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Research Article

Hankel transform of linear combination of three consecutive Catalan numbers

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Abstract

In this paper, we consider the Hankel determinants of the linear combination of three consecutive Catalan numbers, and then three consecutive shifted Catalan numbers. For their computing, we apply known methods based on the connections between continued fractions, orthogonal polynomials and moment-determinants. The properties of orthogonal polynomials enable us to evaluate the generating function of the corresponding sequence Hankel determinants in closed form.

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1. Introduction

The Hankel transformation of sequences has garnered growing interest among mathematicians, particularly within the context of moment theory and orthogonal polynomials. In the paper [8] attention is focused on the research of integer sequences, while papers such as [3] rather study certain Hankel transformations of classical sequences. Several additional Hankel transform evaluations are provided in references [5].

Hankel transformations of sequences involving the Catalan numbers have been the subject of research published in many papers. For instance, the studies [3, 12] explore the Hankel transform of sequences related to both Catalan and modified (adjusted) Catalan numbers. In both papers the authors use methods based on orthogonal polynomials. The generalization of these results is given by Chamberland and French [2], who deal with the sequence where n-th term is obtained by adding n-th and $n+1$-th generalized Catalan numbers, and computed the Hankel transform. They used different techniques to those in [3]. In the paper [10] the authors Mu, Wang and Yeh have examined Hankel transform of linear combinations of two consecutive Catalan-like numbers. Another computation of the Hankel transform of the sequence involving Catalan numbers is given in [1]. Some other properties of polynomials based on Catalan and similar numbers are given in the recent papers [9, 11].

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2. Catalan numbers

The Catalan numbers [8,14] is the sequence of natural numbers frequently encountered in various counting problems, particularly those involving recursive definitions.

It is well-known that the n-th Catalan number can be represented directly using central binomial coefficients as follows:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n} \frac{n+k}{k}.$$

It is referred to as A000108 in the (OEIS) [13] and has its first few members equal to 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, ...

The series of Catalan numbers has an even wider application in connection with many problems in probability theory, physics, engineering and an very interesting application can be found in cryptography [7]. Numerous combinatorial problems are based on the Catalan sequence, including the ballot problem, lattice path problems involving road networks, and the paired parentheses problem [14]. Catalan numbers are predominantly utilized in geometry and also play a crucial role in solving a wide range of combinatorial challenges.

The problem of determining the number of triangulations of a convex n-angle was also solved and it was shown that it is equal to the $(n-2)$-th Catalan number. This problem finds applications in computer graphics and across various fields like engineering, robotics, and pattern recognition. In [6], an alternative recursive definition of Catalan numbers is provided through a combinatorial approach, incorporating the concept of forbidden combinations.

Moreover, according to [3], the recurrence relation for the Catalan numbers C_n can be expressed as

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \qquad \text{for } n \in \mathbb{N}_0 \ , \tag{2.1}$$

with $C_0 = 1$ and its ordinary generating function is given by [3]

$$c(x) = \frac{1-\sqrt{1-4x}}{2x} \ . \tag{2.2}$$

The sequence we study in this paper is defined by the following linear combination

$$a_n = \alpha C_n + \beta C_{n+1} + C_{n+2}, \qquad \text{for } n \in \mathbb{N}_0; \tag{2.3}$$

where $\alpha, \beta \in \mathbb{N}$ are natural numbers. The first few members of the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ are given by:

$$\{a_n\}_{n\in\mathbb{N}_0} = \{2+\alpha+\beta,\ 5+\alpha+2\beta,\ 14+2\alpha+5\beta,\ 42+5\alpha+14\beta,\ 132+14\alpha+42\beta,\ \cdots\}$$

It is known that the sequence of Catalan numbers has the generating function of the following form:

$$C_n = \frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\, dx, \qquad n \in \mathbb{N}_0.$$

Therefore, the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ has generating function given by:

$$a_n = \frac{1}{2\pi}\int_0^4 x^n \cdot \left(\alpha + \beta x + x^2\right) \sqrt{\frac{4-x}{x}}\, dx, \qquad n \in \mathbb{N}_0.$$

We consider the given quadratic trinomial $x^2 + \beta x + \alpha$. This can be written in product form:

$$x^2 + \beta x + \alpha = (x - x_1) \cdot (x - x_2) \ ,$$

where Vieta's formulas imply that

$$x_1 + x_2 = -\beta \qquad \text{and} \qquad x_1 \cdot x_2 = \alpha \ .$$

3. The Hankel transform and the main result

Let us start with the following definition of the Hankel transform.

