Journal of New Results in Science 13 (3) (2024) 221-231

Journal of New Results in Science
https://dergipark.org.tr/en/pub/jnrs
Research Article
Open Access

E-ISSN: 1304-7981 https://doi.org/10.54187/jnrs.1569468

The basis property of generalized eigenfunctions for one boundary value problem with discontinuities at two interior points

Hayati Olğar[1], Oktay Mukhtarov[2]

Keywords
Spectral problem, Boundary conditions, Jump conditions, Eigenvalues, Weak eigenfunctions

Abstract — In this study, we consider a spectral problem for one boundary value problem with discontinuities at two interior points. The boundary conditions involve a spectral parameter. We consider some compact, positive, self-adjoint operators to reduce the spectral problem to an operator-pencil equation. Then, it was proven that this operator-pencil is positive definite, the spectrum is discrete, and the system of weak eigenfunctions forms a Riesz basis of the appropriate Sobolev space.

Subject Classification (2020): 34L10, 34L15

1. Introduction

It's well-known that the spectral method is used to solve many problems in natural science. Sturm-Liouville boundary value problems (SLBVPs) lie as a theoretical basis for the spectral theory of linear differential operators. The development of many areas of physics and technology and the need to find solutions to new BVPs in these areas increase the significance of the Sturm-Liouville theory (SLT) and its relevance. Due to the many SLTs in scientific research and practical engineering, this theory has developed in different directions and found new application areas for about two centuries.

Typically, classical SLBVPs included a spectral parameter only in the second-order linear differential equation. However, SLBVPs, which include a spectral parameter in both differential equations, and BCs appear in the modeling of many significant problems of applied mathematics, physics, chemistry, aerodynamics, fluid dynamics, diffusion, engineering, biotechnology finance, etc. [1–6]. An effective and efficient approach to the new type of SLBVPs with $\lambda-$ dependent BCs together with supplementary jump conditions at some interior points of interaction is provided by Mukhtarov and his colleagues; for example, see [7–19].

Meanwhile, the basis properties in various function spaces of the eigenfunction of the SLBVPs with λ-dependent BCs have been considered by many mathematicians [20–24]. Lately, there has been a significant increase in appeal to the polynomial pencils in Hilbert spaces, see [25–29]. The famous study of Keldysh [25] contains some fundamental results in the spectral theory of polynomial pencils. Thanks to the concept of weak eigenfunctions, a Sturm-Liouville eigenvalue problem is reduced to a polynomial

[1]hayatiolgar@gmail.com (Corresponding Author); [2]omukhtarov@yahoo.com
[1,2]Department of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, Tokat, Türkiye
Article History: Received: 17 Oct 2024 - Accepted: 24 Nov 2024 - Published: 31 Dec 2024

operator equation. In the study published in 1985, the concept of generalized eigenfunctions in Hilbert space was defined by Ladyzhenskaia [27]. A SL type BVP with an eigenvalue parameter in only one of the BCs was examined in [30]. In these studies, Belinskiy et al. showed that the weak eigenfunctions of this class of problems form a Riesz basis. In [22, 23], the authors presented a new method to prove that the generalized eigenfunctions form a Riesz basis for multi-interval SLBVP together with λ-dependent BCs and additional jump conditions at the interaction points.

In this study, we investigate the SLBVP consisting of the SL equation

$$-\frac{d}{dx}\left\{p(x)\frac{df}{dx}\right\}+q(x)f(x)=\lambda r(x)f(x) \tag{1.1}$$

on the interval $0\leq x\leq\pi$ with $\lambda-$ dependent BC at the end-point $x=0$ given by

$$p(x)\frac{d(\ln f)}{dx}|_{x=0}=\frac{-\lambda\cos\alpha}{\cos\alpha+\lambda\sin\alpha}, 0<\alpha<\pi \tag{1.2}$$

with two symmetric discontinuities at $x=c_1=c,$ with $0<c<\frac{\pi}{2},$ and $x=c_2=\pi-c$ satisfying the jump conditions

$$f(c_1^+)-af(c_1^-)=0 \tag{1.3}$$

$$\frac{df}{dx}|_{x=c_1^+}=a^{-1}\frac{df}{dx}|_{x=c_1^-}+bf(c_1^-) \tag{1.4}$$

$$f(c_2^-)-af(c_2^+)=0 \tag{1.5}$$

$$\frac{df}{dx}|_{x=c_2^-}=a^{-1}\frac{df}{dx}|_{x=c_2^+}-bf(c_2^+) \tag{1.6}$$

and BC at $x=\pi$

$$p(x)\frac{d(\ln f)}{dx}|_{x=\pi}=-\cot\beta 0<\beta<\pi \tag{1.7}$$

The coefficients of the SLBVP (1.1)-(1.7) under examination will be assumed to satisfy the following conditions:

