Konuralp Journal of Mathematics, 13 (1) (2025) 1-13

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

New Results on the Perfect Fluid Lorentzian Para-Sasakian Spacetimes

Sunil Kumar Yadav[1*] **and Ahmet Yıldız**[2]

[1] *Department of Applied Science and Humanities, United Collage of Engineering & Research, UPSIDC, Industrial Area, Naini, Prayagraj, Uttar Predesh, India.*

[2] *Education Faculty, Department of Mathematics, Inonu University, Malatya, 44280, Türkiye*

[*] *Corresponding author*

Abstract

In this article, we investigate $\mathscr{LP}$-Sasakian spacetimes attached with perfect fluid whose metrics are $(\text{CERY})_4$-soliton admitting $\mathscr{Z}$-tensor. Also we discuss the application of such soliton to cosmology and general relativity. Besides this, we deduce a modified Poisson equation and modified Liouville equation from the $(\text{CERY})_4$-soliton on $\mathscr{LP}$-Sasakian spacetimes . In addition, we light up the harmonic aspect of such soliton on perfect fluid $\mathscr{LP}$-Sasakian spacetimes. Moreover, we conclude a necessary and sufficient condition for a 1-form $\eta^\sharp$, which is the $g^\star$-dual of the vector field ξ on such a spacetime to be a solution of the Schrödinger-Ricci equation. In conclusion, we present an instance of a 4-dimensional $\mathscr{LP}$-Sasakian spacetime with the $(\text{CERY})_4$-soliton equipped with $\mathscr{Z}$-tensor.

Keywords: *$\mathscr{LP}$-Sasakian manifold, generalized $\mathscr{Z}$ tensor, conformal η-Ricci-Yamabe soliton. Schrödinger-Ricci equation, modified Poisson equation and modified Liouville equation.*

2010 Mathematics Subject Classification: 53B30, 53C43, 53C50, 53C80, 53C25.

1. Introduction

Let $(\Omega^{\widetilde{m}}, g^\star)$ is a Riemannian manifold and $\mathscr{T}_2^s(\Omega)$ is a linear space of its symmetric tensor fields for $(0,2)$-type and $\widetilde{Riem}(\Omega) \subsetneq \mathscr{T}_2^s(\Omega)$ is infinite space of its Riemannian metrics. A Riemannian flow (briefly, RF) is a smooth map on $(\Omega^{\widetilde{m}}, g^\star)$, is defined by

$$g^\star : I^\star \subseteq \Re \to \widetilde{Riem}(\Omega),$$

where, $I^\star$ is an open interval [11]. The map $\widetilde{\mathscr{RY}}^{(\rho^\star,\rho^\star,g^\star)} : I^\star \to \mathscr{T}_2^s(\Omega)$, which is light up by

$$\widetilde{\mathscr{RY}}^{(\rho^\star,\rho^\star,g^\star)} := 2\rho^\star \mathscr{S}(t) + \rho^\star \tau(t) g^\star(t) + \frac{\partial}{\partial t} g^\star(t),$$

is $(\rho^\star,\rho^\star)$-Ricci-Yamabe map of $(\Omega^{\widetilde{m}}, g^\star)$, where $\rho^\star$, $\rho^\star$ are some scalars [11]. If $\widetilde{\mathscr{RY}}^{(\rho^\star,\rho^\star,g^\star)} \equiv 0$, then we call it $g^\star(\cdot)$ an $(\rho^\star,\rho^\star)$-Ricci-Yamabe flow (briefly, RYF). Also, Ricci Yamabe flow is said to be (i) Ricci flow (RF) if $\rho^\star = 1$ [12], $\rho^\star = 0$; (ii) Yamabe flow (YF) [13] if $\rho^\star = 0$, $\rho^\star = 1$ and (iii) Einstein flow (EF) [5] if $\rho^\star = 1$, $\rho^\star = -1$.
A Riemannain manifold $(\Omega^{\widetilde{m}}, g^\star)$, $\widetilde{m} > 2$ admit $(\rho^\star,\rho^\star)$-Ricci-Yamabe soliton $(g^\star, \mathscr{K}, \lambda^\flat, \rho^\star, \rho^\star)$, or (briefly, $(\rho^\star,\rho^\star)$-RYS) if it satisfies

$$(\mathfrak{L}_{\mathscr{K}} g^\star)(\mathscr{F},\mathscr{G}) + 2\rho^\star \mathscr{S}(\mathscr{F},\mathscr{G}) + [2\lambda^\flat - \rho^\star \tau] g(\mathscr{F},\mathscr{G}) = 0, \tag{1.1}$$

where $\mathscr{S}$ is the Ricci tensor, $\rho^\star$, $\lambda^\flat$ and $\rho^\star$ are the real scalars, $\mathfrak{L}_{\mathscr{K}} g^\star$ denotes the Lie derivative of the metric $g^\star$ along $\mathscr{K}$ and τ is the scalar curvature. As a result, depending on whether the soliton is diminishing, increasing, or stable, $\lambda^\flat > 0$, $\lambda^\flat < 0$, or $\lambda^\flat$=0, respectively. If $\lambda^\flat$, $\rho^\star$ and $\rho^\star$ become smooth functions, then (1.1) is called almost Ricci-Yamabe soliton (briefly, ARYS). The notion of η-Ricci Yamabe soliton on $(\Omega^{\widetilde{m}}, g^\star)$ is given by

$$(\mathfrak{L}_{\mathscr{K}} g^\star)(\mathscr{F},\mathscr{G}) + 2\psi^\flat \eta^\sharp(\mathscr{F}) \eta^\sharp(\mathscr{G}) + 2\rho^\star \mathscr{S}(\mathscr{F},\mathscr{G}) + [2\lambda^\flat - \rho^\star \tau] g^\star(\mathscr{F},\mathscr{G}) = 0, \tag{1.2}$$

Email addresses: **prof_sky16@yahoo.com (Sunil Kumar Yadav), a.yildiz@inonu.edu.tr (Ahmet Yıldız)**

where $\eta^{\sharp}$ is a 1-form on Ω and $\psi^{\flat}$ is a constant [30]. Moreover, in [28] author defined the idea of a conformal Ricci-Yamabe soliton (briefly, CRYS) as

$$(\mathfrak{L}_{\mathscr{K}}g^{\star})(\mathscr{F},\mathscr{G})+2\rho^{\star}\mathscr{S}(\mathscr{F},\mathscr{G})+[2\mu^{\star}-(\widetilde{p}+\frac{2}{\widetilde{m}})-\rho^{\star}\tau]g^{\star}(\mathscr{F},\mathscr{G})=0, \tag{1.3}$$

where, $\widetilde{p}$ is a time dependent scalar field. Also the author introduced concept of conformal η-Ricci-Yamabe soliton (briefly, CERYS) which is given by

$$(\mathfrak{L}_{\mathscr{K}}g^{\star})(\mathscr{F},\mathscr{G})+2\psi^{\flat}\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})+2\rho^{\star}\mathscr{S}(\mathscr{F},\mathscr{G})+\left[2\mu^{\star}-\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)-\rho^{\star}\tau\right]g^{\star}(\mathscr{F},\mathscr{G})=0, \tag{1.4}$$

where, $\mu^{\star}$, $\rho^{\star}$, $\rho^{\star}$, and $\psi^{\flat}$ are real scalars [28]. If $\mathscr{K}=grad(f)$, where f is a smooth function of $(\Omega^{\widetilde{m}},g^{\star})$, then (1.4) is known as gradient conformal η-Ricci-Yamabe soliton (briefly, GCERYS), for more details (see, [37, 38]):
According to [18] a generalized symmetric $\mathscr{Z}$ tensor on $(\Omega^{\widetilde{m}},g^{\star})$ is outlined to as

$$\mathscr{Z}(\mathscr{F},\mathscr{G})=\mathscr{S}(\mathscr{F},\mathscr{G})+\psi g^{\star}(\mathscr{F},\mathscr{G}), \tag{1.5}$$

where, ψ is an arbitrary scalar function. In Refs. [31, 19, 20, 21, 2, 39, 40], several $\mathscr{Z}$-tensor attributes were highlighted. From (1.5), the scalar z^{*} one can get

$$z^{*}=\tau+\widetilde{m}\psi. \tag{1.6}$$

In light of the works mentioned above, we propose in our study to investigate several geometric aspects of an $\mathscr{LP}$-Sasakian spacetime admitting a CERYS. Now, we recall the following new definition with the help of 1.5 that will be use in the next sections.
A Riemannian manifold $(\Omega^{\widetilde{m}},g^{\star})$ admit

(i) generalized $\mathscr{Z}$-η-Ricci soliton (briefly, GZ(ERS) if

$$(\mathfrak{L}_{\xi}g^{\star})(\mathscr{F},\mathscr{G})+2\psi^{\flat}\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})+2\lambda^{\flat}g(\mathscr{F},\mathscr{G})+2\mathscr{Z}(\mathscr{F},\mathscr{G})=0, \tag{1.7}$$

(ii) generalized $\mathscr{Z}$ conformal η-Ricci soliton (briefly, GZ(CERS) if

$$(\mathfrak{L}_{\xi}g^{\star})(\mathscr{F},\mathscr{G})+2\mathscr{Z}(\mathscr{F},\mathscr{G})+\left[2\mu^{\star}-\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)\right]g(\mathscr{F},\mathscr{G})+2\psi^{\flat}\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})=0, \tag{1.8}$$

(iii) generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton (briefly, GZ(CERYS) if

$$(\mathfrak{L}_{\xi}g^{\star})(\mathscr{F},\mathscr{G})+2\psi^{\flat}\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})+2\rho^{\star}\mathscr{Z}(\mathscr{F},\mathscr{G})+\left[2\mu^{\star}-\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)-\rho^{\star}z^{*}\right]g^{\star}(\mathscr{F},\mathscr{G})=0, \tag{1.9}$$

(iv) gradient generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton (briefly, gradient GZ(CERYS) if

$$Hess(f)+\psi^{\flat}\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})+\rho^{\star}\mathscr{Z}(\mathscr{F},\mathscr{G})+\left[\mu^{\star}-\frac{1}{2}\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)-\frac{\rho^{\star}z^{*}}{2}\right]g^{\star}(\mathscr{F},\mathscr{G})=0, \tag{1.10}$$

where, $Hess(f)$ is the Hessian of the smooth function f.

