Proceedings of International Mathematical Sciences
ISSN: 2717-6355, URL: https://dergipark.org.tr/tr/pub/pims
Volume 6 Issue 2 (2024), Pages 100-108.
Doi: https://doi.org/10.47086/pims.1567406

MODIFIED PELL MATRIX TECHNIQUE FOR SOLVING OPTIMAL CONTROL PROBLEMS

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Abstract. The orthogonal polynomial basis functions are used to solve different mathematical problems, especially for optimal control and many other engineering problems, which attract many researchers to work on. In this study, the modified Pell polynomials (MPPs) are presented and their new properties are investigated to be used for solution approximation of optimal control problems. Some formulas for MPPs are derived by matrices. A new exact formula expressing the derivatives of MPPs explicitly of any degree is constructed. The main advantage of the presented formulas is that the new properties of MPPs greatly simplify the original problems and the result will lead to easy calculation of the coefficients of expansion, it also increases the accuracy and reduces the computational time. A new computational method along with the MPPs is proposed to solve one of the optimal control problems. Numerical results are included to demonstrate the validity of this new technique. It shows an important improvement in error approximation when the polynomial degree is increased. The contribution in this work is based on the idea of the approximate algorithm in terms of MPPs and their new properties to treat the optimal control problem numerically with less number of terms and unknown parameters with a satisfactory accuracy.

1. Introduction

Recently, there has been increased interest among scientists and engineers to use orthogonal polynomial basis functions along with approximate solutions to solve different problems. The main advantage of such a technique is the ability to reduce a complicated problem to another simple one [1, 2, 3, 4]. The authors of [5] extended using orthogonal polynomials to solve problems in the calculus of variation numerically. They applied the generalized Vieta- Pell polynomials for numerical treatment of variation calculus problems while improved Chebyshev polynomials were applied in [6] for solving optimal control

2020 *Mathematics Subject Classification.* Primary: 49M05; 33F05.
Key words and phrases. Modified Pell polynomials; orthogonal polynomial; optimal control problem; oscillatory systems; numerical approximation.

Submitted on 15.10.2024, Accepted on 26.11.2024.
Communicated by Mehmet Dik.

problems. Furthermore; Boubaker polynomials [7, 8] and Hermite functions [9] were utilized to solve approximately optimal control problems and fractional calculus of variation respectively.

Stabilizing nonlinear chaotic and dynamical systems represents the main core for controlling such systems [10, 11, 12, 13, 14]. The modified Pell polynomial particularly can be used to perform such stabilization. The utilization of this modified polynomial by offering good computational efficiency and high accuracy follows the same scenario in solving the control problem.

The motivation in this work deals with the new application of a modified Pell polynomial for numerical solutions of an optimal control problem. The study of numerical solutions of special classes in optimal control has provided an interesting field for mathematical sciences researchers. For some work concerned with both Vieta-Pell and Vieta Pell Lucas polynomial basis functions can be found in [15, 16].

Motivated by the above presentation, we are interested in suggesting a new iterative algorithm to solve optimal control problems numerically together with modified Pell basis functions to parameterize the state variables. We aim to obtain the accuracy and efficiency simultaneously. Hence the first goal of this work is to introduce MPP with some important properties and then use it to perform the parameterization of state variables in order to solve some problems in optimal control.The work in this article is organized as follows: modified Pell polynomials definition is presented first in section 2. Their new properties are also included in section 2. The new contribution of the modified Pell polynomials is discussed in section 3 to solve a special application in optimal control problems. The conclusion is listed in section 4.

2. Definition of Modified Pell Polynomials

The following modified Pell basis polynomials are obtained
The modified Pell basis polynomials of degree n are defined by:

$$q_n(x) = \frac{n}{2}\sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(n-r-1)!}{(n-2r)!r!}(2x)^{n-2r} \tag{1.1}$$

with special values
$q_n(1) = q_n(-1)$ for even n and $q_n(1) = -q_n(-1)$ for odd n
Define $\Psi(x) = \begin{bmatrix} q_0(x) & q_1(x) & \dots & q_n(x) \end{bmatrix}^T$, where one can get:

$$\Psi(x) = AT_n(x). \tag{1.2}$$

The matrix A is a lower triangular matrix of order $(n+1)\times(n+1)$. The element of A can be listed as below:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 2 & \cdots & 0 \\ 0 & 3 & 0 & 4 & 0 \\ 1 & 0 & 8 & 0 & 8 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{1.3}$$

and

$$a_{ij} = \begin{cases} 1, & \text{if } j = 1, \\ 2a_{i-1,j-1} + a_{i-2,j,}, & \text{if } j \geq i, \text{i is odd}, i, j = 2, 3, \cdots \\ 0, & j > i. \end{cases} \tag{1.4}$$

and the vector $T_n(x)$ of dimension $(n+1) \times 1$ is defined by:

$$T_n(\mathrm{x}) = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix}^T \tag{1.5}$$