C1. The functions $p(x), q(x)$ and $r(x)$ are bounded, positive definite and Lebesgue integrable on $(0, c_1^-), (c_1^+, c_2^-)$ and (c_2^+, π),

C2. $a, b\in\mathbb{R}$ and $a>0$ and $\theta=\cos^2\alpha>0$. The assumption is that $\theta=\cos^2\alpha>0$ is required for the problem to be self-adjoint [3], and therefore for all eigenvalues to be real and bounded below.

C3. λ is a complex eigenvalue parameter.

In this study, we investigate a new type of SLP consisting of many-interval SL equation, four jump conditions, and λ-dependent BCs. It is shown that the generalized eigenfunctions forms a Riesz basis for modified Lebesgue space.

2. Some Auxiliary Facts

This section defines some of the modified Lebesgue spaces and provide some inequalities needed for the examination of the noted SLBVP (1.1)-(1.7).

Definition 2.1. $L_2(a,b)$ is Hilbert space consisting of all Lebesgue measurable functions $f(x)$ on (a,b) for which

$$\int_a^b |f(x)|^2dx)^{\frac{1}{2}}<\infty$$

with the scalar product given by $\langle f,g\rangle_{L_2(a,b)} := \int_a^b f(x)\overline{g(x)}dx$ and the norm $\|f\|^2_{L_2(a,b)}=\langle f,f\rangle_{L_2(a,b)}$.

Let $\oplus L_2 := L_2(0,c_1)\oplus L_2(c_1,c_2)\oplus L_2(c_2,\pi)$ with the scalar product

$$\langle f\, g\rangle_{\oplus L_2} := \int_0^{c_1^-} f(x)\overline{g(x)}dx + \int_{c_1^+}^{c_2^-} f(x)\ \overline{g(x)}dx + \int_{c_2^+}^{\pi} f(x)\overline{g(x)}dx$$

and the corresponding norm

$$\|f\|^2_{\oplus L_2} = \langle f, f\rangle_{\oplus L_2}$$

Definition 2.2. The Sobolev space $W_2^1(a,b)$ is the Hilbert space consisting of all elements $f\in L_2(a,b)$ having generalized derivatives $f'\in L_2(a,b)$ with the scalar product given by

$$\langle f, g\rangle_{W_2^1(a,b)} = \int_a^b (f(x)\overline{g(x)} + f'(x)\overline{g'(x)})dx$$

and the corresponding norm

$$\|f\|^2_{W_2^1(a,b)} = \langle f, f\rangle_{W_2^1(a,b)}.$$

Introduce the appropriate inner-product space $\oplus\, W_2^1$ by

$$\oplus \overset{1}{\underset{2}{W}} = \left\{\ f\in\oplus\, L_2\,|f\in W_2^1(0,c_1)\oplus W_2^1(c_1,c_2)\oplus W_2^1(c_2,\pi), f(c_1^+) = af(c_1^-), f(c_2^-) = af(c_2^+)\ \right\}$$

with the inner-product

$$\langle f, g\rangle_{\oplus\, W_2^1} := \langle f, g\rangle_{\oplus L_2} + \langle f', g'\rangle_{\oplus L_2}$$

and corresponding norm

$$\|f\|^2_{\oplus\, W_2^1} = \langle f, f\rangle_{\oplus\, W_2^1}$$

In the same linear space $\oplus W_2^1$, we define a new inner-product as

$$\langle f, g\rangle_{\oplus W_{2,p,q}^1} := \langle qf, g\rangle_{\oplus L_2} + \langle pf', g'\rangle_{\oplus L_2} \tag{2.1}$$

with corresponding norm

$$\|f\|^2_{\oplus W_{2,p,q}^1} = \langle f, f\rangle_{\oplus W_{2,p,q}^1} \tag{2.2}$$

Lemma 2.3. There exist $0 < M_1 < M_2$, such that

$$M_1\|f\|_{\oplus W_2^1} \le \|f\|_{\oplus W_{2,p,q}^1} \le M_2\|f\|_{\oplus W_2^1}$$

for all $f\in\oplus W_2^1$.