Definition 3.1. The **Hankel transform** of a given sequence $c = \{c_0, c_1, c_2, \ldots\}$ is the sequence $h = \{h_0, h_1, h_2, \ldots\}$, defined by $h_n = \det[c_{i+j-2}]_{i,j=0}^{n-1}$, i.e

$$h_n = \det \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{bmatrix}. \tag{3.1}$$

The element h_n is known as **Hankel determinant** of order n. We also denote $h = \mathcal{H}(c)$.

The first few members of the Hankel transform $h = \mathcal{H}(a) = \{h_n\}_{n\in\mathbb{N}_0}$ of the sequence $a = \{a_n\}_{n\in\mathbb{N}_0}$ defined by (2.3), are

$$\begin{aligned} h_0 &= 2 + \alpha + \beta, \\ h_1 &= 3 + 8\alpha + \alpha^2 + 4\beta + 3\alpha\beta + \beta^2, \\ h_2 &= 4 + 34\alpha + 14\alpha^2 + \alpha^3 + 10\beta + 30\alpha\beta + 5\alpha^2\beta + 6\beta^2 + 6\alpha\beta^2 + \beta^3, \\ h_3 &= 5 + 104\alpha + 101\alpha^2 + 20\alpha^3 + \alpha^4 + 20\beta + 159\alpha\beta + 80\alpha^2\beta + 21\beta^2 + 72\alpha\beta^2 + 15\alpha^2\beta^2 \\ &\quad + 8\beta^3 + 10\alpha\beta^3 + \beta^4, \\ &\ \ldots \end{aligned}$$

The main result of this paper is the expression of the n-th term of $\mathcal{H}(a)$, where $a = \{a_n\}_{n\in\mathbb{N}_0}$ is defined by (2.3) (i.e. linear combination of C_n, C_{n+1} and C_{n+2}). It is formulated by the following theorem.

Theorem 3.1. The generating function $G(x) = \sum_{n=0}^{\infty} h_{n-1}x^n$, where $h = \{h_n\}_{n\in\mathbb{N}_0}$ is the Hankel transform of the sequence (2.3), can be written by

$$G(x) = \frac{(1-x)^2 - \beta x}{\left((1-x)^2 - \beta x\right)^2 - \alpha x\,(1+x)^2} \tag{3.2}$$

where we additionally defined $h_{-1} = 1$.

In what follows, we use methods based on orthogonal polynomials to prove Theorem 3.1.

4. Proof of the Theorem 3.1

We start with the formulation of two important lemmas which describe the connection of the three-term recurrence coefficients, satisfied by the sequences of monic orthogonal polynomials.

Let $\{S_n(x)\}_{n\in\mathbb{N}_0}$ represent the sequence of monic polynomials, orthogonal with respect to the weight function $\rho(x)$. Then it satisfies

$$S_{n+1}(x) = (x - p_n)S_n(x) - q_n S_{n-1}(x), \tag{4.1}$$

which is known as three-term recurrence relation. Similarly, denote by $\left\{\tilde{S}_n(x)\right\}_{n\in\mathbb{N}_0}$ another sequence of orthogonal monic polynomials, corresponding to the weight function $\tilde{\rho}(x)$ and the three-term recurrence relation

$$\tilde{S}_{n+1}(x) = (x - \tilde{p}_n)\tilde{S}_n(x) - \tilde{q}_n\tilde{S}_{n-1}(x), \tag{4.2}$$

The transformation formulas we introduce, relate the coefficients $\tilde{p}_n$ and $\tilde{q}_n$ of the original weight function to those of the transformed weight function, as stated in the following two well-known lemmas.

Lemma 4.1 ([12]). Let $\rho(x)$ and $\tilde{\rho}(x)$ represent weight functions, with $\{S_n(x)\}_{n\in\mathbb{N}_0}$ and $\left\{\tilde{S}_n(x)\right\}_{n\in\mathbb{N}_0}$ denoting their respective orthogonal polynomials sequences. Additionally, let $\{p_n\}_{n\in\mathbb{N}_0}$, $\{q_n\}_{n\in\mathbb{N}_0}$, and $\{\tilde{p}_n\}_{n\in\mathbb{N}_0}$, $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ be the coefficients of the three-term recurrence relations associated with $\rho(x)$ and $\tilde{\rho}(x)$, respectively. The following statements are true:

(1) If $\tilde{\rho}(x) = C\rho(x)$ where $C > 0$ then we have $\tilde{p}_n = p_n$ for $n \in \mathbb{N}_0$ and $\tilde{q}_0 = Cq_0$, $\tilde{q}_n = q_n$ for $n \in \mathbb{N}$. Additionally there holds $\tilde{S}_n(x) = S_n(x)$ for all $n \in \mathbb{N}_0$.