2. Preliminaries

Let $\Omega^{\widetilde{m}}$ be a manifold with a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field $\eta^{\sharp}$ and a Lorentzian metric $g^{\star}$ of type $(0,2)$ such that for each point $p\in\Omega^{\widetilde{m}}$ the tensor $g^{\star}_{p}:\mathscr{T}_{p}\Omega\times\mathscr{T}_{p}\Omega\rightarrow\Re$ is an inner product of signature $(-,+,+,.....,+)$, where $\mathscr{T}_{p}\Omega$ is the tangent space of $\Omega^{\widetilde{m}}$ at p and $\Re$ is the real number space which satisfies

$$\phi^{2}(\mathscr{F})=\mathscr{F}+\eta^{\sharp}(\mathscr{F})\xi,\quad \eta^{\sharp}(\xi)=-1, \tag{2.1}$$

$$g^{\star}(\mathscr{F},\xi)=\eta^{\sharp}(\mathscr{F}),\ \ g^{\star}(\phi\mathscr{F},\phi\mathscr{G})=g^{\star}(\mathscr{F},\mathscr{G})+\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G}), \tag{2.2}$$

$\forall\ \ \mathscr{F},\mathscr{G}$ on $\Omega^{\widetilde{m}}$. Then $\Omega^{\widetilde{m}}$ with the structure $(\phi,\xi,\eta^{\sharp},g^{\star})$ is called the Lorentzian almost paracontact manifold (briefly, $(\text{LAPCM})_{\widetilde{m}}$) [22]. Also we have

$$\phi\xi=0,\ \ \eta^{\sharp}(\phi\mathscr{F})=0,\ \ \Omega^{\flat}(\mathscr{F},\mathscr{G})=\Omega^{\flat}(\mathscr{F},\mathscr{G}), \tag{2.3}$$

where $\Omega^{\flat}(\mathscr{F},\mathscr{G})=g^{\star}(\mathscr{F},\phi\mathscr{G})$ [22]. If the $(\text{LAPCM})_{\widetilde{m}}$ satisfies

$$(\nabla_{\mathscr{H}}\Omega^{\flat})(\mathscr{F},\mathscr{G})=\alpha[\{g^{\star}(\mathscr{F},\mathscr{H})+\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{H})\}\eta^{\sharp}(\mathscr{G})+\{g^{\star}(\mathscr{G},\mathscr{H})+\eta^{\sharp}(\mathscr{G})\eta^{\sharp}(\mathscr{H})\}\eta^{\sharp}(\mathscr{F})], \tag{2.4}$$

$$\Omega^{\flat}(\mathscr{F},\mathscr{G})=\frac{1}{\alpha}(\nabla_{\mathscr{F}}\eta^{\sharp})(\mathscr{G}), \tag{2.5}$$

for all vector fields $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ on $(\Omega^{\widetilde{m}},g^{\star})$, where α is a non-zero scalar function, then $(\Omega^{\widetilde{m}},g^{\star})$ is called an $\mathscr{LP}$-Sasakian manifold with the coefficient α [8]. Also a vector field $\mathscr{V}$ satisfies

$$\nabla_{\mathscr{F}}\mathscr{V}=\alpha\mathscr{F}+\mathscr{A}(\mathscr{F})\mathscr{V}, \tag{2.6}$$

where $\mathscr{V}$ is referred to as a torse-forming vector field (briefly, TFVF) and $\mathscr{A}$ is a non-zero 1-form [32].
Particularly, if ξ is a unit TFVF on $(\Omega^{\widetilde{m}}, g^{\star})$, then we have

$$\nabla_{\mathscr{F}}\xi = \alpha\mathscr{F} + \mathscr{A}(\mathscr{F})\xi. \tag{2.7}$$

Moreover, $g^{\star}(\xi,\xi) = -1$, which means that $g^{\star}(\nabla_{\mathscr{F}}\xi,\xi) = 0$. So, from (2.7), we yield

$$\mathscr{A}(\mathscr{F}) = \alpha\eta^{\sharp}(\mathscr{F}), \tag{2.8}$$

and

$$(\nabla_{\mathscr{F}}\eta^{\sharp})(\mathscr{G}) = \nabla_{\mathscr{F}}\eta^{\sharp}(\mathscr{G}) - \eta^{\sharp}(\nabla_{\mathscr{F}}\mathscr{G}) = g^{\star}(\mathscr{G},\nabla_{\mathscr{F}}\xi). \tag{2.9}$$

In light of (2.7)-(2.8), we obtain

$$(\nabla_{\mathscr{F}}\eta^{\sharp})(\mathscr{G}) = \alpha[g^{\star}(\mathscr{F},\mathscr{G}) + \eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})]. \tag{2.10}$$

Especially, if $\eta^{\sharp}$ satisfies

$$(\nabla_{\mathscr{F}}\eta^{\sharp})(\mathscr{G}) = \varepsilon\{g^{\star}(\mathscr{F},\mathscr{G}) + \eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})\}, \quad \varepsilon^2 = 1, \tag{2.11}$$

then $(\Omega^{\widetilde{m}}, g^{\star})$ is called an $\mathscr{LSP}$-Sasakian manifold [8]. In particular, if α satisfies (2.10) and along with

$$\nabla_{\mathscr{F}}\alpha = d\alpha(\mathscr{F}) = \sigma\eta^{\sharp}(\mathscr{F}), \tag{2.12}$$

where σ is a smooth function then ξ is called a CVF. Also on $\Omega^{\widetilde{m}}(\phi,\xi,\eta^{\sharp},g^{\star})$ with a coefficient α [16], we have

$$\eta^{\sharp}(\mathscr{R}(\mathscr{F},\mathscr{G})\mathscr{H}) = (\alpha^2-\sigma)[g^{\star}(\mathscr{G},\mathscr{H})\eta^{\sharp}(\mathscr{F}) - g^{\star}(\mathscr{F},\mathscr{H})\eta^{\sharp}(\mathscr{G})], \tag{2.13}$$

$$\mathscr{S}(\mathscr{F},\xi) = (\widetilde{m}-1)(\alpha^2-\sigma)\eta^{\sharp}(\mathscr{F}), \tag{2.14}$$

$$\mathscr{R}(\mathscr{F},\mathscr{G})\xi = (\alpha^2-\sigma)[\eta^{\sharp}(\mathscr{G})\mathscr{F} - \eta^{\sharp}(\mathscr{F})\mathscr{G}], \tag{2.15}$$

$$\mathscr{R}(\xi,\mathscr{G})\mathscr{F} = (\alpha^2-\sigma)[g(\mathscr{F},\mathscr{G})\xi - \eta^{\sharp}(\mathscr{F})\mathscr{G}], \tag{2.16}$$

$$(\nabla_{\mathscr{F}}\phi)(\mathscr{G}) = \alpha[g^{\star}(\mathscr{F},\mathscr{G})\xi + 2\eta^{\sharp}(\mathscr{F})\eta^{\sharp}(\mathscr{G})\xi + \eta^{\sharp}(\mathscr{G})\mathscr{F}], \tag{2.17}$$

$$\mathscr{S}(\phi\mathscr{F},\phi\mathscr{G}) = \mathscr{S}(\mathscr{F},\mathscr{G}) + (\widetilde{m}-1)(\alpha^2-\sigma)g^{\star}(\mathscr{F},\mathscr{G}), \tag{2.18}$$

$\forall\,\mathscr{F},\mathscr{G},\mathscr{H}$ on $(\Omega^{\widetilde{m}}, g^{\star})$.
The author of [23] talked about how semi-Riemannian geometry is used in the theory of relativity. Kaigorodov studies the curvature structure of spacetime [15]. Raychaudhary et al. elaborate on these concepts from the general theory of spacetime [26]. Chaki and Roy [6] investigated the covariant constant EMT in spacetime. There are many authors analyzed the features of PFST with different types of solitons in this sequel [1, 9, 40, 41, 14, 34, 35, 36].
A vector field $\mathscr{F}$ on $(\Omega^{\widetilde{m}}, g^{\star})$ is refered to as infinitesimal transformation (briefly, IT) if there exists a function υ, which obeys [3]

$$(\mathfrak{L}_{\mathscr{F}}\eta^{\sharp})(\mathscr{G}) = \upsilon\eta^{\sharp}(\mathscr{G}). \tag{2.19}$$

Specifically, $\mathscr{F}$ is known as a strict infinitesimal transformation (briefly, SIT) on $(\Omega^{\widetilde{m}}, g^{\star})$ if υ=0.
A vector field $\mathscr{K}$ on $(\Omega^{\widetilde{m}}, g^{\star})$ is said to be a conformal vector field (briefly, CVF) if

$$\mathfrak{L}_{\mathscr{K}}g^{\star}(\mathscr{F},\mathscr{G}) = 2\psi g^{\star}(\mathscr{F},\mathscr{G}),$$

where ψ is a smooth function [33]. Thus for the CVF, $\mathscr{K}$ on $(\Omega^{\widetilde{m}}, g^{\star})$ we have

$$(\mathfrak{L}_{\mathscr{K}}\mathscr{S})(\mathscr{F},\mathscr{G}) = -(\widetilde{m}-2)g^{\star}(\nabla_{\mathscr{F}}\mathscr{D}^{\flat}\psi,\mathscr{G}) + (\triangle^{\flat}\psi)g^{\star}(\mathscr{F},\mathscr{G}), \tag{2.20}$$

$$\mathfrak{L}_{\mathscr{K}}\tau = -2\psi r + 2(\widetilde{m}-1)\triangle^{\flat}\psi, \tag{2.21}$$

$\forall\,\mathscr{F},\mathscr{G}$ on $(\Omega^{\widetilde{m}}, g^{\star})$, where $\mathscr{D}^{\flat}$ and $\triangle^{\flat}$ denote the gradient operator and the Laplacian operator [33], respectively .

3. Generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton on $\mathscr{L}\mathscr{P}$-Sasakian manifold

A generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton on $(\Omega^{\widetilde{m}}, g^\star)$ are discuss in this section and deduce the remarkable results. From (1.5) and (1.9), we have

$$\mathfrak{L}_\xi g^\star(\mathscr{F},\mathscr{G}) + 2\psi^\flat \eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) + 2\rho^\star[\mathscr{S}(\mathscr{F},\mathscr{G}) + \psi g^\star(\mathscr{F},\mathscr{G})] + \left[2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right] g^\star(\mathscr{F},\mathscr{G}) = 0. \tag{3.1}$$

With reference to (2.7) and (2.8), equation (3.1) reduces

$$\mathscr{S}(\mathscr{F},\mathscr{G}) = \left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right] g^\star(\mathscr{F},\mathscr{G}) - \frac{1}{\rho^\star}(\alpha+\psi^\flat)\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}), \tag{3.2}$$

and

$$\mathscr{S}(\mathscr{F},\xi) = \left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right] \eta^\sharp(\mathscr{F}) + \frac{1}{\rho^\star}(\alpha+\psi^\flat)\eta^\sharp(\mathscr{F}). \tag{3.3}$$

As per equation (3.3), we can easily get

$$\mathscr{S}(\xi,\xi) = \left[\psi + \frac{\psi^\flat}{\rho^\star} + \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right], \tag{3.4}$$

$$\widetilde{\mathscr{Q}}\mathscr{F} = \left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right] \mathscr{F} - \frac{1}{\rho^\star}(\alpha+\psi^\flat)\eta^\sharp(\mathscr{F})\xi, \tag{3.5}$$

$$\widetilde{\mathscr{G}}\xi = \left[-\psi + \frac{\psi^\flat}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right]\xi, \tag{3.6}$$

$$\tau = \left[-\frac{\widetilde{m}(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{\widetilde{m}}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\} + \frac{1}{\rho^\star}(\alpha+\psi^\flat)\right], \tag{3.7}$$

$$\mu^\star + \psi^\flat = -\left[\psi\rho^\star + \rho^\star(\widetilde{m}-1)(\alpha^2-\sigma) - \frac{1}{2}\left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) + \frac{\rho^\star z^*}{2}\right], \tag{3.8}$$

where τ is the scalar curvature of $(\Omega^{\widetilde{m}}, g^\star)$ and $\alpha^2 - \sigma \neq 0$. As per above result, We assert that:

Theorem 3.1. *Let $(\Omega^{\widetilde{m}}, g^\star)$ admit a GZ(CERY)$_{\widetilde{m}}$-soliton, where ξ is a unit TFVF then the soliton factor $\mu^\star$ and $\psi^\flat$ are given by the equation* (3.8).

Also, we exhibit the following results.

Theorem 3.2. *Let $(\Omega^{\widetilde{m}}, g^\star)$ admit a symmetric and a skew-symmetric tensor $\Psi^\flat$ and ϕ of type $(0,2)$ respectively. If $\Psi^\flat$ is parallel with respect to ∇ on $(\Omega^{\widetilde{m}}, g^\star)$, then the structure $(\phi,\xi,\eta^\sharp,g^\star)$ possesses a GZ(CERY)$_{\widetilde{m}}$-soliton.*

Proof We think about the following

$$\Psi^\flat(\mathscr{F},\mathscr{G}) = (\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G}) + 2\rho^\star \mathscr{Z}(\mathscr{F},\mathscr{G}). \tag{3.9}$$

In view of (1.5) and (1.9), equation (3.9) reduces

$$\Psi^\flat(\mathscr{F},\mathscr{G}) = -\left[2\mu^\star - \left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right] g^\star(\mathscr{F},\mathscr{G}) - 2\psi^\flat\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) \tag{3.10}$$

For fix $\mathscr{F} = \mathscr{G} = \xi$ in (3.10), we yield

$$\Psi^\flat(\xi,\xi) = 2\left[\mu^\star - \frac{1}{2}\left(\widetilde{p} + \frac{2}{\widetilde{m}}\right) - \frac{\rho^\star z^*}{2} - \psi^\flat\right]. \tag{3.11}$$

Using (3.8) in (3.11), we get

$$\Psi^\flat(\xi,\xi) = -\left[4\psi^\flat + 2\rho^\star\psi + 2\rho^\star(\widetilde{m}-1)(\alpha^2-\sigma)\right] \neq 0. \tag{3.12}$$

Thus, the Theorem 3.2 is derive from the results (see Theorem 2.1, p-384) in [27]), as well as the preceding results (3.12). So proof is completed.

Theorem 3.3. *Let $\Omega^{\widetilde{m}}(\phi,\xi,\eta^\sharp,g^\star)$, $\widetilde{m} > 1$, with coefficient α admit a GZ(CERY)$_{\widetilde{m}}$-soliton, then soliton will be (i) shrinking if, $\alpha^2 > \sigma$, (ii) expanding if, $\alpha^2 < \sigma$.*

For the case $\alpha = 1$, we get from (3.12) that $\mu^\star < 0$, for $\widetilde{m} > 1$. We conclude that

Theorem 3.4. *A GZ(CERY)$_{\widetilde{m}}$-soliton on an $\mathscr{L}\mathscr{P}$-Sasakian manifold $(\Omega^{\widetilde{m}}, g^\star)$, $\widetilde{m} > 1$, is always shrinking.*

4. Generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton on $\mathscr{L}\mathscr{P}$-Sasakian spacetimes

Let $(\Omega^4, g^\star)$ is an $\mathscr{L}\mathscr{P}$-Sasakian manifold with a constant coefficient α. Since α is a constant, we get from (2.12) that σ=0 and so from (2.14), we have

$$\mathscr{S}(\mathscr{F},\xi) = 3\alpha^2\eta^\sharp(\mathscr{F}). \tag{4.1}$$

If ℓ^2 is the square of length of the Ricci tensor, then

$$\ell^2 = \sum_{i=1}^{\widetilde{m}} \mathscr{S}(\mathscr{Q}\vec{e}_i,\vec{e}_i), \tag{4.2}$$

Now taking, $\mathscr{F}=\mathscr{G}=\vec{e}_i$, $1\le i\le \widetilde{m}$, in (3.2), we get

$$\tau = 4a_1^* + b_1^*, \tag{4.3}$$

where, $a_1^* = \left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p}+\frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right]$ and b_1^* =-$\frac{1}{\rho^\star}(\alpha+\psi^\flat)$.
With the help of (3.2) and (4.1), we obtain

$$\mathscr{S}(\xi,\xi) = b_1^* - a_1^* = \left[\psi + \frac{\psi^\flat}{\rho^\star} + \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p}+\frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\}\right] = -3\alpha^2, \tag{4.4}$$

which implies that

$$\mu^\star = \left[\frac{1}{2}\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right) + \frac{\rho^\star z^*}{2} - \psi^\flat - \psi\rho^\star - 3\alpha^2\right]. \tag{4.5}$$

In view of (3.2), (4.2) and (4.4), we yield

$$\ell^2 = 36\alpha^2. \tag{4.6}$$

Thus, ℓ^2 is a constant so $\mathfrak{L}_{\mathscr{F}}\ell^2 = 0$, that is, $(\Omega^4, g^\star)$ is an Einstein spacetime [33]. So we infer that:

Theorem 4.1. *If $(\Omega^4, g^\star)$ coupled with GZ(CERY)$_4$-soliton admits an infinitesimal non-isometric conformal transformation, then $(\Omega^4, g^\star)$ reduces to an Einstein spacetime.*

Again, from a result of Mantica et al. [17]. We get the following.

Corollary 4.2. *An $\mathscr{L}\mathscr{P}$-Sasakian spacetime with a GZ(CERY)$_4$-soliton satisfying an infinitesimal non-isometric conformal transformation the spacetime is a GRW spacetime.*

In light of (4.3) and the Theorem 4.1, we state the followings:

Theorem 4.3. *An η-Einstein $\mathscr{L}\mathscr{P}$-Sasakian spacetime with an infinitesimal non-isometric conformal transformation does not admit a proper GZ(CERY)$_4$-soliton.*

Corollary 4.4. *An $\mathscr{L}\mathscr{P}$-Sasakian spacetime with a GZ(CERY)$_4$-soliton possesses a constant scalar curvature.*

Again, the Einstein's field equations (EFEs) [23]:

$$\mathscr{S}(\mathscr{F},\mathscr{G}) - \frac{\tau}{2}g^\star(\mathscr{F},\mathscr{G}) + \pi^\flat g^\star(\mathscr{F},\mathscr{G}) = \theta^\flat\mathscr{T}(\mathscr{F},\mathscr{G}), \tag{4.7}$$

where $\theta^\flat$ is the gravitational constant and $\mathscr{T}$ is the EMT of type $(0,2)$ is defined as

$$\mathscr{T}(\mathscr{F},\mathscr{G}) = (\mathscr{E}_{df} + \mathscr{I}_{pf})\mathscr{A}(\mathscr{F})\mathscr{A}(\mathscr{G}) + \mathscr{I}_{pf}g^\star(\mathscr{F},\mathscr{G}),$$

where, $\mathscr{E}_{df}$ is the energy density function, $\mathscr{I}_{pf}$ is the isotropic pressure function of the fluid and $\mathscr{A}$ is a non-zero 1-form such that $g^\star(\mathscr{F},\mathscr{V})$=$\mathscr{A}(\mathscr{G})$ for all $\mathscr{F}$, $\mathscr{V}$ being the flow vector field of the fluid [23].
In an $\mathscr{L}\mathscr{P}$-Sasakian spacetime, if ξ is the flow vector field of the fluid, then

$$\mathscr{T}(\mathscr{F},\mathscr{G}) = (\mathscr{E}_{df} + \mathscr{I}_{pf})\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) + \mathscr{I}_{pf}g^\star(\mathscr{F},\mathscr{G}). \tag{4.8}$$

In view of (3.2), (4.3), (4.4) and (4.8), we get

$$\mathscr{T}(\mathscr{F},\mathscr{G}) = \frac{1}{\theta^\flat}\left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p}+\frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right\} - \frac{\tau}{2} + \pi^\flat\right]g^\star(\mathscr{F},\mathscr{G}) - \frac{1}{\rho^\star\theta^\flat}(\alpha+\psi^\flat)\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}). \tag{4.9}$$

As per above, we assert the following:

Theorem 4.5. *If $(\Omega^4, g^\star)$ attached with GZ(CERY)$_4$-soliton satisfies the EFEs with the cosmological term $\pi^\flat$, then the EMT of the space is given by* (4.9).