By using the well-known embedding theorems [27], we get

$$|f(x_j)|^2 \le \ell||f'||^2_{\oplus L_2} + \frac{2}{\ell}||f||^2_{\oplus L_2}\ j\in\{1,2,3,4\} \tag{2.3}$$

$$|f(\xi)| \le C||f||_{\oplus W_2^1} \tag{2.4}$$

for any $f\in\oplus W_2^1$ where $x_1 = 0, x_2 = c_1^{\mp}, x_3 = c_2^{\mp}, x_4 = \pi$ and ℓ is any positive real number which is small enough and $\xi\in[0,\pi]$, $C>0$ is a constant.

According to the following result, the function $q(x)$ can be assumed to be positive.

Remark 2.4. Suppose that $\frac{q(x)}{r(x)}$ is bounded below, then the shift $\lambda\to\lambda-h$ transforms (1.1) into a new equation with $\widetilde{q}(x) = q(x)\ + hr(x)$. By taking $h > \sup_x \frac{-q(x)}{r(x)}$(where $x\in[0,c_1^-]\cup[c_1^+,c_2^-]\cup[c_2^+,\pi]$) the function $q(x)$ can be assumed to be positive on $[0,c_1^-]\cup[c_1^+,c_2^-]\cup[c_2^+,\pi]$.

Denote by $\mathbb{H}$ the Hilbert space consisting of all vector functions $(\varphi,\varphi_1)\in\oplus W_2^1\oplus\mathbb{C}$. The inner product of this space is defined by

$$\langle\Psi,\Phi\rangle_{\mathbb{H}} := \langle\varphi,\phi\rangle_{\oplus W_2^1} + \varphi_1\overline{\phi_1}$$

Here, $\Psi = (\varphi, \varphi_1) = (f(x), w_1)$ and $\Phi = (\phi, \phi_1) = (g(x), w_2) \in \mathbb{H}$ such that $f(x), g(x) \in \oplus W_2^1$ and $w_1, w_2 \in \mathbb{C}$.

Recall some definitions of basicity.

Definition 2.5. [26] A set $\{f_n\}$ in a Hilbert space $\mathbb{H}$ is called a basis if for an arbitrary function $f \in \mathbb{H}$ there is an unique expansion $f = \sum_{k=1}^{\infty} a_k f_k$.

If$\langle f_n, f_m\rangle_{\mathbb{H}} = 0$, for $n \neq m$, then a basis $\{f_n\}$ is said to be a orthogonal basis. A orthogonal basis $\{f_n\}$ is said to be an orthonormal basis of $\langle f_n, f_m\rangle = \delta_{nm}$, where δ_{nm} is the kronecker delta [26].

Definition 2.6. [26] A basis $\{f_n\}_{n=0,1,2,...}$ is called a Riesz basis in $\mathbb{H}$ if the series $\sum_{n=0}^{\infty} a_n f_n$ converges in $\mathbb{H}$ iff $\sum_{n=0}^{\infty} a_n^2 < \infty$.

Theorem 2.7. Let $A : \mathbb{H} \to \mathbb{H}$ be any bounded invertible operator and let $\{f_n\}$ be any orthonormal basis in $\mathbb{H}$. Then, the system of elements $\{Af_n\}$ forms a Riesz basis.

3. Operator-Polynomial Treatment of the Problem

Definition 3.1. The element $(f(x), \kappa) \in \mathbb{H}$ is called a generalized solution of the Sturm-Liouville problem (1.1)-(1.7) if the equations

$$\begin{aligned}\langle f, \varrho\rangle_{\oplus W^1_{2,p,q}} &+ \cot\beta f(\pi)\overline{\varrho}(\pi) - \cot\alpha f(0)\overline{\varrho}(0) + \frac{b}{a} f(c_1^+)\overline{\varrho}(c_1^+) + a.bf(c_2^+)\overline{\varrho}(c_2^+) \\ &- \kappa \csc\alpha \overline{\varrho}(0) = \lambda\langle rf, \varrho\rangle_{\oplus L_2}\end{aligned}$$