(2) If $\tilde{\rho}(x) = \rho(ax+b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ there holds $\tilde{p}_n = \frac{p_n - b}{a}$ for $n \in \mathbb{N}_0$ and $\tilde{q}_0 = \frac{q_0}{|a|}$ and $\tilde{q}_n = \frac{q_n}{a^2}$ for $n \in \mathbb{N}$. Additionally we have $\tilde{S}_n(x) = \frac{1}{a^n} S_n(ax+b)$.

Lemma 4.2 ([12]). Under the same notation as in the Lemma 4.1 assume that the sequence $\{r_n\}_{n\in\mathbb{N}_0}$ is defined by

$$r_0 = c - p_0, \qquad r_n = c - p_n - \frac{q_n}{r_{n-1}} \qquad (n \in \mathbb{N}_0). \tag{4.3}$$

If $\tilde{\rho}(x) = (x-c)\rho(x)$ where $c < \inf \operatorname{supp}(\rho)$, there holds

$$\begin{aligned} &\tilde{q}_0 = \int_{\mathbb{R}} \tilde{\rho}(x)\, dx, \qquad \tilde{q}_n = q_n \frac{r_n}{r_{n-1}}, \qquad (n \in \mathbb{N}), \\ &\tilde{p}_n = p_{n+1} + r_{n+1} - r_n, \quad (n \in \mathbb{N}_0). \end{aligned} \tag{4.4}$$

Now we can proceed with the actual proof. The first step is to apply above lemmas in the following concrete case.

Lemma 4.3. The required coefficients $\tilde{p}_n$ and $\tilde{q}_n$ of the expression (4.2) are equal to

$$\tilde{p}_0 = 1, \quad \tilde{p}_n = 2 \qquad (n \in \mathbb{N}); \qquad\qquad \tilde{q}_0 = 2\pi, \quad \tilde{q}_n = 1 \quad (n \in \mathbb{N}).$$

Proof. For the original weight function $\rho(x)$, we take one corresponding to the monic Chebyshev polynomials of the third kind $V_n(x) = P^{(-1/2,\,1/2)}(x)$. It is defined by $\rho(x) = \sqrt{\dfrac{1+x}{1-x}}$ on the interval $(-1, 1)$. The polynomials $V_n(x)$ satisfy the tree-term recurrence relation

$$V_{n+1}(x) = (x - p_n)V_n(x) - q_n V_{n-1}(x), \qquad V_0(x) = 1, \qquad V_1(x) = x - \frac{1}{2}, \tag{4.5}$$

with coefficients

$$p_0 = \frac{1}{2}, \quad p_n = 0 \quad (n \in \mathbb{N})\ , \qquad\qquad q_0 = \pi, \quad q_n = \frac{1}{4} \quad (n \in \mathbb{N})\ .$$

The transformed weight function is given by

$$\tilde{\rho}(x) = \rho\left(-\frac{1}{2}x + 1\right) = \sqrt{\frac{4-x}{x}}\ .$$

It can be noted that the last expression represents the weight function of the sequence of the Catalan numbers.

By change $\tilde{\rho}(x) = \sqrt{\dfrac{4-x}{x}}$ and applying the Lemma 4.1, we get:

$$\tilde{q}_0 = 2\pi, \quad \tilde{q}_n = 1 \quad (n \in \mathbb{N}), \qquad \tilde{p}_0 = 1 \qquad \tilde{p}_n = 2 \qquad (n \in \mathbb{N})\ , \tag{4.6}$$

and

$$\tilde{V}_{n+1}(x) = (x - \tilde{p}_n)\tilde{V}_n(x) - \tilde{q}_n \tilde{V}_{n-1}(x), \qquad \tilde{V}_0(x) = 1, \qquad \tilde{V}_1(x) = x - 1 \tag{4.7}$$

completing the proof of the lemma. □

It is very useful that $\{\tilde{p}_n\}_{n\in\mathbb{N}_0}$ and $\{\tilde{q}_n\}_{n\in\mathbb{N}_0}$ are constant sequences. Hence the recurrence relation (4.5) becomes the difference equation

$$\tilde{V}_{n+1}(x) - (x-2)\tilde{V}_n(x) + \tilde{V}_{n-1}(x) = 0, \qquad (n \in \mathbb{N}) \ . \tag{4.8}$$