Also, by virtue of (4.4), (4.7) and (4.9), we obtain

$$\mu^\star = \frac{1}{4}\left[2(\widetilde{p}+\frac{2}{\widetilde{m}}) + 2\rho^\star z^* - 6\alpha^2\theta^\flat + \tau\theta^\flat + 2\theta^\flat\pi^\flat\right] + \frac{1}{4}\left[\frac{2(\alpha+\psi^\flat)}{\rho^\star} - \frac{2(\alpha+\psi\rho^\star)}{\rho^\star} - \tau + 2\pi^\flat\right]. \tag{4.10}$$

Thus we have the followings:

Theorem 4.6. *If the* $(\Omega^4, g^\star)$ *equipped with GZ(CERY)*$_4$*-soliton satisfies the EFE with the cosmological term* $\pi^\flat$*, then the soliton constant is given by* (4.10)*.*

Theorem 4.7. *If* $(\Omega^4, g^\star)$ *attached with GZ(CERY)*$_4$*-soliton satisfies the EFEs, then* (Ω^4, g) *reduces to a quasi Einstein. Moreover, the spacetime to be dust iff the Lie-derivative of the EMT with respect to* ξ *vanish.*

Proof. In view of (4.8) and (4.9), we get

$$\frac{1}{\theta^\flat}\left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star}-\frac{1}{2\rho^\star}\left\{2\mu^\star-\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)-\rho^\star z^*\right\}-\frac{\tau}{2}+\pi^\flat-\mathscr{I}_{pf}\right]g^\star(\mathscr{F},\mathscr{G})=[\mathscr{E}_{df}+\mathscr{I}_{pf}+\frac{1}{\rho^\star\theta^\flat}(\alpha+\psi^\flat)]\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) \quad (4.11)$$

If we put $\mathscr{F}=\mathscr{G}=\xi$ in (4.11), we obtain

$$\pi^\flat=\left[\frac{(\alpha+\psi\rho^\star)}{\rho^\star}+\frac{1}{2\rho^\star}\left\{2\mu^\star-\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)-\rho^\star z^*\right\}+\frac{\tau}{2}\right]-\frac{1}{\rho^\star}(\alpha+\psi^\flat)-\theta^\flat\mathscr{E}_{df} \quad (4.12)$$

Also, contracting (4.11), we have

$$\pi^\flat=\left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star}-\frac{1}{2\rho^\star}\left\{2\mu^\star-\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right)-\rho^\star z^*\right\}-\frac{\tau}{2}\right]+\frac{1}{4\rho^\star}(\alpha+\psi^\flat)+\frac{\theta^\flat}{4}(\mathscr{E}_{df}-3\mathscr{I}_{pf}). \quad (4.13)$$

Adding (4.12) and (4.13), we get

$$\mathscr{E}_{df}+\mathscr{I}_{pf}=\frac{1}{\theta^\flat}\left[-\frac{8\pi^\flat}{3}-\frac{(\alpha+\psi^\flat)}{\rho^\star}\right] \quad (4.14)$$

Using (4.8) and (4.14) in (4.7), we have

$$\mathscr{S}(\mathscr{F},\mathscr{G})=\left[\frac{\tau}{2}+\theta^\flat\mathscr{I}_{pf}-\pi^\flat\right]g^\star(\mathscr{F},\mathscr{G})-\left[\frac{8\pi^\flat}{3}+\frac{(\alpha+\psi^\flat)}{\rho^\star}\right]\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}). \quad (4.15)$$

Taking $\mathscr{F}=\mathscr{G}=e_i$, $1\le i\le 4$, in (4.15), we obtain

$$\tau=\left[4\pi^\flat-4\theta^\flat\mathscr{I}_{pf}-\frac{8\pi^\flat}{3}-\frac{(\alpha+\psi^\flat}{\rho^\star}\right]. \quad (4.16)$$

In consequence of (4.15) and (4.16), we find

$$\mathscr{S}(\mathscr{F},\mathscr{G})=\left[\pi^\flat-\theta^\flat\mathscr{I}_{pf}-\frac{4\pi^\flat}{3}-\frac{(\alpha+\psi^\flat)}{2\rho^\star}\right]g^\star(\mathscr{F},\mathscr{G})-\left[\frac{8\pi^\flat}{3}+\frac{(\alpha+\psi^\flat)}{\rho^\star}\right]\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}), \quad (4.17)$$

which means that $(\Omega^4, g^\star)$ is a (QE)$_4$ spacetime. So, the proof is finished.
Also from (4.8) and (4.14), we yield

$$\mathscr{T}(\mathscr{F},\mathscr{G})=\mathscr{I}_{pf}g^\star(\mathscr{F},\mathscr{G})+\frac{1}{\theta^\flat}\left[-\frac{8\pi^\flat}{3}-\frac{(\alpha+\psi^\flat}{\rho^\star}\right]\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}). \quad (4.18)$$

After taking the Lie derivative of (4.18) along with ξ, we have

$$\begin{aligned}(\mathfrak{L}_\xi\mathscr{T})(\mathscr{F},\mathscr{G}) &= \mathscr{I}_{pf}(\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G})-\frac{1}{\theta^\flat}\left[\frac{8\pi^\flat}{3}+\frac{(\alpha+\psi^\flat)}{\rho^\star}\right](\mathfrak{L}_\xi g^\star)(\mathscr{F},\xi)g^\star(\mathscr{G},\xi)\\ &+ g^\star(\mathscr{F},\mathfrak{L}_\xi\xi)g^\star(\mathscr{G},\xi)+g^\star(\mathscr{F},\xi)(\mathfrak{L}_\xi g^\star)(\mathscr{G},\xi)\\ &+ g^\star(\mathscr{F},\xi)g^\star(\mathscr{G},\mathfrak{L}_\xi\xi)\},\end{aligned}$$

which implies that, if $(\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G})=\frac{2}{\theta^\flat}\left[-\frac{8\pi^\flat}{3}-\frac{(\alpha+\psi^\flat}{\rho^\star}\right]\{g^\star(\mathscr{F},\mathscr{G})+\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{F})\}$ and $\mathfrak{L}_\xi\xi$=0, then it turn up

$$(\mathfrak{L}_\xi\mathscr{T})(\mathscr{F},\mathscr{G})=\mathscr{I}_{pf}(\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G}). \quad (4.19)$$

Since in general, $g^\star(\phi\mathscr{F},\phi\mathscr{G})\neq 0$ on $(\widetilde{\Omega^4},g)$ and therefore $\mathfrak{L}_\xi g\neq 0$. Thus from (4.19), we see that ($\mathfrak{L}_\xi\mathscr{T})(\mathscr{F},\mathscr{G})$=0 if and only if $\mathscr{I}_{pf}=0$, provided $\theta^\flat\neq 0$. So, Theorem 4.7 is finished.

Theorem 4.8. *If* $(\Omega^4, g^\star)$ *attached with GZ(CERY)*$_4$*-soliton satisfies the EFEs with the cosmological constant and the Lie-derivative of the EMT with respect to* ξ *vanishes, then the acceleration vector and the expansion scalar of the fluid vanishes .*

Proof. According to [23], for PFS

$$\xi\mathscr{E}_{df}=-(\mathscr{I}_{pf}+\mathscr{E}_{df})div\xi,$$

and

$$(\mathscr{I}_{pf}+\mathscr{E}_{df})\nabla_\xi\xi=-grad\,\mathscr{I}_{pf}-(\xi\mathscr{I}_{pf})\xi.$$

With the help of (4.14) and (4.19) along with above equation, we get $div\xi=0$ and $\nabla_\xi\xi=0$. So the Theorem 4.8 is justified.

Theorem 4.9. *Let $(\Omega^4, g^\star)$ attached with GZ(CERY)$_4$-soliton satisfies the EFEs with a cosmological term and the Lie-derivative of the EMT with respect to ξ vanishes, we have*

$$(\nabla_{\mathscr{F}}\mathscr{S})(\mathscr{G},\mathscr{H}) = \theta^\flat(\nabla_{\mathscr{F}}\mathscr{T})(\mathscr{G},\mathscr{H}).$$

Proof. Taking the covariant derivative of (4.15), we get

$$(\nabla_{\mathscr{F}}\mathscr{S})(\mathscr{G},\mathscr{H}) = -\alpha\left[\frac{8\pi^\flat}{3}+\frac{(\alpha+\psi^\flat)}{\rho^\star}\right][g^\star(\mathscr{F},\mathscr{G})\eta^\sharp(\mathscr{H})+g^\star(\mathscr{F},\mathscr{H})\eta^\sharp(\mathscr{G})+2\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G})\eta^\sharp(\mathscr{H})].$$

Also by the similar way from (4.18), we yield

$$(\nabla_{\mathscr{F}}\mathscr{T})(\mathscr{G},\mathscr{H}) = -\frac{\alpha}{\theta^\flat}\left[\frac{8\pi^\flat}{3}+\frac{(\alpha+\psi^\flat}{\rho^\star}\right][g^\star(\mathscr{F},\mathscr{G})\eta^\sharp(\mathscr{H})+g^\star(\mathscr{F},\mathscr{H})\eta^\sharp(\mathscr{G})+2\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G})\eta^\sharp(\mathscr{H})].$$

So from above two equations, one can easily get

$$(\nabla_{\mathscr{F}}\mathscr{S})(\mathscr{G},\mathscr{H}) = \theta^\flat(\nabla_{\mathscr{F}}\mathscr{T})(\mathscr{G},\mathscr{H}). \tag{4.20}$$

Therefore the proof is finished.
Next, using the definitions of cyclic parallel and Codazzi type Ricci tensor on $(\Omega^{\widetilde{m}}, g^\star)$ along with Theorem 4.9, we state the result:

Theorem 4.10. *Let $(\Omega^4, g^\star)$ admit a GZ(CERY)$_4$-soliton satisfies the EFEs with a cosmological term. If the Lie-derivative of EMT with respect to ξ vanishes, then the Ricci tensor is of Codazzi type iff the EMT is also Codazzi type.*

Corollary 4.11. *Let $(\Omega^4, g^\star)$ attached with a GZ(CERY)$_4$-soliton satisfies the EFEs with a cosmological term and the Lie-derivative of EMT with respect to ξ vanishes, then the necessary and sufficient condition for the Ricci tensor is of cyclic parallel iff the EMT is also cyclic parallel.*

5. Existence of generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton on $(\Omega^4, g^\star)$

In this segment, we demonstrate the existence of GZ(CERY)$_4$-soliton on $(\Omega^4, g^\star)$ for $\mathscr{K} = \xi$. Now we prove:

Theorem 5.1. *There exists a GZ(CERY)$_4$-soliton on an $\mathscr{LP}$-Sasakian spacetime.*

Proof. If possible, let $(\Omega^4, g^\star)$ admit a GZ(CERY)$_4$-soliton, then from (1.9), we have

$$\frac{1}{2}(\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G}) = -\psi^\flat\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) - \rho^\star\mathscr{Z}(\mathscr{F},\mathscr{G}) - \left[\mu^\star - \frac{1}{2}\left(\widetilde{p}+\frac{2}{\widetilde{m}}\right) - \frac{\rho^\star z^*}{2}\right]g^\star(\mathscr{F},\mathscr{G}). \tag{5.1}$$

Using (2.7) and (2.8) in (5.1), we yield

$$\alpha\{g^\star(\mathscr{F},\mathscr{G})+\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G})\} = -2\psi^\flat\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) - 2\rho^\star\mathscr{Z}(\mathscr{F},\mathscr{G}) - \left[2\mu^\star - \left(\widetilde{p}+\frac{2}{\widetilde{m}}\right) - \rho^\star z^*\right]g^\star(\mathscr{F},\mathscr{G}). \tag{5.2}$$

Putting $\mathscr{F} = \mathscr{G} = \xi$ in (5.2), using (1.5) and (4.1), we get

$$\mu^\star = [\frac{1}{2}(\widetilde{p}+\frac{2}{\widetilde{m}}) - (3\alpha^2+\psi)\rho^\star + \psi^\flat + \frac{\rho^\star z^*}{2}]. \tag{5.3}$$

In view of (5.1) and (5.3), we yield

$$\begin{aligned}(\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G}) \quad &+ \quad 2\psi^\flat\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) + 2\rho^\star\mathscr{Z}(\mathscr{F},\mathscr{G}) \\ &+ \quad 2\left[\psi^\flat - (3\alpha^2+\psi)\rho^\star\right]g^\star(\mathscr{F},\mathscr{G}) = 0.\end{aligned} \tag{5.4}$$

It indicates that $(\Omega^4, g^\star)$ admit a GZ(CERY)$_4$-soliton. Hence the Theorem 5.1 is proved.