2.1 Function Approximation. A function $u(x)$ as is square integrable in $[-1, 1]$, It can be expanded in terms of modified Pell basis as below:

$$u(x) = \sum_{i=0}^{n} a_i q_i(x) = a^T q(x) \tag{1.6}$$

where
$a^T = \begin{bmatrix} a_0 & a_1 & \dots & a_n \end{bmatrix}$, then $a = Q^{-1}\langle u(x), q(x)\rangle$, then the matrix Q of order $n \times n$ is called a dual matrix which is derived later.

2.2 Some properties of Modified Pell Sequence: Theorem 1. For $n \geq 1$, the following formulas can be obtained
(i) $\sum_{i=1}^{n} q_i(x) = \frac{1}{2x}\left[q_{n+1}(x) + q_n(x) - (q_1(x) + q_0(x))\right]$,
(ii) $\sum_{i=1}^{n} q_{2i}(x) = \frac{1}{2x}\left(q_{2n+1}(x) + q_1(x)\right)$,
(iii) $\sum_{i=1}^{n} q_{2i+1}(x) = \frac{1}{2x}\left(q_{2n}(x) + q_0(x)\right)$.

Proof. The mathematical induction is suggested to prove (i)
Take $n = 1$, then $q_1(x) = x = 1/2x\left[2x^2 + 1 + x - x - 1\right]$.
Let the result in (i) be valid for $n = k$, then

$$\sum_{i=1}^{k} q_i(x) = \frac{1}{2x}\left[q_{k+1}(x) + q_k(x) - (q_1(x) + q_0(x))\right]$$

$$\text{or } 2\sum_{i=1}^{k+1} q_i(x) = q_{k+1}(x) + q_k(x) - (q_1(x) + q_0(x)) + 2xq_{k+1}(x)$$

Apply the recursive relation related with modified Pell polynomials to obtain
$q_{n+1}(x) = 2xq_n(x) + q_{n-1}(x)$,
Therefore; $2\sum_{i=1}^{k+1} q_i(x) = q_{k+2}(x) + q_{k+1}(x) - (q_1(x) + q_0(x))$
or $\sum_{i=1}^{k+1} q_i(x) = \frac{1}{2x}\left[q_{k+2}(x) + q_{k+1}(x) - (q_1(x) + q_0(x))\right]$
This is the same result in Eq.(1.6).
Note that the identities (ii) and (iii) are a direct result of i.

2.3 Dual Operational Matrix of Modified Pell Polynomials. This section illustrates the building of a modified Pell dual operational matrix. The cross-product integration of two modified Pell basis vectors is taken as below

$$\begin{aligned} Q &= \int_{-1}^{1} \left\langle AT_n(x), (AT_n(x))^T \right\rangle dx \\ &= A\int_{-1}^{1} \left(T_n(x)T_n(x)^T\right) dx = AHA^T \end{aligned}$$

The matrix A is defined in Eq.(1.3) and the matrix H is given for the particular case with $n = 3$ as below

$$H = \begin{pmatrix} \int_{-1}^{1} q_0(x)q_0(x)dx & \int_{-1}^{1} q_0(x)q_1(x)dx & \int_{-1}^{1} q_0(x)q_2(x)dx \\ \int_{-1}^{1} q_1(x)q_0(x)dx & \int_{-1}^{1} q_1(x)q_1(x)dx & \int_{-1}^{1} q_1(x)q_2(x)dx \\ \int_{-1}^{1} q_2(x)q_0(x)dx & \int_{-1}^{1} q_2(x)q_1(x)dx & \int_{-1}^{1} q_2(x)q_2(x)dx \end{pmatrix}$$

Hence

$$H = \begin{pmatrix} 2 & 0 & \frac{10}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{10}{3} & 0 & \frac{94}{15} \end{pmatrix}$$