Then by multiplying into the equation (4.8), we get the value for $\tilde{V}_2(x)$:

$$\tilde{V}_2(x) = (x-2)\cdot\tilde{V}_1(x) - \tilde{V}_0(x) = (x-2)\cdot(x-1) - 1 = x^2 - 3x + 1 \ .$$

The characteristic equation for (4.8) takes the form:

$$\lambda^2 - (x-2)\ \lambda + 1 = 0$$

and there are solutions

$$\lambda_1(x) = \frac{x - 2 - \sqrt{x(x-4)}}{2} \ , \qquad \lambda_2(x) = \frac{x - 2 + \sqrt{x(x-4)}}{2} \ .$$

Notice $\lambda_1(x)\cdot\lambda_2(x) = 1$. Then holds

$$\tilde{V}_n(x) = E_1(x)\lambda_1^n(x) + E_2(x)\lambda_2^n(x) \tag{4.9}$$

including the initial values

$$E_1(x) + E_2(x) = \tilde{V}_0 = 1 \qquad \text{and} \qquad E_1(x)\cdot\lambda_1(x) + E_2(x)\cdot\lambda_2(x) = \tilde{V}_1(x) = x - 1$$

we get to the coefficients

$$E_1(x) = \frac{\lambda_2(x) - \tilde{V}_1(x)}{\lambda_2(x) - \lambda_1(x)}, \qquad E_2(x) = \frac{\lambda_1(x) - \tilde{V}_1(x)}{\lambda_2(x) - \lambda_1(x)} \ .$$

According to Dini and Maroni [4], it is valid

$$h_n = \frac{D_{n+1}}{\gamma D_1} \tag{4.10}$$

where

$$D_n = \det\begin{bmatrix} \tilde{V}_n(x_1) & \tilde{V}_{n+1}(x_1), \\ \tilde{V}_n(x_2) & \tilde{V}_{n+1}(x_2) \end{bmatrix}, \qquad \gamma = \frac{C_0}{C_2 + \beta C_1 + \alpha C_0} = \frac{1}{2 + \beta + \alpha} \ . \tag{4.11}$$

and x_1 and x_2 are solutions of the quadratic equation $x^2 + \beta x + \alpha = 0$, i.e. $x_{1,2} = (-\beta \pm \sqrt{\beta^2 - 4\alpha})/2$. Since

$$D_1 = \det\begin{bmatrix} \tilde{V}_1(x_1) & \tilde{V}_2(x_1) \\ \tilde{V}_1(x_2) & \tilde{V}_2(x_2) \end{bmatrix} \ , \tag{4.12}$$

i.e

$$\begin{aligned} D_1 &= \tilde{V}_1(x_1)\tilde{V}_2(x_2) - \tilde{V}_1(x_2)\tilde{V}_2(x_1) \\ &= (x_1 - 1)(x_2^2 - 3x_2 + 1) - (x_2 - 1)(x_1^2 - 3x_1 + 1) \\ &= (x_2 - x_1)(2 - x_1 - x_2 + x_1\cdot x_2) \\ &= \sqrt{\beta^2 - 4\alpha}\cdot(2 + \beta + \alpha) \end{aligned}$$

Also since

$$x_2 - x_1 = \sqrt{\beta^2 - 4\alpha} \ , \tag{4.13}$$

and then

$$h_n = \frac{D_{n+1}}{\gamma\cdot D_1} = \frac{D_{n+1}}{\dfrac{1}{2+\beta+\alpha}\cdot\sqrt{\beta^2 - 4\alpha}\cdot(2+\beta+\alpha)} \ , \tag{4.14}$$

i.e

$$h_n = \frac{D_{n+1}}{\sqrt{\beta^2 - 4\alpha}} \ . \tag{4.15}$$

Note that the last expression is satisfied even for $n=-1$, since $D_0 = x_2 - x_1 = \sqrt{\beta^2-4\alpha}$ and $h_{-1}=1$. The required generating function of the sequence $\{h_{n-1}\}_{n\in\mathbb{N}_0}$ is now given by