Theorem 5.2. *Any infinitesimal contact transformation which admit GZ(CERY)$_4$-soliton on $(\Omega^{\widetilde{m}}, g^\star)$, is an infinitesimal strict contact transformation.*

Proof. Let $(\Omega^{\widetilde{m}}, g^\star)$ is a $\mathscr{LP}$-Sasakian spacetime with constant coefficient α. Then from (4.16) the scalar curvature is constant. Thus from (2.20) and (2.21) we have

$$(\mathfrak{L}_{\mathscr{K}}\mathscr{S})(\mathscr{G},\mathscr{H}) = -(\widetilde{m}-2)g^\star(\nabla_{\mathscr{G}}\mathscr{D}^\flat\kappa,\mathscr{H}) + (\triangle^\flat\kappa)g^\star(\mathscr{G},\mathscr{H}), \tag{5.5}$$

and

$$\psi = \kappa - \lambda.$$

From (5.5), we find

$$(\mathfrak{L}_{\mathscr{K}}\mathscr{S})(\mathscr{G},\mathscr{H}) = 0. \tag{5.6}$$

Putting $\mathscr{H} = \xi$ in (5.6), we yield

$$(\mathfrak{L}_{\mathscr{K}}\mathscr{S})(\mathscr{G},\xi) = 0. \tag{5.7}$$

Also, we have

$$(\mathfrak{L}_{\mathscr{K}}\mathscr{S})(\mathscr{G},\xi)=\mathfrak{L}_{\mathscr{K}}(\mathscr{S}(\mathscr{G},\xi))-\mathscr{S}(\mathfrak{L}_{\mathscr{K}}\mathscr{G},\xi)-\mathscr{S}(\mathscr{G},\mathfrak{L}_{\mathscr{K}}\xi).$$

By the use of (2.19) and (4.1), the above equation reduces to

$$\mathscr{S}(\mathscr{G},\mathfrak{L}_{\mathscr{K}}\xi)=3\alpha^2(\mathfrak{L}_{\mathscr{K}}\eta^\sharp)(\mathscr{G})=3\alpha^2\upsilon\eta^\sharp(\mathscr{G}). \tag{5.8}$$

Again, putting $\mathscr{G}=\xi$ in (5.8), we have

$$\mathscr{S}(\xi,\mathfrak{L}_{\mathscr{K}}\xi)=-3\upsilon\alpha^2. \tag{5.9}$$

Keeping in mind (4.1) and (5.9), we get

$$\eta^\sharp(\mathfrak{L}_{\mathscr{K}}\xi)=-\upsilon. \tag{5.10}$$

Again (2.19) and (5.10) yield

$$(\mathfrak{L}_{\mathscr{K}}\eta^\sharp)(\xi)=\upsilon, \tag{5.11}$$

which implies that

$$\mathfrak{L}_{\mathscr{K}}(\eta^\sharp(\xi))-\eta^\sharp(\mathfrak{L}_{\mathscr{K}}\xi)=\upsilon. \tag{5.12}$$

In view of (5.10) and (5.12), we get $\upsilon=0$. Thus from (2.19) together with (5.12), the Theorem 5.2 is proved.

Next, let potential vector field $\mathscr{K}=\beta\xi$, where β is a smooth function. Then we obtain

$$\nabla_{\mathscr{F}}\mathscr{K}=\nabla_{\mathscr{F}}(\beta\xi)=(\mathscr{F}\beta)\xi+\alpha\beta\phi^2(\mathscr{F}). \tag{5.13}$$

In view of (2.1), (2.2), (2.7) and (5.13), we obtain

$$(\mathfrak{L}_{\mathscr{K}}g^\star)(\mathscr{F},\mathscr{G})=(\mathscr{F}\beta)\eta^\sharp(\mathscr{G})+(\mathscr{G}\beta)\eta^\sharp(\mathscr{F})+2\alpha\beta\{g^\star(\mathscr{F},\mathscr{G})+\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G})\}. \tag{5.14}$$

Using (1.9) in (5.14), we yield

$$\begin{aligned}(\mathscr{F}\beta)\eta^\sharp(\mathscr{G})+(\mathscr{G}\beta)\eta^\sharp(\mathscr{F})+2\alpha\beta\{g^\star(\mathscr{F},\mathscr{G})+\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G})\} &= -2\psi^\flat\eta^\sharp(\mathscr{F})(\eta^\sharp\mathscr{G})-2\rho^\star\mathscr{Z}(\mathscr{F},\mathscr{G})\\ &- \left[2\mu^\star-\left(\tilde{p}+\frac{2}{\tilde{\tilde{m}}}\right)-\rho^\star z^*\right]g^\star(\mathscr{F},\mathscr{G}).\end{aligned} \tag{5.15}$$

After, contracting (5.15) over $\mathscr{F}$ and $\mathscr{G}$, it follows that

$$\xi\beta=-3\alpha\beta+\psi^\flat-\rho^\star z^*-2\left[2\mu^\star-\left(\tilde{p}+\frac{2}{\tilde{\tilde{m}}}\right)-\rho^\star z^*\right]. \tag{5.16}$$

Putting $\mathscr{G}=\xi$ in (5.15) and using (5.16), it implies

$$-\mathscr{F}\beta=\left[2\mu^\star-\left(\tilde{p}+\frac{2}{\tilde{\tilde{m}}}\right)-\rho^\star z^*+\psi^\flat+\rho^\star z^*-2\rho^\star(3\alpha^2+\psi)+3\alpha\beta\right]\eta^\sharp(\mathscr{F}). \tag{5.17}$$

Again replacing $\mathscr{F}=\xi$ in (5.17), we find

$$\xi\beta=\left[2\mu^\star-\left(\tilde{p}+\frac{2}{\tilde{\tilde{m}}}\right)-\rho^\star z^*+\psi^\flat+\rho z^*-2\rho^\star(3\alpha^2+\psi)+3\alpha\beta\right]. \tag{5.18}$$

In view of (5.16) and (5.18), we get

$$\mu^\star=\left[\left(\tilde{p}+\frac{2}{\tilde{\tilde{m}}}\right)-\rho^\star z^*-\frac{\rho^\star z^*}{3}-\alpha\beta+\frac{(3\alpha^2+\psi)}{3}\right]. \tag{5.19}$$

By using (5.19) in (5.17), we have

$$\mathscr{F}\beta=\left[\frac{(4\alpha^2+\psi)\rho^\star}{3}-\alpha\beta-\frac{\rho^\star z^*}{3}\right]\eta^\sharp(\mathscr{F}),$$

which implies that

$$g(D\beta,\mathscr{F})=\left[\frac{(4\alpha^2+\psi)\rho^\star}{3}-\alpha\beta-\frac{\rho^\star z^*}{3}\right]g^\star(\mathscr{F},\xi),$$

that is,

$$D\beta=\left[\frac{(4\alpha^2+\psi)\rho^\star}{3}-\alpha\mathscr{K}-\frac{\rho^\star z^*}{3}\right].$$

Thus we state:

Theorem 5.3. *If $(\Omega^4,g^\star)$ admits a GZ(CERY)$_4$-soliton, then the potential vector field $\mathscr{K}$ and the gradient of function β are linearly dependent.*

Theorem 5.4. *Let $(\Omega^4,g^\star)$ admit a GZ(CERY)$_4$-soliton and the potential vector field $\mathscr{K}$ is point-wise collinear with ξ, then $(\Omega^4,g^\star)$ is a space of constant curvature.*

Moreover, as per [25], we state the finding.

Corollary 5.5. *An $\mathscr{LP}$-Sasakian spacetime attached with GZ(CERY)$_4$-soliton, where the pneotential vector field $\mathscr{K}$ is point-wise collianear with ξ, then $(\Omega^4,g^\star)$ is of Petrov type O.*

6. Physical significance of conformal pressure on $\mathscr{LP}$-Sasakian spacetimes

According to [10] and Eq. (4.10), we state the followings:

Theorem 6.1. *If an $\mathscr{LP}$-Sasakian spacetime admits the GZ(CERY)$_4$-soliton and satisfies the EFEs with cosmologocal constant $\pi^\flat$, then the conformal pressure is*

$$\widetilde{p}=\frac{1}{2}\left[4\mu^\star-2\rho^\star z^*+2(3\alpha^2-\pi^\flat)\theta^\flat+\tau(1-\theta^\flat)+\frac{2}{\rho^\star}(\psi^\flat(\rho^\star-1))-(2\pi^\flat+1)\right].$$

Theorem 6.2. *If an $\mathscr{LP}$-Sasakian spacetime satisfies the EFEs with cosmologocal constant $\pi^\flat$ admits GZ(CERY)$_4$-soliton then the metric $g^\star$ is an equilibrium point or Einstein iff*

$$4\mu^\star-2\rho^\star z^*+2(3\alpha^2-\pi^\flat)\theta^\flat+\tau(1-\theta^\flat)+\frac{2}{\rho^\star}(\psi^\flat(\rho^\star-1))=(2\pi^\flat+1).$$

Likewise, we turn th following corollary in view of dynamical system.