In general, the element of the constant matrix h_{ij} can be determined as follows

$$H_{ij} = \begin{cases} \sum_{r=0}^{\lfloor j/2 \rfloor} 2^{j-2r-1} \frac{(j-r-1)!}{(j-2r+1)!r!}, & \text{if } |i-j| \text{is even} \\ 0, & \text{otherwise.} \end{cases}$$

2.4 Derivative for Modified Pell Polynomials Matrix: consider the vector $\Psi(x)$ in Eq.(1.2) can be written in matrix form as follows

$\Psi(x) = AT(x)$

One can get

$\dot{\Psi}(x) = A\dot{T}_n(x)$

where $\dot{T}_n(x) = \begin{bmatrix} 0 & 1 & 2x & \dots & nx^{n-1} \end{bmatrix}^T$, this equation can be reformulated as:

$\dot{T}_n(x) = A_1 T_n(x)$

Then

$\dot{\Psi}(x) = AA_1 T_n(x)$,

where: $A_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \end{pmatrix}$.

and A is given in Eq.(1.3).

2.5 Operational Matrix of Integration for Modified Pell Polynomials. Let M be an operational matrix of integration of order $(n+1) \times (n+1)$, then

$$\int_{-1}^{x} q(t)dt = MX$$

where

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{5}{3} & 1 & 0 & \frac{2}{3} & 0 & 0 \\ \frac{-10}{4} & 0 & \frac{3}{2} & 0 & 1 & 0 \\ 79 & 1 & 0 & \frac{8}{3} & 0 & \frac{8}{5} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and $X = \begin{bmatrix} x & x^2 & \dots & x^{n+1} \end{bmatrix}^T$.

This can be evaluated below

$$m_{ij} = i\begin{cases} \sum_{r=0}^{[i/2]} \frac{(i-r-1)!2^{i-2r-1}}{r!(i-2r)!(i-2r+1)}, & j=1, \text{ even } i \\ \sum_{r=0}^{ii/2]} \frac{(i-r-1)!2^{i-2r-1}}{r!(i-2r)!(i-2r+1)}, & j=1, \text{ odd } i \\ \frac{1}{i}\left(m_{i-1,j} + 2m_{i-1,j-1}\right), & j>1 \end{cases}$$

2.6. Initial and Final Values. Lemma 1. Let $\sigma_1 = n\sum_{r=0}^{\lfloor n/2\rfloor} \frac{1}{n-r}\binom{n-r}{r}2^{n-2r-1}$,
Then

$$q_n(1) = \sigma_1, \text{ and } q_n(-1) = \begin{cases} \sigma_1, & \text{if } n \text{ even} \\ -\sigma_1, & \text{if } n \text{ odd} \end{cases}$$

Lemma 2. We have $q_{n+1}(-1)\left(q_n(1) - q_n(-1)\right) \neq 0$,
If n is even, then
$q_{2n+1}(-1)q_{2n}(1) - q_{2n+1}(1)q_{2n}(-1) = \sigma_1 \cdot (-\sigma_1) \cdot \sigma_1 = -2\sigma_1{}^2 \neq 0.$
If n is odd, then

$$q_{2(n+1)}(-1)q_{2n+1}(1) - q_{2(n+1)}(1)q_{2n+1}(-1) = \sigma_1 \cdot \sigma_1 - (-\sigma_1 \cdot \sigma_1) = 2\sigma_1^2 \neq 0$$

3. Application of Modified Pell Polynomial in optimal control

Consider the following optimal control problem which describes many important oscillating phenomena in some dynamic [12, 13, 14, 15, 16], engineering and physical systems $J = \int_{-\tau}^{0} \frac{1}{2}u^2(t)dt$, governed by the following system $\ddot{x}(t) + \sigma\dot{x}(t) + \omega^2 x(t) + \in x^3(t) = f\cos(\alpha t) - u(t), -\tau \leq t \leq 0$,
where τ is known, $\sigma \geq 0$, is the viscous damping coefficient, f and α are the amplitude ω is the stiffness parameter, and frequency of the external input, respectively.