$$G(t) = \sum_{n=0}^{\infty} h_{n-1}t^n = \frac{1}{\sqrt{\beta^2-4\alpha}} \sum_{n=0}^{\infty} D_n t^n \ .$$

since

$$\sum_{n=0}^{\infty} D_n t^n = \frac{(\lambda_1(x_2)-\lambda_1(x_1))\, E_1(x_1)E_1(x_2)}{1-\lambda_1(x_1)\lambda_1(x_2)\, t} + \frac{(\lambda_2(x_2)-\lambda_2(x_1))\, E_2(x_1)E_2(x_2)}{1-\lambda_2(x_1)\lambda_2(x_2)\, t}$$
$$+ \frac{(\lambda_2(x_2)-\lambda_1(x_1))\, E_1(x_1)E_2(x_2)}{1-\lambda_1(x_1)\lambda_2(x_2)\, t} + \frac{(\lambda_1(x_2)-\lambda_2(x_1))\, E_1(x_2)E_2(x_1)}{1-\lambda_1(x_2)\lambda_2(x_1)\, t} \ .$$

So, we finally have

$$\sum_{n=0}^{\infty} D_n t^n =$$
$$= \frac{\sqrt{\beta^2-4\alpha}\cdot(1-(2+\beta)t+t^2)}{1-(4-2x_1-2x_2+x_1x_2)\ t + \left(6-4x_1+x_1^2-4x_2+x_2^2\right)\ t^2 - (4-2x_1-2x_2+x_1x_2)\ t^3 + t^4} \ .$$

Since $x_1+x_2=-\beta$ and $x_1x_2=\alpha$, we can write

$$G(t) = \frac{1-(2+\beta)t+t^2}{1-(4+2\beta+\alpha)\ t(1+t^2) + (6+4\beta+\beta^2-2\alpha)\ t^2 + t^4} \ .$$

By writing numerator and denominator of the previous expression in the compact form, we get the result of the Theorem 3.1.

5. Shifted Catalan numbers

We now consider the sequence of shifted Catalan numbers:

$$\{C_n^*\}_{n\in\mathbb{N}_0} = \{C_{n+1}\}_{n\in\mathbb{N}_0} = \{1,\, 2,\, 5,\, 15,\, \ldots\} \ ,$$

and the corresponding sequence analogous to a_n, defined by:

$$a_n^* = \alpha C_{n+1} + \beta C_{n+2} + C_{n+3}. \tag{5.1}$$

The first few members of this sequence a_n^* are:

$$5+\alpha+2\beta,\ 14+2\alpha+5\beta,\ 42+5\alpha+14\beta,\ 132+14\alpha+42\beta,\ 429+42\alpha+132\beta, \cdots$$

while the generating function of (5.1) has the following form:

$$a_n^* = \frac{1}{2\pi}\int_0^4 \left(\alpha x^{n+1} + \beta x^{n+2} + x^{n+3}\right)\cdot\sqrt{\frac{4-x}{x}}\, dx$$
$$= \frac{1}{2\pi}\int_0^4 x^{n+2}\cdot\left(\alpha+\beta x+x^2\right)\cdot\sqrt{\frac{4-x}{x}}\, dx \ .$$

The Hankel transform $\{h_n^*\}_{n\in\mathbb{N}_0}$ of the observed sequence (5.1) start with terms:

$$h_0 = 5+\alpha+2\beta \ ,$$
$$h_1 = 14+\alpha^2+11\alpha+3\beta^2+14\beta+4\alpha\beta) \ ,$$
$$h_2 = 30+\alpha^3+17\alpha^2+63\alpha+4\beta^3+27\beta^2+54\beta+6\alpha^2\beta+52\alpha\beta+10\alpha\beta^2 \ , \cdots$$

Now we can write and prove the following result, which is analogous of the Theorem 3.1 for the sequence $\{a_n^*\}_{n\in\mathbb{N}_0}$.