and

$$-\csc\alpha f(0) - \kappa\cot\alpha\sec^2\alpha = \lambda\kappa\sec^2\alpha$$

are satisfied for any $\varrho \in \oplus W_2^1$ where

$$\kappa := -\cos\alpha f(0) - \sin\alpha(pf')(0) \tag{3.1}$$

Lemma 3.2. Let $f \in C^2[0, c_1) \oplus C^2(c_1, c_2) \oplus C^2(c_2, \pi], p \in C^1[0, c_1) \oplus C^1(c_1, c_2) \oplus C^1(c_2, \pi]$, q and r belongs to $C[0, c_1)\oplus C(c_1, c_2)\oplus C(c_2, \pi]$ and there exists finite limit values $f^{(k)}(c_i \pm 0), p^{(s)}(c_i \pm 0), q(c_i \pm 0), r(c_i \pm 0)$ for $k = 0, 1, 2; i = 1, 2; s = 0, 1$. Then, the weak (generalized) eigenfunction of the SLP satisfies equations $(1.1) - (1.7)$ in the classical sense.

Consider the following linear forms in $\oplus W_2^1$.

$$\Lambda_0(f, \varrho) =: \cot\beta f(\pi)\overline{\varrho}(\pi) - \cot\alpha f(0)\overline{\varrho}(0) + \frac{b}{a} f(c_1^+)\overline{\varrho}(c_1^+) + a.bf(c_2^+)\overline{\varrho}(c_2^+) \tag{3.2}$$

$$\Lambda_1(f, \varrho) =: \left\{\int_0^{c_1^-} + \int_{c_1^+}^{c_2^-} + \int_{c_2^+}^{\pi}\right\} r(x)f(x)\overline{\varrho}(x)dx$$

$$\Lambda_2(\kappa, \varrho) =: -\kappa\csc\alpha\overline{\varrho}(0) \tag{3.3}$$

where $f \in \oplus W_2^1, \kappa \in \mathbb{C}$.

Theorem 3.3. *i.* The linear functionals$\Lambda_0(f, \varrho)$ and $\Lambda_1(f, \varrho)$ are continuous in $\varrho \in \oplus W_2^1$ for each $f \in \oplus W_2^1$,

ii. The linear functionals$\Lambda_2(\kappa, \varrho)$ is are continuous in ϱ for each $\kappa \in \mathbb{C}$.

Proof. Using (3.2) and (3.3), we get

$$|\Lambda_0(f, \varrho)| \leq C_1 \left\{|f(\pi)|\ |\varrho(\pi)| + |f(0)||\varrho(0)| + |f(c_1^+)|\ |\varrho(c_1^+)| + |f(c_2^+)|\ |\varrho(c_2^+)|\right\} \tag{3.4}$$

$$|\Lambda_1(f,\varrho)| \leq C_2\|f\|\|\varrho\|$$

and

$$|\Lambda_2(\kappa,\varrho)| \leq C_3|\kappa||\varrho(0)|$$

respectively. The inequality

$$\|f\| \leq C_4\|f\|_{\oplus W^1_{2,p,q}} \tag{3.5}$$

follows immediately from (2.1) and (2.2). By using the interpolation inequalities (2.3), (2.4), and (3.5), we have the following inequalities

$$|\Lambda_0(f,\varrho)| \leq C_5\|f\|_{\oplus W^1_{2,p,q}}\ \|\varrho\|_{\oplus W^1_{2,p,q}}$$
$$|\Lambda_1(f,\varrho)| \leq C_6\|f\|_{\oplus W^1_{2,p,q}}\ \|\varrho\|_{\oplus W^1_{2,p,q}}$$
$$|\Lambda_2(\kappa,\varrho)| \leq C_7|\kappa|\ \|\varrho\|_{\oplus W^1_{2,p,q}}$$

□

Theorem 3.4. There are linear bounded operators $\mathcal{T}_k : \oplus W^1_2 \to \oplus W^1_2 (k=0,1)$ and $\mathcal{T}_2 : \mathbb{C} \to \oplus W^1_2$ satisfying the following representations:

$$\Lambda_k(f,\varrho) = \langle \mathcal{T}_k f, \varrho\rangle_{\oplus W^1_{2,p,q}} (k=0,1) \text{ and } \Lambda_2(\kappa,\varrho) = \langle \mathcal{T}_2\kappa, \varrho\rangle_{\oplus W^1_{2,p,q}} \tag{3.6}$$

Proof. The proof follows immediately from the well-known Riesz representation theorem (see, for example, [31]). □