The initial and boundary conditions can be given as follows:

$$x(-\tau) = \alpha, x(0) = 0, \dot{x}(-\tau) = \beta, \dot{x}(0) = 0$$

To obtain the optimal performance index $J(\cdot)$, the following steps are suggested:
In order to use Modified Pell polynomials, the transformation $\tau = \frac{1}{2}\tau(t-1)$ is used to obtain the following restated optimal control problem

$$J = \frac{\tau}{2}\int_{-1}^{1} \frac{1}{2}u^2(t)dt \tag{1.7}$$

governed by the following system

$$\ddot{x}(t) = \frac{1}{2}\tau^2\left[-\sigma\dot{x}(t) - \omega^2 x(t) - \in x^3(t) + f\cos(\alpha t) - u(t)\right], t \in [-1, 1] \tag{1.8}$$

The initial and boundary conditions can be given as follows:

$$x(-1) = \alpha, x(1) = 0, \dot{x}(-1) = \beta, \dot{x}(1) = 0 \tag{1.9}$$

To illustrate the present method for obtaining the optimal performance index $J(\cdot)$, the following steps are suggested:

Step 1. Assume that the approximate solution of the state variables $x(t)$ in terms of MPPs that satisfy the conditions given in Eq.(1.9) below:
$x^1(t) = a_0q_0(t) + a_1q_1(t) + a_2q_2(t) + a_3q_3(t) + a_4q_4(t)$
It is worth mentioning that the function $x^1(t)$ is chosen to satisfy the conditions in Eq.(1.9).
Therefore, the following equations are obtained:
$a_0 + a_1 + 3a_2 + 7a_3 + 17a_4 = 0$

$a_0 - a_1 + 3a_2 - 7a_3 + 17a_4 = \alpha$
$a_1 + 4a_2 + 15a_3 + 48a_4 = 0$
$a_1 - 4a_2 + 15a_3 - 48a_4 = \beta$
Step 2. Eliminate the unknown a_0, a_1, a_2 and a_3 to get
$a_0 = \frac{1}{2}\left[\left(\alpha + \frac{3}{4}\beta\right) - 33.5a_4\right], a_1 = \frac{1}{2}(-2\alpha - \beta), a_2 = \frac{1}{8}(-\beta - 96a_4),$
$a_3 = \frac{1}{14}(\alpha + \beta)$

As a result $x^1(t) = \frac{1}{2}\left[\left(\alpha + \frac{3}{4}\beta\right) - 33.5a_4\right]q_0(t) + \frac{1}{2}(-2\alpha - \beta)q_1(t)$

$$+\frac{1}{8}(-\beta - 96a_4)\,q_2(t) + \frac{1}{14}(\alpha + \beta)q_3(t) + a_4q_4(t)$$

and then obtain the first approximation to $u(t)$ using Eq.(1.8)

$$u^1(t) = f\left(x^1(t), \ddot{x}^1(t)\right)$$

Step 3. Obtain J as a function of the unknown a_4 by determining

$$J^1(a_4) = \int_{-1}^{1} F(u_1(t))\,dt$$

Step4. Minimize $J_1(a^*)$ is the solution to the problem in Eq's.(1.7-1.9).
Step 5. Calculate $x^1(t)$ and $u^1(t)$ from a^* approximately. The procedure is repeated until an acceptable accuracy is obtained.

Note that the approximate solution in the n step is given by
$x_n(t) = x_{n-1}(t) + \sum_{i=n-1}^{n+1} a_iq_i(t),$
with

$$a_{n-1} = \frac{q_{n+1}(-1)q_{n+2}(1) - q_{n+1}(1)q_{n+2}(-1)}{q_n(-1)q_{n+1}(1) - q_n(1)q_{n+1}(-1)}a_{n+1}$$
$$a_n = \frac{q_n(-1)q_{n+2}(1) - q_n(1)q_{n+2}(-1)}{q_n(1)q_{n+1}(-1) - q_n(-1)q_{n+1}(1)}a_{n+1}$$

The optimal control problem in Eq's.(1.7-1.9) is solved with the following choice of numerical values of parameters in a certain standard case: $\omega = 1, \sigma = 1, \in = 1, f = 0, \tau = 2, \alpha = 0.5, \beta = -0.5$. Figures 1-5 illustrate the values of the state and the control for different values of n while the relative absolute errors are plotted in Figure 6.

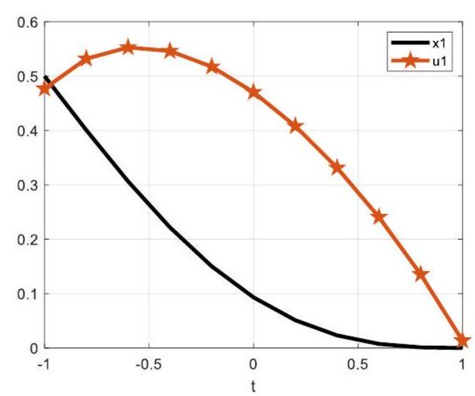

FIGURE 1. The approximate $x(t)$ and $u(t)$ with $n = 5$.