Theorem 5.1. Denote by $\{h_n^*\}_{n\in\mathbb{N}_0}$ the Hankel transform of the sequence (5.1) and let $h_{-1}^* = 1$. The sequence $\{h_{n-1}^*\}_{n\in\mathbb{N}_0}$ has the generating function given by

$$G^*(t) = \sum_{n=0}^{\infty} h_{n-1}t^n = \frac{1+t}{\left((1-t)^2-\beta t\right)^2 - \alpha t\,(1+t)^2}. \tag{5.2}$$

Proof. We once again utilize transformations of the weight function, starting with $\rho^*(x) = x\cdot\tilde{\rho}(x)$. The associated coefficients are provided in (4.6):

$$\tilde{q}_0 = 2\pi, \quad \tilde{q}_n = 1 \quad (n\in\mathbb{N}), \qquad \tilde{p}_0 = 1 \qquad \tilde{p}_n = 2 \qquad (n\in\mathbb{N}) \ .$$

Based on Lemma 4.2, we need to examine the temporary sequence $\{r_n^*\}_{n\in\mathbb{N}_0}$, which is defined as follows:

$$r_0^* = c - \tilde{p}_0, \qquad r_n^* = c - \tilde{p}_n - \frac{\tilde{q}_n}{r_{n-1}^*} \qquad (n\in\mathbb{N}) \ ,$$

i.e

$$r_0^* = 0-1 = -1, \qquad r_1^* = 0-2-\frac{1}{-1} = -1, \quad \cdots \quad \Rightarrow r_n^* = 0-2-\frac{1}{-1} = -1 \qquad (n\in\mathbb{N}) \ .$$

By direct evaluation we find

$$\begin{aligned} q_0^* &= \int_{\mathbb{R}} \rho^*(x)\ dx = 2\pi, \qquad q_n^* = \tilde{q}_n \frac{r_n^*}{r_{n-1}^*} = \tilde{q}_n = 1, \qquad (n\in\mathbb{N}) \ , \\ p_n^* &= \tilde{p}_{n+1} + r_{n+1}^* - r_n^* = \tilde{p}_{n+1} = 2, \qquad (n\in\mathbb{N}_0) \ . \end{aligned} \tag{5.3}$$

Let us denote by $\{V_n^*(x)\}_{n\in\mathbb{N}_0}$ the orthogonal polynomials corresponding to the weight functions $\rho^*(x)$ with calculated coefficients (5.3). The three-term recurrence for these polynomials is

$$V_{n+1}^*(x) = (x-p_n^*)\,V_n^*(x) - q_n^*\,V_{n-1}^*(x) \tag{5.4}$$

i.e

$$V_{n+1}^*(x) = (x-2)\,V_n^*(x) - V_{n-1}^*(x) \tag{5.5}$$

with the initial value

$$x\cdot V_0^*(x) = -\frac{1}{\tilde{V}_0(0)}\cdot\det\begin{bmatrix}\tilde{V}_0(x) & \tilde{V}_1(x)\\ \tilde{V}_0(0) & \tilde{V}_1(0)\end{bmatrix} = -\frac{1}{1}\cdot\det\begin{bmatrix}1 & (x-1)\\ 1 & -1\end{bmatrix} = x$$

implying $V_0^*(x) = 1$ and

$$\begin{aligned} x\cdot V_1^*(x) &= -\frac{1}{\tilde{V}_1(0)}\cdot\det\begin{bmatrix}\tilde{V}_1(x) & \tilde{V}_2(x)\\ \tilde{V}_1(0) & \tilde{V}_2(0)\end{bmatrix} = -\frac{1}{-1}\cdot\det\begin{bmatrix}(x-1) & (x^2-3x+1)\\ -1 & 1\end{bmatrix} \\ &= x^2-2x \end{aligned}$$

implying $V_1^*(x) = x-2$. Using the relation (5.5), we can compute and $V_2^*(x)$ as:

$$V_2^*(x) = (x-2)\cdot V_1^*(x) - V_0^*(x) = (x-2)\cdot(x-2) - 1 = x^2-4x+3 \ .$$

The characteristic equation for (5.5) takes the form:

$$\lambda^2 - (x-2)\ \lambda + 1 = 0$$

and there are solutions

$$\lambda_1(x) = \frac{x-2-\sqrt{x(x-4)}}{2} \ , \qquad \lambda_2(x) = \frac{x-2-\sqrt{x(x-4)}}{2} \ .$$

Then we have

$$V_n^*(x) = K_1(x)\lambda_1^n(x) + K_2(x)\lambda_2^n(x) \tag{5.6}$$

including the initial values

$$K_1(x) + K_2(x) = V_0^* = 1 \qquad \text{and} \qquad K_1(x)\cdot\lambda_1(x) + K_2(x)\cdot\lambda_2(x) = V_1^*(x) = x-2$$

we get to the coefficients

$$K_1(x) = \frac{\lambda_2(x) - V_1^*(x)}{\lambda_2(x) - \lambda_1(x)}, \qquad K_2(x) = \frac{\lambda_1(x) - V_1^*(x)}{\lambda_2(x) - \lambda_1(x)} \,.$$