Lemma 3.5. The operators $\mathcal{T}_0 : \oplus W^1_2 \to \oplus W^1_2$ $\mathcal{T}_1 : \oplus W^1_2 \to \oplus W^1_2$ and $\mathcal{T}_2 : \mathbb{C} \to \oplus W^1_2$ $\mathcal{T}^*_2 : \oplus W^1_2 \to \mathbb{C}$ are completely continuous (i.e., compact), $\mathcal{T}_0$ and $\mathcal{T}_1$ are selfadjoint, $\mathcal{T}_1$ is positive, where $\mathcal{T}^*_2$ is the conjugated of $\mathcal{T}_2$(where $\mathcal{T}^*_2 f := -\csc\alpha f(0)$).

Proof. Since the function $r(x)$ is bounded and positive definite, it follows that $\mathcal{T}_1$ is selfadjoint and positive. Let Let $f,\varrho \in \oplus W^1_2$. From (3.2) and (3.6) it follows that

$$\langle f, \mathcal{T}_0\varrho\rangle_{\oplus W^1_{2,p,q}} = \overline{\langle \mathcal{T}_0\varrho, f\rangle}_{\oplus W^1_{2,p,q}} = \overline{\Lambda_0(\varrho,f)} = \Lambda_0(f,\varrho) = \langle \mathcal{T}_0 f, \varrho\rangle_{\oplus W^1_{2,p,q}}$$

i.e. the operator $\mathcal{T}_0 : \oplus W^1_2 \to \oplus W^1_2$ is selfadjoint. We prove that $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$ are compact operators. Let $\{f_k\}$ be weakly convergent sequence in the Hilbert space $\oplus W^1_2$ and $f = \lim f_k$ in the sense of weak convergence. Show that the sequence $\{\mathcal{T}_0 f_k\}$ converges strongly in the Hilbert space $\oplus W^1_2$. Since the operator $\mathcal{T}_0 : \oplus W^1_2 \to \oplus W^1_2$ is bounded, the sequence $\{\mathcal{T}_0 f_k\}$ converges weakly to $\mathcal{T}_0 f$ in $\oplus W^1_2$. Since the embedding operator $J : \oplus W^1_2 \subset \oplus L_2$ is compact, it follows that

$$\|f_k - f\|_{\oplus L_2} \to 0 \text{ and } \|\mathcal{T}_0 f_k - \mathcal{T}_0 f\|_{\oplus L_2} \to 0 \text{ as } k \to \infty.$$

Further, the compactness of the embedding operator $W^1_2[a,b] \subseteq C[a,b]$ ($[a,b]$ is arbitrary closed interval) implies the strong convergence of the sequences $\{f_k(0)\}$, $\{f_k(c_1^-)\}$, $\{f_k(c_1^+)\}$, $\{f_k(c_2^-)\}$, $\{f_k(c_2^+)\}$, $\{f_k(\pi)\}$ and $\{(\mathcal{T}_0 f_k)(0)\}$, $\{(\mathcal{T}_0 f_k)(c_1^-)\}$, $\{(\mathcal{T}_0 f_k)(c_1^+)\}$, $\{(\mathcal{T}_0 f_k)(c_2^-)\}$, $\{(\mathcal{T}_0 f_k)(c_2^+)\}$, $\{(\mathcal{T}_0 f_k)(\pi)\}$ in $\mathbb{C}$ to $f(0), f(c_1^-), f(c_1^+), f(c_2^-), f(c_2^+), f(\pi)$ and $(\mathcal{T}_0 f)(0), (\mathcal{T}_0 f)(c_1^-), (\mathcal{T}_0 f)(c_1^+), (\mathcal{T}_0 f)(c_2^-), (\mathcal{T}_0 f)(c_2^+), (\mathcal{T}_0 f)(\pi)$, respectively.

