}, z_{1,y^2} z_{1,1} z_{1,y^2}),$
$(4) \cap (5) : z_{x,1}^2 z_{1,y^2}, \quad (z_{x,1} z_{1,y^2}, z_{x,1} z_{1,y^2} z_{x,1}),$
$(4) \cap (5) : z_{x,1}^2 z_{1,y}, \quad (z_{x,1} z_{1,y}, z_{x,1} z_{1,y} z_{x,1}),$
$(4) \cap (5) : z_{x,1}^2 z_{1,1}, \quad (z_{x,1} z_{1,1}, z_{x,1} z_{1,1} z_{x,1}),$
$(4) \cap (5) : z_{x,y}^2 z_{x,1}, \quad (z_{x,y} z_{x,1}, z_{x,y} z_{x,1} z_{x,y}),$
$(4) \cap (5) : z_{x,y}^2 z_{1,y^2}, \quad (z_{x,y} z_{1,y^2}, z_{x,y} z_{1,y^2} z_{x,y}),$
$(4) \cap (5) : z_{x,y}^2 z_{1,y}, \quad (z_{x,y} z_{1,y}, z_{x,y} z_{1,y} z_{x,y}),$
$(4) \cap (5) : z_{x,y}^2 z_{1,1}, \quad (z_{x,y} z_{1,1}, z_{x,y} z_{1,1} z_{x,y}),$
$(4) \cap (5) : z_{x,y^2}^2 z_{x,y}, \quad (z_{x,y^2} z_{x,y}, z_{x,y^2} z_{x,y} z_{x,y^2}),$
$(4) \cap (5) : z_{x,y^2}^2 z_{x,1}, \quad (z_{x,y^2} z_{x,1}, z_{x,y^2} z_{x,1} z_{x,y^2}),$
$(4) \cap (5) : z_{x,y^2}^2 z_{1,y^2}, \quad (z_{x,y^2} z_{1,y^2}, z_{x,y^2} z_{1,y^2} z_{x,y^2}),$
$(4) \cap (5) : z_{x,y^2}^2 z_{1,y}, \quad (z_{x,y^2} z_{1,y}, z_{x,y^2} z_{1,y} z_{x,y^2}),$
$(4) \cap (5) : z_{x,y^2}^2 z_{1,1}, \quad (z_{x,y^2} z_{1,1}, z_{x,y^2} z_{1,1} z_{x,y^2}),$

Overlaps of form (4) with form (6):
$(4) \cap (6) : z_{1,1}^2 y, \quad (z_{1,1} y, z_{1,1} y z_{1,y}),$
$(4) \cap (6) : z_{1,y}^2 y, \quad (z_{1,y} y, z_{1,y} y z_{1,y^2}),$
$(4) \cap (6) : z_{1,y^2}^2 y, \quad (z_{1,y^2} y, z_{1,y^2} y z_{1,1}),$
$(4) \cap (6) : z_{x,1}^2 y, \quad (z_{x,1} y, z_{x,1} y z_{x,y}),$
$(4) \cap (6) : z_{x,y}^2 y, \quad (z_{x,y} y, z_{x,y} y z_{x,y^2}),$
$(4) \cap (6) : z_{x,y^2}^2 y, \quad (z_{x,y^2} y, z_{x,y^2} y z_{x,1}),$