Corollary 6.3. *If an $\mathscr{LP}$-Sasakian spacetime satisfies the EFEs with cosmologocal constant $\pi^\flat$ admits GZ(CERY)$_4$-soliton then the metric $g^\star$ is an equilibrium point and acts as a nonlinear restoring force.*

7. Modified Liouville equation on $\mathscr{LP}$-Sasakian spacetimes

Let $(\Omega^4,g^\star)$ is an $\mathscr{LP}$-Sasakian spacetime admit GZ(CERY)$_4$-soliton. Then from (1.9), (1.5) and (4.17), we have

$$\begin{aligned}\frac{1}{2}[g^\star(\nabla_{\mathscr{F}}\xi,\mathscr{G}) &+ g^\star(\mathscr{F},\nabla_{\mathscr{G}}\xi)]\\ &+ \left[\mu^\star+\rho^\star(\pi^\flat+\psi^\flat)-\rho^\star\theta^\flat\mathscr{I}_{pf}-\frac{4\pi^\flat\rho^\star}{3}-\frac{\alpha+\psi^\flat}{3}-\frac{1}{2}(\widetilde{p}+\frac{1}{2})-\frac{\sigma^\star z^\star}{2}\right]g(\mathscr{F},\mathscr{G})\\ &- (\alpha+\frac{8\pi^\flat\rho^\star}{3})\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G})=0, \end{aligned}\tag{7.1}$$

for any $\mathscr{F},\mathscr{G}\in\chi(\Omega^4)$. On contracting (7.1), we get

$$\mathbb{D}iv(\xi) = -\left[4\mu^\star+4\rho^\star(\pi^\flat+\psi^\flat)-4\rho^\star\theta^\flat\mathscr{I}_{pf}-\frac{16\pi^\flat\rho^\star}{3}-\frac{4(\alpha+\psi^\flat)}{3}-2(\widetilde{p}+\frac{1}{2})-2\sigma^\star z^\star\right]-(\alpha+\frac{8\pi^\flat\rho^\star}{3}),\tag{7.2}$$

Remark: In particular, for $\Phi^\star\in C^\infty(\Omega)$, then for the vector field ξ, we have

$$\mathbb{D}iv(\Phi^\star\xi)=\xi(d\Phi^\star)+\Phi^\star\mathbb{D}iv(\xi),$$

which implies that, $\Phi^\star\in C^\infty(\Omega)$ is a last multiplier of ξ with respect to $g^\star$ if $\mathbb{D}iv(\Phi^\star\xi)$=0. So it gives

$$\xi(d\,ln\Phi^\star)=-\mathbb{D}iv(\xi),$$

is called the Liouville's equation of the vector field ξ with respect to $g^\star$[24]. So, utilizing this fact and from (7.2), we sate the outcome:

Theorem 7.1. *Let (Ω^4,g) admit a GZ(CERY)$_4$-soliton with a unit time-like vector field ξ and $\Phi^\star$ is the last multiplier of ξ and if $\eta^\sharp$ be the $g^\star$-dual 1-form of ξ, then the modified Liouville equation satisfying by $\Phi^\star$ and ξ is*

$$\begin{aligned}\xi(d\,ln\Phi^\star) &= \left[4\mu^\star+4\rho^\star(\pi^\flat+\psi^\flat)-4\rho^\star\theta^\flat\mathscr{I}_{pf}-\frac{16\pi^\flat\rho^\star}{3}-\frac{4(\alpha+\psi^\flat)}{3}-2(\widetilde{p}+\frac{1}{2})-2\sigma^\star z^\star\right]\\ &+ (\alpha+\frac{8\pi^\flat\rho^\star}{3})\end{aligned}\tag{7.3}$$

8. Gradient generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton on $\mathscr{LP}$-Sasakian spacetimes

Let the soliton vector $\mathscr{F}=\mathscr{D}f$, where f is a smooth function and $\mathscr{D}$ stands for gradient operator of $g^\star$ on $(\Omega^4,g^\star)$. So from (1.5) and (1.10) we have

$$\rho^\star\mathscr{S}+Hess f+[\rho^\star\psi+\mu^\star-\frac{1}{2}(\widetilde{p}+\frac{1}{2})]g+\psi^\flat\eta^\sharp\otimes\eta^\sharp=0,\tag{8.1}$$

which is equivalent to,

$$\nabla\mathscr{D}f=-[\rho^\star\psi+\mu^\star-\frac{1}{2}(\widetilde{p}+\frac{1}{2})-\frac{\sigma^\star z^\star}{2}]I-\rho^\star\mathscr{Q}-\psi^\flat\eta\otimes\xi.\tag{8.2}$$

After contracting (8.2) and using (4.1), we yield

$$\triangle f=-\rho^\star\psi-4\mu^\star+2(\widetilde{p}+\frac{1}{2})+2\rho^\star z^\star-12\alpha^2\rho^\star+\psi^\flat.\tag{8.3}$$

So, we the outcome

Theorem 8.1. *Let (Ω^4,g) be a $\mathscr{LP}$-Sasakian spacetime admit a gradient GZ(CERY)$_4$-soliton, then the potential function f satisfies the Poisson's equation*

$$\triangle f = -\rho^\star\psi - 4\mu^\star + 2(\widetilde{p}+\frac{1}{2}) + 2\rho^\star z^\star - 12\alpha^2\rho^\star + \psi^\flat.$$

Further, if the conformal pressure $\widetilde{p} = \frac{1}{2}[-4(\mu^\star+\rho^\star\psi)+2\sigma^\star z^\star - 12\rho^\star\alpha^2 - (1+\psi^\flat)]$, then $\triangle f = 0$, i.e., the Laplace equation.

Corollary 8.2. *Let (Ω^4,g) admit a gradient GZ(CERY)$_4$-soliton. If the conformal pressure of a perfect fluid matter is $\widetilde{p} = \frac{1}{2}[-4(\mu^\star+\rho^\star\psi)+2\sigma^\star z^\star - 12\rho^\star\alpha^2 - (1+\psi^\flat)]$, then the potential function f satisfies the Laplace equation.*

Also, from the Theorem 8.1, we state that the following:

Corollary 8.3. *If the potential function f of a gradient GZ(CERY)$_4$-soliton on (Ω^4,g) satisfies the Laplace equation then the soliton is expanding, stable, or declining according to*
(i) $(\widetilde{p}+\frac{1}{2}) > \frac{1}{2}[\rho^\star(\psi - 2z^\star + 12\alpha^2) - \psi^\flat]$,
(ii) $(\widetilde{p}+\frac{1}{2}) = \frac{1}{2}[\rho^\star(\psi - 2z^\star + 12\alpha^2) - \psi^\flat]$,
(iii) $(\widetilde{p}+\frac{1}{2}) < \frac{1}{2}[\rho^\star(\psi - 2z^\star + 12\alpha^2) - \psi^\flat]$, *respectively.*

Theorem 8.4. *Let (Ω^4,g) admit a gradient GZ(CERY)$_4$-soliton with potential function f. Then f is harmonic, subharmonic and superharmonic if the conformal pressure is*
(i) $\widetilde{p} = \frac{1}{2}[-4(\mu^\star+\rho^\star\psi)+2\sigma^\star z^\star - 12\rho^\star\alpha^2 - (1+\psi^\flat)]$,
(i) $\widetilde{p} \geq \frac{1}{2}[-4(\mu^\star+\rho^\star\psi)+2\sigma^\star z^\star - 12\rho^\star\alpha^2 - (1+\psi^\flat)]$,
(i) $\widetilde{p} \leq \frac{1}{2}[-4(\mu^\star+\rho^\star\psi)+2\sigma^\star z^\star - 12\rho^\star\alpha^2 - (1+\psi^\flat)]$.

9. Harmonic aspect of generalized $\mathscr{Z}$ conformal η-Ricci-Yamabe soliton on $\mathscr{LP}$-Sasakian spacetimes

Let $\eta^\sharp$ is a $g^\star$-dual 1-form of ξ, such that $g^\star(\mathscr{F},\xi) = \eta^\sharp(\mathscr{F})$ and $g^\star(\xi,\xi) = -1$. Then, ξ is called a solution of the Schrödinger-Ricci equation (briefly, SRE) if it satisfies

$$\mathbb{D}iv(\mathfrak{L}_\xi g^\star) = 0. \tag{9.1}$$

According to [7], we have

$$\mathbb{D}iv(\mathfrak{L}_\xi g^\star) = (\Gamma + \mathscr{S})(\xi) + d(\mathbb{D}iv(\xi)), \tag{9.2}$$

where Γ is the Laplace-Hodge operator with respect to the metric $g^\star$ and $\mathscr{S}$ is the Ricci tensor. Now, form (1.9), we yield

$$(\mathfrak{L}_\xi g^\star)(\mathscr{F},\mathscr{G}) + 2\psi^\flat\eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}) + 2\rho^\star\mathscr{Z}(\mathscr{F},\mathscr{G}) + \left[2\mu^\star - \left(\widetilde{p}+\frac{1}{2}\right) - \rho^\star z^\star\right]g^\star(\mathscr{F},\mathscr{G}) = 0. \tag{9.3}$$

Taking trace of (9.3), we get

$$\mathbb{D}iv(\xi) + (\rho^\star - 2\sigma^\star)z^\star + 4\mu^\star - 2(\widetilde{p}+\frac{1}{2}) + \psi^\flat|\xi|^2 = 0. \tag{9.4}$$

On the other hand, we reflect

$$\mathbb{D}iv(\eta^\sharp\otimes\eta^\sharp) = \mathbb{D}iv(\xi)\eta^\sharp + \nabla_\xi\eta^\sharp. \tag{9.5}$$

From (9.3) and (9.5), we have

$$\mathbb{D}iv(\mathfrak{L}_\xi g^\star) + 2\rho^\star d(\tau + 4\psi) + 2\psi^\flat[div(\xi)\eta^\sharp + \nabla_\xi\eta^\sharp] = 0. \tag{9.6}$$

Also, for Schrödinger-Ricci solution, 1-form π satisfies

$$(\Gamma + \mathscr{S})(\pi) + d(\mathbb{D}iv(\pi)) = 0. \tag{9.7}$$

Thus we state:

Theorem 9.1. *Let $(\Omega^4,g^\star)$ attached with a GZ(CERY)$_4$-soliton and $\eta^\sharp$ being the $g^\star$-dual of ξ, then $\eta^\sharp$ is a solution of the Schrödinger-Ricci equation iff,*

$$d(\tau+4\psi) \quad = \quad \frac{\psi^\flat}{\rho^\star}[\{(\rho^\star - 2\sigma^\star)z^\star + 4\mu^\star - 2(\widetilde{p}+\frac{1}{2}) + \psi^\flat|\xi|^2\}\eta^\sharp - \nabla_\xi\eta^\sharp] \tag{9.8}$$