Again according to Dini and Maroni [4], it is valid

$$h_n^* = \frac{D_{n+1}^*}{\gamma^* D_1^*} \tag{5.7}$$

where

$$D_n^* = \det \begin{bmatrix} V_n^*(x_1) & V_{n+1}^*(x_1), \\ V_n^*(x_2) & V_{n+1}^*(x_2) \end{bmatrix}, \qquad \gamma^* = \frac{C_1}{C_3 + \beta C_2 + \alpha C_1} = \frac{1}{5 + 2\beta + \alpha} \tag{5.8}$$

and $x_{1,2} = (-\beta \pm \sqrt{\beta^2 - 4\alpha})/2$. Since

$$D_1^* = \det \begin{bmatrix} V_1^*(x_1) & V_2^*(x_1) \\ V_1^*(x_2) & V_2^*(x_2) \end{bmatrix} , \tag{5.9}$$

i.e

$$\begin{aligned} D_1^* &= V_1^*(x_1)V_2^*(x_2) - V_1^*(x_2)V_2^*(x_1) \\ &= (x_1 - 2)(x_2^2 - 4x_2 + 3) - (x_2 - 2)(x_1^2 - 4x_1 + 3) \\ &= (x_2 - x_1)(5 - 2(x_1 + x_2) + x_1 \cdot x_2) \\ &= \sqrt{\beta^2 - 4\alpha} \cdot (5 + 2\beta + \alpha) \end{aligned}$$

If again we take that $x_2 > x_1$ and use (4.13), we get:

$$h_n^* = \frac{D_{n+1}^*}{\gamma^* \cdot D_1^*} = \frac{D_{n+1}^*}{\frac{1}{5 + 2\beta + \alpha} \cdot \sqrt{\beta^2 - 4\alpha} \cdot (5 + 2\beta + \alpha)} , \tag{5.10}$$

i.e

$$h_n^* = \frac{D_{n+1}^*}{\sqrt{\beta^2 - 4\alpha}} \,. \tag{5.11}$$

The generating function of the sequence $\{h_n^*\}_{n \in \mathbb{N}_0}$ is

$$G^*(t) = \sum_{n=0}^{\infty} h_{n-1}^* t^n = \frac{1}{\sqrt{\beta^2 - 4\alpha}} \sum_{n=0}^{\infty} D_n^* t^n \,.$$

since

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^* t^n = &\frac{(\lambda_1(x_2) - \lambda_1(x_1))\, K_1(x_1)K_1(x_2)}{1 - \lambda_1(x_1)\lambda_1(x_2)\, t} + \frac{(\lambda_2(x_2) - \lambda_2(x_1))\, K_2(x_1)K_2(x_2)}{1 - \lambda_2(x_1)\lambda_2(x_2)\, t} \\ &+ \frac{(\lambda_2(x_2) - \lambda_1(x_1))\, K_1(x_1)K_2(x_2)}{1 - \lambda_1(x_1)\lambda_2(x_2)\, t} + \frac{(\lambda_1(x_2) - \lambda_2(x_1))\, K_1(x_2)K_2(x_1)}{1 - \lambda_1(x_2)\lambda_2(x_1)\, t} \,. \end{aligned}$$

Thus, we finally have

$$\sum_{n=0}^{\infty} D_n^* t^n =$$

$$= \frac{\sqrt{\beta^2 - 4\alpha} \cdot (1 + t)}{1 - (4 - 2x_1 - 2x_2 + x_1 x_2)\, t + (6 - 4x_1 + x_1^2 - 4x_2 + x_2^2)\, t^2 - (4 - 2x_1 - 2x_2 + x_1 x_2)\, t^3 + t^4} \,.$$

Since $x_1 + x_2 = -\beta$ and $x_1 x_2 = \alpha$, the preceding expression takes the form

$$G^*(t) = \frac{(1 + t)}{1 - (4 + 2\beta + \alpha)\, t(1 + t^2) + (6 + 4\beta + \beta^2 - 2\alpha)\, t^2 + t^4} \,.$$

Grouping the factors in the numerator and denominator of the previous expression in the required way, we finish the proof. □

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