Overlappings of the relation (5) with form (4):
$(5) \cap (4) : z_{x,y^2} z_{x,y}^2, \quad (z_{x,y} z_{x,y^2} z_{x,y}, z_{x,y^2} z_{x,y}),$
$(5) \cap (4) : z_{x,y^2} z_{x,1}^2, \quad (z_{x,1} z_{x,y^2} z_{x,1}, z_{x,y^2} z_{x,1}),$
$(5) \cap (4) : z_{x,y^2} z_{1,y^2}^2, \quad (z_{1,y^2} z_{x,y^2} z_{1,y^2}, z_{x,y^2} z_{1,y^2}),$
$(5) \cap (4) : z_{x,y^2} z_{1,y}^2, \quad (z_{1,y} z_{x,y^2} z_{1,y}, z_{x,y^2} z_{1,y}),$
$(5) \cap (4) : z_{x,y^2} z_{1,1}^2, \quad (z_{1,1} z_{x,y^2} z_{1,1}, z_{x,y^2} z_{1,1}),$
$(5) \cap (4) : z_{x,y} z_{x,1}^2, \quad (z_{x,1} z_{x,y} z_{x,1}, z_{x,y} z_{x,1}),$
$(5) \cap (4) : z_{x,y} z_{1,y^2}^2, \quad (z_{1,y^2} z_{x,y} z_{1,y^2}, z_{x,y} z_{1,y^2}),$
$(5) \cap (4) : z_{x,y} z_{1,y}^2, \quad (z_{1,y} z_{x,y} z_{1,y}, z_{x,y} z_{1,y}),$
$(5) \cap (4) : z_{x,y} z_{1,1}^2, \quad (z_{1,1} z_{x,y} z_{1,1}, z_{x,y} z_{1,1}),$
$(5) \cap (4) : z_{x,1} z_{1,y^2}^2, \quad (z_{1,y^2} z_{x,1} z_{1,y^2}, z_{x,1} z_{1,y^2}),$
$(5) \cap (4) : z_{x,1} z_{1,y}^2, \quad (z_{1,y} z_{x,1} z_{1,y}, z_{x,1} z_{1,y}),$
$(5) \cap (4) : z_{x,1} z_{1,1}^2, \quad (z_{1,1} z_{x,1} z_{1,1}, z_{x,1} z_{1,1}),$
$(5) \cap (4) : z_{1,y^2} z_{1,y}^2, \quad (z_{1,y} z_{1,y^2} z_{1,y}, z_{1,y^2} z_{1,y}),$
$(5) \cap (4) : z_{1,y^2} z_{1,1}^2, \quad (z_{1,1} z_{1,y^2} z_{1,1}, z_{1,y^2} z_{1,1}),$
$(5) \cap (4) : z_{1,y} z_{1,1}^2, \quad (z_{1,1} z_{1,y} z_{1,1}, z_{1,y} z_{1,1}),$

Overlappings of the relation (5) with itself:
$(5) \cap (5) : z_{x,y^2} z_{x,y} z_{x,1}, \quad (z_{x,y} z_{x,y^2} z_{x,1}, z_{x,y^2} z_{x,1} z_{x,y}),$
$(5) \cap (5) : z_{x,y^2} z_{x,y} z_{1,y^2}, \quad (z_{x,y} z_{x,y^2} z_{1,y^2}, z_{x,y^2} z_{1,y^2} z_{x,y}),$
$(5) \cap (5) : z_{x,y^2} z_{x,y} z_{1,y}, \quad (z_{x,y} z_{x,y^2} z_{1,y}, z_{x,y^2} z_{1,y} z_{x,y}),$
$(5) \cap (5) : z_{x,y^2} z_{x,y} z_{1,1}, \quad (z_{x,y} z_{x,y^2} z_{1,1}, z_{x,y^2} z_{1,1} z_{x,y}),$
$(5) \cap (5) : z_{x,y^2} z_{x,1} z_{1,y^2}, \quad (z_{x,1} z_{x,y^2} z_{1,y^2}, z_{x,y^2} z_{1,y^2} z_{x,1}),$

$(5)\cap(5)\ : z_{x,y^2}z_{x,1}z_{1,y},\quad (z_{x,1}z_{x,y^2}z_{1,y}, z_{x,y^2}z_{1,y}z_{x,1}),$

$(5)\cap(5)\ : z_{x,y^2}z_{x,1}z_{1,1},\quad (z_{x,1}z_{x,y^2}z_{1,1}, z_{x,y^2}z_{1,1}z_{x,1}),$

$(5)\cap(5)\ : z_{x,y^2}z_{1,y^2}z_{1,y},\quad (z_{1,y^2}z_{x,y^2}z_{1,y}, z_{x,y^2}z_{1,y}z_{1,y^2}),$

$(5)\cap(5)\ : z_{x,y^2}z_{1,y^2}z_{1,1},\quad (z_{1,y^2}z_{x,y^2}z_{1,1}, z_{x,y^2}z_{1,1}z_{1,y^2}),$