Proof. In view of (2.8),(9.3), (9.4), (9.5) and the formula $2\mathbb{D}iv(\mathscr{Z}) = d(z^\star)$, it implies that $\eta^\sharp$ is a solution of SRE iff (9.6) holds. □

Again, If 1-form π is a Schrödinger-Ricci harmonic form [4], then

$$(\Gamma + \mathscr{S})(\pi) = 0. \tag{9.9}$$

Moreover, if $\psi^\flat = 0$, then $(\Omega^4,g^\star)$ has genelaized conformal $\mathscr{Z}$-RYS, or

$$\nabla_\xi\eta^\sharp = \{(\rho^\star - 2\sigma^\star)z^\star + 4\mu^\star - 2(\widetilde{p}+\frac{1}{2})\}\eta^\sharp,$$

where $\psi^\flat = (\rho^\star - 2\sigma^\star)z^\star + 4\mu^\star - 2(\widetilde{p}+\frac{1}{2})$. Now we claim that

Corollary 9.2. *If a GZ(CERY)$_4$-soliton on $(\Omega^4, g^\star)$ with $\eta^\sharp$ being the $g^\star$-dual of the time-like vector field ξ, then $\eta^\sharp$ is the Schrödinger-Ricci harmonic form if and only if $\psi^\flat$=0, which produces GZ(CRY)$_4$-soliton or,*

$$\nabla_\xi \eta^\sharp = \{(\rho^\star - 2\sigma^\star)z^\star + 4\mu^\star - 2(\widetilde{p} + \frac{1}{2})\}\eta^\sharp,$$

where, $\psi^\flat = (\rho^\star - 2\sigma^\star)z^\star + 4\mu^\star - 2(\widetilde{p} + \frac{1}{2})$.

10. An Example

Let us consider a 4-dimensional differentiable manifold $\Omega^4 = \{(u,v,w,t) \in \Re^4 : (u,v,w,t) \neq 0\}$, where (u,v,w,t) is the standard coordinate in $\Re^4$ and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and e_4 on $(\Omega^4, g^\star)$ given by

$$\vec{e}_1 = e^{u-at}\frac{\partial}{\partial u}, \quad \vec{e}_2 = e^{v-at}\frac{\partial}{\partial v}, \quad \vec{e}_3 = e^{w-at}\frac{\partial}{\partial w}, \quad \vec{e}_4 = \frac{\partial}{\partial t},$$

where $a \neq 0$. Let the metric $g^\star$ on $(\Omega^4, g^\star)$ is defined as

$$g^\star_{ij} = g^\star(\vec{e}_i, \vec{e}_j) = \begin{cases} 0, & i \neq j \\ -1, & i = j = 4 \\ 1, & \text{elsewhere.} \end{cases}.$$

Let $\eta^\sharp$ be the 1-form coupled to the metric as

$$\eta^\sharp(\mathscr{F}) = g^\star(\mathscr{F}, \vec{e}_4)$$

for any $\mathscr{F} \in \Gamma(\mathscr{T}\Omega)$. The $(1,1)$-tensor field ϕ is defined by

$$\phi(\vec{e}_1) = \vec{e}_1, \qquad \phi(\vec{e}_2) = \vec{e}_2, \qquad \phi(\vec{e}_3) = \vec{e}_3, \qquad \phi(\vec{e}_4) = 0,$$

Using the linearity properties of ϕ and g, we yield

$$\eta^\sharp(\vec{e}_4) = -1, \;\; \phi^2(\mathscr{F}) = \mathscr{F} + \eta^\sharp(\mathscr{F})\vec{e}_4, \;\; g^\star(\phi\mathscr{F}, \phi\mathscr{G}) = g^\star(\mathscr{F}, \mathscr{G}) + \eta^\sharp(\mathscr{F})\eta^\sharp(\mathscr{G}),$$

for any $\mathscr{F}, \mathscr{G} \in \Gamma(\mathscr{T}\Omega)$.
Thus for $\vec{e}_4 = \xi$, the structure $(\phi, \xi, \eta^\sharp, g^\star)$ leads to the (LPCM)$_4$ of dimension 4 (or four dimensional spacetime.) Now, the existing components of the Lie bracket are as

$$[\vec{e}_1, \vec{e}_4] = a\vec{e}_1, \;\; [\vec{e}_2, \vec{e}_4] = a\vec{e}_2, \;\; [\vec{e}_3, \vec{e}_4] = a\vec{e}_3.$$

Also for $\vec{e}_4 = \xi$, the Koszul's formula gives

$$\begin{array}{llll} \nabla_{\vec{e}_1}\vec{e}_1 = a\vec{e}_4, & \nabla_{\vec{e}_1}\vec{e}_2 = 0, & \nabla_{\vec{e}_1}\vec{e}_3 = 0, & \nabla_{\vec{e}_1}\vec{e}_4 = a\vec{e}_1, \\ \nabla_{\vec{e}_2}\vec{e}_1 = 0, & \nabla_{\vec{e}_2}\vec{e}_2 = a\vec{e}_4, & \nabla_{\vec{e}_2}\vec{e}_3 = 0, & \nabla_{\vec{e}_2}\vec{e}_4 = a\vec{e}_2, \\ \nabla_{\vec{e}_3}\vec{e}_1 = 0, & \nabla_{\vec{e}_3}\vec{e}_2 = 0, & \nabla_{\vec{e}_3}\vec{e}_3 = a\vec{e}_4, & \nabla_{\vec{e}_3}\vec{e}_4 = a\vec{e}_3, \\ \nabla_{\vec{e}_4}\vec{e}_1 = 0, & \nabla_{\vec{e}_4}\vec{e}_2 = 0, & \nabla_{\vec{e}_4}\vec{e}_3 = 0, & \nabla_{\vec{e}_4}\vec{e}_4 = 0. \end{array}$$

If $\mathscr{F} \in \chi(\Omega^4)$, then we write it as $\mathscr{F}$=$a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 + a_4\vec{e}_4$, where $a_i \in \Re$, $i = 1,2,3,4$. Then one can notice that $\nabla_{\mathscr{F}}\vec{e}_4$=$a\{\mathscr{F} + \eta^\sharp(\mathscr{F})\vec{e}_4\}$ holds for each $\mathscr{F} \in \chi(\Omega^4)$. So the (LPCM)$_4$ is an $\mathscr{LP}$-Sasakian manifold of dimension 4 with $a \neq 0$.
Using above relations, the existing components of the curvature tensor

$$\begin{array}{lll} \mathscr{R}(\vec{e}_1,\vec{e}_2)\vec{e}_1 = -a^2\vec{e}_2, & \mathscr{R}(\vec{e}_1,\vec{e}_3)\vec{e}_1 = -a^2\vec{e}_3, & \mathscr{R}(\vec{e}_1,\vec{e}_4)\vec{e}_1 = -a^2\vec{e}_4, \\ \mathscr{R}(\vec{e}_1,\vec{e}_2)\vec{e}_2 = a^2\vec{e}_1, & \mathscr{R}(\vec{e}_2,\vec{e}_3)\vec{e}_2 = -a^2\vec{e}_3, & \mathscr{R}(\vec{e}_2,\vec{e}_4)\vec{e}_2 = -a^2\vec{e}_4, \\ \mathscr{R}(\vec{e}_1,\vec{e}_3)\vec{e}_3 = a^2\vec{e}_1, & \mathscr{R}(\vec{e}_2,\vec{e}_3)\vec{e}_3 = a^2\vec{e}_2, & \mathscr{R}(\vec{e}_3,\vec{e}_4)\vec{e}_3 = -a^2\vec{e}_4, \\ \mathscr{R}(\vec{e}_1,\vec{e}_4)\vec{e}_4 = -a^2\vec{e}_1, & \mathscr{R}(\vec{e}_2,\vec{e}_4)\vec{e}_4 = -a^2\vec{e}_2, & \mathscr{R}(\vec{e}_3,\vec{e}_4)\vec{e}_4 = -a^2\vec{e}_3. \end{array}$$

Also, $\mathscr{S}(\mathscr{F},\mathscr{G}) = \sum_{i=1}^{4} \varepsilon_i g^\star(\mathscr{R}(\vec{e}_i,\mathscr{F})\mathscr{G},\vec{e}_i)$, where $\varepsilon_i = g^\star(\vec{e}_i,\vec{e}_i)$, $i,j = 1,2,3,4$. So we have

$$\mathscr{S}(\vec{e}_i,\vec{e}_j) = \begin{bmatrix} 3a^2 & 0 & 0 & 0 \\ 0 & 3a^2 & 0 & 0 \\ 0 & 0 & 3a^2 & 0 \\ 0 & 0 & 0 & -3a^2 \end{bmatrix}.$$

Also from (3.2), we yield

$$\mathscr{S}(\vec{e}_i,\vec{e}_i) = \left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p}+\frac{1}{2}\right) - \rho^\star z^*\right\}\right] g^\star(\vec{e}_i,\vec{e}_i) - \frac{1}{\rho^\star}(\alpha+\psi^\flat)\eta^\sharp(\vec{e}_i)\eta^\sharp(\vec{e}_i), \tag{10.1}$$

for all $i \in \{1,2,3,4.\}$. Hence we acquire

$$\mathscr{S}(\vec{e}_i,\vec{e}_i) = \left[-\frac{(\alpha+\psi\rho^\star)}{\rho^\star} - \frac{1}{2\rho^\star}\left\{2\mu^\star - \left(\widetilde{p}+\frac{1}{2}\right) - \rho^\star z^*\right\}\right] g^\star(\vec{e}_i,\vec{e}_i) - \frac{1}{\rho^\star}(\alpha+\psi^\flat)\delta_{i4}, \tag{10.2}$$

for all $i \in \{1,2,3,4.\}$. Thus from above, we can easily get

$$\mu^\star = (\alpha^2+\alpha+\psi)\rho^\star - \frac{1}{2}\left(\widetilde{p}-\frac{1}{2}\right) - \frac{\rho^\star z^*}{2},\ \psi^\flat = (3\alpha^2+\psi)\rho^\star + \mu^\star - \frac{1}{2}\left(\widetilde{p}+\frac{1}{2}\right) - \frac{\rho^\star z^*}{2}. \tag{10.3}$$

So the data $(g^\star,\xi,\mu^\star,\psi^\flat)$ is a GZ(CERYS)$_4$ on $(\Omega^4,g^\star)$, which is expanding if $(\widetilde{p}+\frac{1}{2}) > 2(\alpha+\alpha^2+\psi)\rho^\star - \rho^\star z^\star$, shrinking if $(\widetilde{p}+\frac{1}{2}) < 2(\alpha+\alpha^2+\psi)\rho^\star - \rho^\star z^\star$, or steady if $(\widetilde{p}+\frac{1}{2}) = 2(\alpha+\alpha^2+\psi)\rho^\star - \rho^\star z^\star$. Thus the Theorem 5.1 is verified.