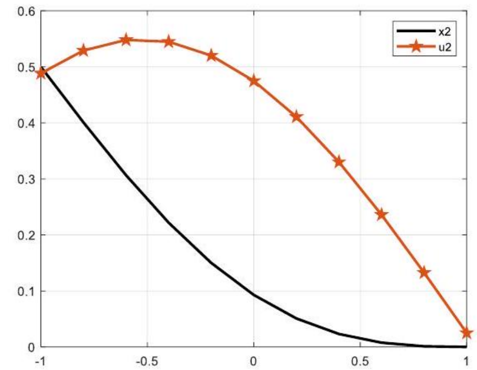

FIGURE 2. The approximate state $x(t)$ and $u(t)$ with $n = 6$.

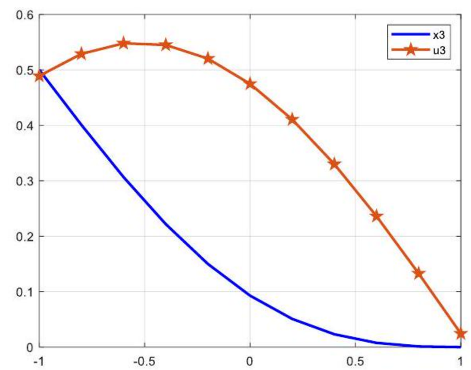

FIGURE 3. The approximate $x(t)$ and $u(t)$ with $n = 7$.

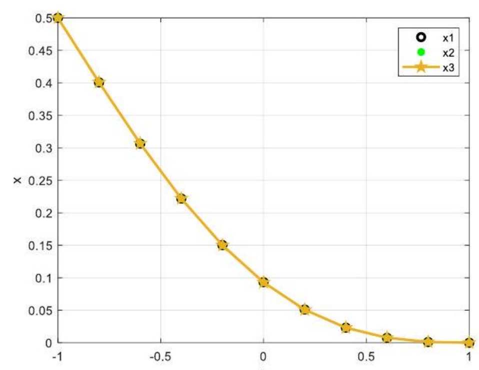

FIGURE 4. The approximate state $x(t)$ with different n.

The relative errors of the optimal cost functional J for $n = 5, 6, 7$ are respectively 00008106925, 0.00000017319, 0.00000001325.

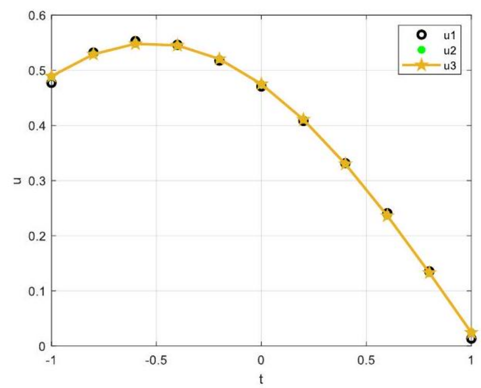

FIGURE 5. The approximate control $u(t)$

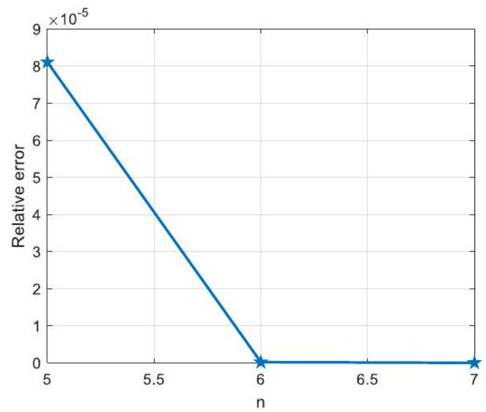

FIGURE 6. The Relative Errors of J with different n.

4. CONCLUSION

This paper proposed the modified Pell polynomial method for the optimal control problem. The new modified parameterization technique has been investigated for the approximate solution based on MPPs, which was our one important highlight. Numerical results are provided to prove the effectiveness of the suggested method. The obtained results show that as the order of the modified Pell polynomial increases the error in the approximate solution will be decreased and exactly close to the exactly one with satisfactory decimal places. This is the main modification of the approach and this small contribution to the assumption of iterative method in terms of MPPs results when obtaining the approximate solution with the minimum number of MPPs terms and satisfactory accuracy.

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