$(5)\cap(5)\ : z_{x,y^2}z_{1,y}z_{1,1},\quad (z_{1,y}z_{x,y^2}z_{1,1}, z_{x,y^2}z_{1,1}z_{1,y}),$

$(5)\cap(5)\ : z_{x,y}z_{x,1}z_{1,y^2},\quad (z_{x,1}z_{x,y}z_{1,y^2}, z_{x,y}z_{1,y^2}z_{x,1}),$

$(5)\cap(5)\ : z_{x,y}z_{x,1}z_{1,y},\quad (z_{x,1}z_{x,y}z_{1,y}, z_{x,y}z_{1,y}z_{x,1}),$

$(5)\cap(5)\ : z_{x,y}z_{x,1}z_{1,1},\quad (z_{x,1}z_{x,y}z_{1,1}, z_{x,y}z_{1,1}z_{x,1}),$

$(5)\cap(5)\ : z_{x,y}z_{1,y^2}z_{1,y},\quad (z_{1,y^2}z_{x,y}z_{1,y}, z_{x,y}z_{1,y}z_{1,y^2}),$

$(5)\cap(5)\ : z_{x,y}z_{1,y^2}z_{1,1},\quad (z_{1,y^2}z_{x,y}z_{1,1}, z_{x,y}z_{1,1}z_{1,y^2}),$

$(5)\cap(5)\ : z_{x,y}z_{1,y}z_{1,1},\quad (z_{1,y}z_{x,y}z_{1,1}, z_{x,y}z_{1,1}z_{1,y}),$

$(5)\cap(5)\ : z_{x,1}z_{1,y^2}z_{1,y},\quad (z_{1,y^2}z_{x,1}z_{1,y}, z_{x,1}z_{1,y}z_{1,y^2}),$

$(5)\cap(5)\ : z_{x,1}z_{1,y^2}z_{1,1},\quad (z_{1,y^2}z_{x,1}z_{1,1}, z_{x,1}z_{1,1}z_{1,y^2}),$

$(5)\cap(5)\ : z_{x,1}z_{1,y}z_{1,1},\quad (z_{1,y}z_{x,1}z_{1,1}, z_{x,1}z_{1,1}z_{1,y}),$

$(5)\cap(5)\ : z_{1,y^2}z_{1,y}z_{1,1},\quad (z_{1,y}z_{1,y^2}z_{1,1}, z_{1,y^2}z_{1,1}z_{1,y}),$

Overlappings of the relation (5) with form (6):

$(5)\cap(6)\ : z_{x,y^2}z_{x,y}y,\quad (z_{x,y}z_{x,y^2}y,\ z_{x,y^2}yz_{x,y^2}),$

$(5)\cap(6)\ : z_{x,y^2}z_{x,1}y,\quad (z_{x,1}z_{x,y^2}y,\ z_{x,y^2}yz_{x,y}),$

$(5)\cap(6)\ : z_{x,y^2}z_{1,y^2}y,\quad (z_{1,y^2}z_{x,y^2}y,\ z_{x,y^2}yz_{1,1}),$

$(5)\cap(6)\ : z_{x,y^2}z_{1,y}y,\quad (z_{1,y}z_{x,y^2}y,\ z_{x,y^2}yz_{1,y^2}),$