Also, $\tau = \sum_{i=1}^4 S(e_i,e_j) = 6a^2$, which means that $(\Omega^4,g^\star)$ possesses a constant scalar curvature and hence the Corollary 4.4 is verified.

So, we conclude that
(i) If $\rho^\star$=1 and $\rho^\star$=0, then $(\Omega^4,g^\star)$ admits Ricci flow, which is expanding if $(\widetilde{p}-\frac{1}{2}) > 2(\alpha+\alpha^2+\psi)$, shrinking if $(\widetilde{p}-\frac{1}{2}) < 2(\alpha+\alpha^2+\psi)$ or, steady if $(\widetilde{p}-\frac{1}{2}) = 2(\alpha+\alpha^2+\psi)$.

(ii) For $\rho^\star$=0 and $\rho^\star$=1, then $(\Omega^4,g^\star)$ admits Yamabe flow, which is expanding if $\widetilde{p} < \frac{1}{2}$, shrinking if $\widetilde{p} > \frac{1}{2}$ or, steady if $\widetilde{p} = \frac{1}{2}$.
(iii) If $\rho^\star = 1$ and $\rho^\star = -1$, then $(\Omega^4,g^\star)$ admits Einstein flow, which is expanding if $(\widetilde{p}-\frac{1}{2}) > 2(\alpha+\alpha^2+\psi)+z^\star$, shrinking if $(\widetilde{p}-\frac{1}{2}) < 2(\alpha+\alpha^2+\psi)+z^\star$ or, steady if $(\widetilde{p}-\frac{1}{2}) = 2(\alpha+\alpha^2+\psi)+z^\star$.

11. Conclusions

A symmetric $\mathscr{Z}$-tensor in pseudo-Riemannian manifolds and spacetimes explores their geometric and general behavior. In addition to improving our understanding of geometric structures with finite symmetries, this study of such manifolds has applications in physics and other disciplines. For instance, in [18] defined $\mathscr{Z}$-tensor and studied its applications in physics. Thereafter many authors study various properties of these tensors [19, 20, 21]. Inspired by these works we light up some geometric and physical aspect of perfect fluid $\mathscr{LP}$-Sasakian spacetimes whose metrics are the (CERY)$_4$- soliton admitting the $\mathscr{Z}$-tensor.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

References

[1] A. H. Alkhaldi, M. D. Siddiqi, M. A. Khan, and L. S. Alqahtani Imperfect Fluid Generalized Robertson Walker Spacetime Admitting Ricci-Yamabe Metric, *Advances in Mathematical Physics* Volume 2021, Article ID 2485804, 10 pages
[2] A. Barman, and Inan Unal, Geometry of Kenmotsu manifolds admitting $\mathscr{Z}$-tensor, *Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science* (2022), 23-40.
[3] D. E. Blair, Riemannian geometry of contact and symmetric manifolds, Birhauser, Bston. 2010.
[4] A. M. Blaga, Harmonic aspects in an η-Ricci soliton, *Int. Electron. J. Geom.*, 13 (2020), 41-49.
[5] G. Catino, L. Mazzieri, Gradient Einstein solitons, *Nonlinear Anal.*, 132 (2016), 66-94.
[6] M. C. Chaki, S. Roy, Spacetime with covariant constant energy momentum tensor, *Int. J. Theor. Physics*, 35 (1996), 1027-1032.
[7] B. Chow, S. C. Chu and D. Glickenstein et al., The Ricci flow:techniques and applications,*in Part I: Geometric Aspects* 135, AMS, 2007.
[8] U. C. De, A. A. Shaikh, and A. Sengupta, On LP-Sasakian manifolds with a coefficient α., *Kyungpook Math. J.*, 42 (2002), 177-186.
[9] K. De, U. C. De, A. A. Syied, N. B. Turki and S. Alsaeed, Perfect fluid spacetimes and Gradient Solitons, *J. Nonlin. Math. Phys.*, 29, 4 (2022), 843-858.
[10] A. E. Fischer, An introduction to conformal Ricci flow, *Classical and Quantum Gravity.*, 21, (2004) 171-218.
[11] S. G*ü*ler, M. Crasmareanu, Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy, *Turk. J. Math.*, 43 (2019), 261-2641.
[12] R. S. Hamilton, Three Manifold with positive Ricci curvature, *J. Differ. Geom.* 17 (1982), 255-306.
[13] R. S. Hamilton, The Ricci flow on surfaces, *Contemp. Math.*, 71 (1988), 237-261.
[14] Y. Li, A. Haseeb and M. Ali, LP-Kenmotsu Manifolds Admitting |*eta*-Ricci Solitons and Spacetime, *Journal of Mathematics*, (2022), Article ID 6605127, 10 pages.
[15] V. R. Kaigorodov, The curvature structure of spacetime, *Prob. Geom.* 14 (1983), 177-202.
[16] H. Kachar, Infinitesimal characterization of Friedmann Universes,*Arch. Math. Basel.*, 38 (1982), 58-64.
[17] C. A. Mantica, U. C. De, Y. J. Suh, and L.G. Molinari,*Perfect fluid spacetimes with harmonic generalized curvature tensor*, Osaka J. Math. 56, (2019), 173-182.
[18] C. A. Mantica L. G. and Molinari, Weakly Z symmetric manifolds,*Acta Math. Hungar.*, 135 (2012), 80-96.
[19] C. A. Mantica and Y. J. Suh, Pseudo $\mathscr{Z}$ symmetric Riemannian manifolds with harmonic curvature tensors, *Int. J. Geom. Methods Mod. Phys.*, 9 1 (2012), 1250004.
[20] C. A. Mantica and Y. J. Suh,, Pseudo $\mathscr{Z}$ symmetric spacetimes, *J. Math. Phys*, 55 (2014), 042502.

[21] C. A. Mantica and Y. J. Suh, Recurrent $\mathscr{Z}$ forms on Riemannian and Kaehler manifolds, *Int. J. Geom. Methods Mod. Phys*., 9 (2012), 1250059.
[22] K. Matsumoto, On Lorentzian paracontact manifolds, *Bull. of Yamagata Univ. Nat. Sci.*, 12 (1989), 151-156.
[23] B. O'Neill, Semi-Riemannian geometry with application to relativity, Pure and applied mathematics, Academic press, New York. 1983
[24] A. G. Popov, *Exact formula for constructing solutions of the Liouville equation* $\triangle_2 u = e^v$ *from solutions of the Laplace equation* $\triangle_2 v=0$. (Russian). Dokl. Akad. Nauk. 1993, 333, 440-441. Translation in Russian Acad. Sci. Dokl. Math. 1994, 2, 55-71.
[25] A. Z. Petro, *Einstein space*, Pergamon Press, Oxford, (1967).
[26] A. K. Raychaudhary, S. Banerji and A. Banerjee , General relatively, Astrophysics and cosmology, Springer-Verlag. 1942.
[27] R. N. Singh, S. K. Pandey and G. Pandey, Second Order Parallel Tensors on LP-Sasakian Manifolds, *Journal of International Academy of Physical Sciences*, 13, (2009), 383-388.
[28] S. Sarkara, S. Dey and X. Chen, Certain Results of Conformal and $*$-Conformal Ricci Soliton on Para-Cosymplectic and Para-Kenmotsu Manifolds, *Filomat*, 35, 15 (2021), 5001-5015.
[29] Y. J. Suh and S. K. Chaubey, Ricci solitons on general relativistic spacetime, *Physica Scripta*, 98 6,(2023).
[30] M. D. Siddiqi, F. Mofarreh, M. A. Akyol and A. H. Hakami, η-Ricci-Yamabe Solitons along Riemannian Submersions, *Axioms*, 12 (2023), 796.
[31] A. Y. Tasci and F. O. Zengin, On $\mathscr{Z}$-symmetric manifold with conharmonic curvature tensor in special conditions, *Kragujevac Journal of Mathematics*, 49, 1 (2025), 65-80.
[32] K. Yano, M. Kon, Structures on manifolds, Series in Pure Math., World Scientific. 1984.
[33] K. Yano, Integral formulas in Riemannian geometry, Pure and Applied Mathematical, Marcel Dekkr, Inc., New York. 1970.
[34] S. K. Yadav, A note on space time in $f(R)$-gravity, *Annals of communications in Mathematics*, 6, 2 (2023), 99-108.
[35] S. K. Yadav, D. L.Suthar and S. K. Chaubey, Almost conformal Ricci soliton on generalized Robertson walker-space time, *Research in Mathematics*, 10, 1 (2023), 1-10.
[36] S. K. Yadav, P. K. Dwivedi and D. L. Suthar, On $(LCS)_n$-manifolds satisfying certain conditions on the concircular curvature tensor,*Thiland Journal of Mathematics*, l, 9 (2011), 597-603.
[37] S. K. Yadav, A. Haseeb and A. Yildiz, Conformal η-Ricci-Yamabe solitons on submanifolds of an $(LCS)_n$-manifold admitting a quarter-symmetric metric connection, *Commun. Fac. Sci. Univ. Ank.Ser. A1 Math. Stat*, 73, 3, (2024),1-19
[38] S. K. Yadav and D. L. Suthar, Kaehlerian Norden spacetime admitting conformal η-Ricci-Yamabe Metric, *Int. J. of Geometric Methods in Modern Physics*, https://doi.org/10.1142/S0219887824502347.
[39] S. K. Yadav and X. Chen, $\mathscr{Z}$-tensor in mixed generalized quasi-Einstein GRW space-time, *Sohag Journal of mathematics*, 11, 1, (2024), 1-9.
[40] S. K. Yadav, Sumesh Senway, N. B. Turki and Rajendra Prasad, The $\mathscr{Z}$-Tensor on Almost Co-Kählerian Manifolds Admitting Riemann Soliton Structure, *Advance in Mathematical Physics*, Volume 2024, Article ID 7445240, 14 pages, https://doi.org/10.1155/2024/7445240
[41] P. Zhang, Y. Li,S. Roy, S. Dey and A. Bhattacharyya , Geometrical Structure in a Perfect Fluid Spacetime with Conformal Ricci-Yamabe Soliton, *Symmetry*, 14, 3 (2022), 594.