$(5)\cap(6)\ : z_{x,y^2}z_{1,1}y,\quad (z_{1,1}z_{x,y^2}y,\ z_{x,y^2}yz_{1,y}),$

$(5)\cap(6)\ : z_{x,y}z_{x,1}y,\quad (z_{x,1}z_{x,y}y,\ z_{x,y}yz_{x,y}),$

$(5)\cap(6)\ : z_{x,y}z_{1,y^2}y,\quad (z_{1,y^2}z_{x,y}y,\ z_{x,y}yz_{1,1}),$

$(5)\cap(6)\ : z_{x,y}z_{1,y}y,\quad (z_{1,y}z_{x,y}y,\ z_{x,y}yz_{1,y^2}),$

$(5)\cap(6)\ : z_{x,y}z_{1,1}y,\quad (z_{1,1}z_{x,y}y,\ z_{x,y}yz_{1,y}),$

$(5)\cap(6)\ : z_{x,1}z_{1,y^2}y,\quad (z_{1,y^2}z_{x,1}y,\ z_{x,1}yz_{1,1}),$

$(5)\cap(6)\ : z_{x,1}z_{1,y}y,\quad (z_{1,y}z_{x,1}y,\ z_{x,1}yz_{1,y^2}),$

$(5)\cap(6)\ : z_{x,1}z_{1,1}y,\quad (z_{1,1}z_{x,1}y,\ z_{x,1}yz_{1,y}),$

$(5)\cap(6)\ : z_{1,y^2}z_{1,y}y,\quad (z_{1,y}z_{1,y^2}y,\ z_{1,y^2}yz_{1,y^2}),$

$(5)\cap(6)\ : z_{1,y^2}z_{1,1}y,\quad (z_{1,1}z_{1,y^2}y,\ z_{1,y^2}yz_{1,y}),$

$(5)\cap(6)\ : z_{1,y}z_{1,1}y,\quad (z_{1,1}z_{1,y}y,\ z_{1,y}yz_{1,y}),$

Overlappings of the relation (6) with form (2):

$(6)\cap(2)\ : z_{1,1}y^3,\quad (yz_{1,y}y^2, z_{1,1}x),$

$(6)\cap(2)\ : z_{1,y}y^3,\quad (yz_{1,y^2}y^2,\ z_{1,y}x),$

$(6)\cap(2)\ : z_{1,y^2}y^3,\quad (yz_{1,1}y^2,\ z_{1,y^2}x),$

$(6)\cap(2)\ : z_{x,1}y^3,\quad (yz_{x,y}y^2,\ z_{x,1}x),$

$(6)\cap(2)\ : z_{x,y}y^3,\quad (yz_{x,y^2}y^2,\ z_{x,y}x),$

$(6)\cap(2)\ : z_{x,y^2}y^3,\quad (yz_{x,1}y^2,\ z_{x,y^2}x),$

Overlappings of the relation (7) with form (4):

$(7)\cap(4)\ : xz^2_{1,1},\quad (z_{1,1}xz_{1,1},\ xz_{1,1}),$

$(7)\cap(4)\ : xz^2_{1,y},\quad (z_{1,y}xz_{1,y},\ xz_{1,y}),$

$(7)\cap(4)\ : xz^2_{1,y^2},\quad (z_{1,y^2}xz_{1,y^2},\ xz_{1,y^2}),$

$(7)\cap(4)\ : xz^2_{x.1},\quad (z_{x,1}xz_{x,1},\ xz_{x,1}),$

$(7)\cap(4)\ : xz^2_{x.y},\quad (z_{x,y}xz_{x,y},\ xz_{x,y}),$

$(7)\cap(4)\ : xz^2_{x.y^2},\quad (z_{x,y^2}xz_{x,y^2},\ xz_{x,y^2}),$

Overlappings of the relation (7) with form (5):

$(7)\cap(5)\ : xz_{1,y}z_{1,1},\quad (z_{1,y}xz_{1,1}, xz_{1,1}z_{1,y}),$

$(7)\cap(5)\ : xz_{1,y^2}z_{1,y},\quad (z_{1,y^2}xz_{1,y}, xz_{1,y}z_{1,y^2}),$

$(7)\cap(5)\ : xz_{1,y^2}z_{1,1},\quad (z_{1,y^2}xz_{1,1}, xz_{1,1}z_{1,y^2}),$

$(7)\cap(5)\ : xz_{x,1}z_{1,y^2},\quad (z_{x,1}xz_{1,y^2}, xz_{1,y^2}z_{x,1}),$

$(7)\cap(5)\ : xz_{x,1}z_{1,y},\quad (z_{x,1}xz_{1,y}, xz_{1,y}z_{x,1}),$

$(7)\cap(5)\ : xz_{x,1}z_{1,1},\quad (z_{x,1}xz_{1,1}, xz_{1,1}z_{x,1}),$

$(7)\cap(5)\ : xz_{x,y}z_{x,1},\quad (z_{x,y}xz_{x,1}, xz_{x,1}z_{x,y}),$

$(7)\cap(5)\ : xz_{x,y}z_{1,y^2},\quad (z_{x,y}xz_{1,y^2}, xz_{1,y^2}z_{x,y}),$

$(7)\cap(5)\ : xz_{x,y}z_{1,y},\quad (z_{x,y}xz_{1,y}, xz_{1,y}z_{x,y}),$

$(7)\cap(5)\ : xz_{x,y}z_{1,1},\quad (z_{x,y}xz_{1,1}, xz_{1,1}z_{x,y}),$

$(7)\cap(5)\ : xz_{x,y^2}z_{x,y},\quad (z_{x,y^2}xz_{x,y}, xz_{x,y}z_{x,y^2}),$

$(7)\cap(5)\ : xz_{x,y^2}z_{x,1},\quad (z_{x,y^2}xz_{x,1}, xz_{x,1}z_{x,y^2}),$

$(7)\cap(5)\ : xz_{x,y^2}z_{1,y^2},\quad (z_{x,y^2}xz_{1,y^2}, xz_{1,y^2}z_{x,y^2}),$

$(7)\cap(5)\ : xz_{x,y^2}z_{1,y},\quad (z_{x,y^2}xz_{1,y}, xz_{1,y}z_{x,y^2}),$

$(7)\cap(5)\ : xz_{x,y^2}z_{1,1},\quad (z_{x,y^2}xz_{1,1}, xz_{1,1}z_{x,y^2}),$

Overlappings of the relation (7) with form (6):

$(7)\cap(6)\ : xz_{1,1}y,\quad (z_{1,1}xy,\ xyz_{1,y}),$

$(7)\cap(6)\ : xz_{1,y}y,\quad (z_{1,y}xy,\ xyz_{1,y^2}),$

$(7)\cap(6)\ : xz_{1,y^2}y,\quad (z_{1,y^2}xy,\ xyz_{1,1}),$

$(7)\cap(6)\ : xz_{x,1}y,\quad (z_{x,1}xy,\ xyz_{x,y^2}),$

$(7)\cap(6)\ : xz_{x,y}y,\quad (z_{x,y}xy,\ xyz_{x,y^2})\,.$

All these above critical pairs are resolved by corresponding reduction steps. We show two of them as examples.

$$(5)\cap(6)\ z_{x,y^2}z_{x,y}y \to \begin{cases} z_{x,y}z_{x,y^2}y \to z_{x,y}yz_{x,1} \to yz_{x,y^2}z_{x,1} \to yz_{x,1}z_{x,y^2} \\ z_{x,y^2}yz_{x,y^2} \to yz_{x,1}z_{x,y^2} \end{cases}$$

$$(7)\cap(5)\ xz_{1,y}z_{1,1} \to \begin{cases} z_{1,y}xz_{1,1} \to z_{1,y}z_{1,1}x \to z_{1,1}z_{1,y}x \\ xz_{1,1}z_{1,y} \to z_{1,1}xz_{1,y} \to z_{1,1}z_{1,y}x \end{cases}$$

Since the rewriting system given with (1)-(7) is Noetherian and confluent, it is complete.

Now we consider normal form structure of an arbitrary word $u \in C_{2_{cp}} \#_{\alpha}^{f} C_3$. It is easily seen that it is of the form;

$y^k W_{z_{a',b'}} x^l \ (0 \le k \le 2,\ 0 \le l \le 1),$

where $W_{z_{a',b'}} \ (a' \in C_2, b' \in C_3)$ is a reduced word obtained by generators $z_{a,b} \ (a \in C_2, b \in C_3)$. For example, the reduced words $y^2 z_{1,1} z_{1,y}$ and $z_{1,y^2} z_{x,y^2} x$ are of the form $y^k W_{z_{a',b'}} x^l$.

Finally, we can say that the number of generators and relations of $C_{2_{cp}} \#_{\alpha}^{f} C_3$ are 8 and 36. We can easily see these results by taking $n = 2$ and $m = 3$ in the generator number formula $mn+2$ and relator number formula $\frac{nm(nm+5)+6}{2}$ in Corollary 2.4.

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