Then, by using Theorem 3.4 and the inequality (3.4) we get that there is a constant $C_8 > 0$ such that

$$\|\mathcal{T}_0 f_k - \mathcal{T}_0 f_m\|^2_{\oplus W^1_{2,p,q}} = \langle \mathcal{T}_0(f_k - f_m), \mathcal{T}_0(f_k - f_m)\rangle_{\oplus W^1_{2,p,q}} = \Lambda_0\Big(f_k - f_m\, \mathcal{T}_0(f_k - f_m)\Big)$$
$$\leq C_8\Big\{ |(f_k(\pi) - f_m(\pi))| \cdot |(\mathcal{T}_0(f_k - f_m))(\pi)| + |(f_k(0) - f_m(0))| \cdot |(\mathcal{T}_0(f_k - f_m))(0)|$$
$$+ \Big|(f_k(c_1^+) - f_m(c_1^+))\Big| \cdot \Big|(\mathcal{T}_0(f_k - f_m))(c_1^+)\Big| + \Big|(f_k(c_2^+) - f_m(c_2^+))\Big| \cdot \Big|(\mathcal{T}_0(f_k - f_m))(c_2^+)\Big| \Big\}$$

and

$$\begin{aligned}
\|\mathcal{T}_1 f_k - \mathcal{T}_1 f_m\|^2_{\oplus W^1_{2,p,q}} &= \langle \mathcal{T}_1(f_k - f_m), \mathcal{T}_1(f_k - f_m)\rangle_{\oplus W^1_{2,p,q}} = \Lambda_1\Big(f_k - f_m\, \mathcal{T}_1(f_k - f_m)\Big) \\
&\leq \left|\left\{\int_0^{c_1^-} + \int_{c_1^+}^{c_2^-} + \int_{c_2^+}^{\pi}\right\} r(x)(f_k(x) - f_m(x)) \cdot \overline{\mathcal{T}_1(f_k - f_m)}dx\right| \\
&\leq C_9\|f_k - f_m\|_{\oplus L_2} \cdot \|\mathcal{T}_1(f_k - f_m)\|_{\oplus W^1_{2,p,q}}
\end{aligned}$$

Therefore, $\|\mathcal{T}_i(f_k - f_m)\|_{\oplus W^1_{2,p,q}} \to 0$ as $k, m \to \infty (i = 0, 1)$, i.e., $\{\mathcal{T}_i f_k\}(i = 0, 1)$ are the Cauchy sequence in the Hilbert space $\oplus W_2^1$. Hence, the sequences $\{\mathcal{T}_i f_k\}(i = 0, 1)$ converges strongly in $\oplus W_2^1$. The compactness of $\mathcal{T}_i (i = 0, 1)$ is proven.

Similarly, one can prove that the operator $\mathcal{T}_2 : \mathbb{C} \to \oplus W_2^1$ is also compact. Moreover, it is easy to show that the operator $\mathcal{T}_2^*$ has the form $\mathcal{T}_2^* f := -\csc\alpha f(0)$. Then, we get

$$|\mathcal{T}_2^* f| \leq C_1 \max\Big\{|f| : x \in [0, c_1) \cup (c_1, c_2) \cup (c_2, \pi]\Big\} \leq \|f\|_{\oplus W_2^1}$$

i.e., the operator $\mathcal{T}_2^* : \oplus W_2^1 \to \mathbb{C}$ is bounded. Since the dimension of the range of $\mathcal{T}_2^*$ is finite and $\mathcal{T}_2^*$ is bounded linear operator, the operator $\mathcal{T}_2^*$ is also compact. □

Applying Lemma 3.5, we get

$$\langle f, \varrho\rangle_{\oplus W^1_{2,p,q}} + \langle \mathcal{T}_0 f, \varrho\rangle_{\oplus W^1_{2,p,q}} + \langle \mathcal{T}_2\kappa, \varrho\rangle_{\oplus W^1_{2,p,q}} = \lambda\langle \mathcal{T}_1 f, \varrho\rangle_{\oplus W^1_{2,p,q}} \tag{3.7}$$
$$\mathcal{T}_2^* f - \kappa\cot\alpha\sec^2\alpha = \lambda\kappa\sec^2\alpha$$

The arbitrariness of $\varrho \in \oplus W_2^1$ in identity (3.7) implies

$$f + \mathcal{T}_0 f + \mathcal{T}_2\kappa = \lambda\mathcal{T}_1 f$$

Define the following two operators

$$\mathcal{R}(f, \kappa) = \Big(f + \mathcal{T}_0 f + \mathcal{T}_2\kappa\, \mathcal{T}_2^* f - \kappa\cot\alpha\sec^2\alpha\Big)$$

and

$$\mathcal{S}(f, \kappa) = \Big(\mathcal{T}_1 f\, \kappa\sec^2\alpha\Big)$$

in the Hilbert space $\mathbb{H}$. Consider the equation

$$\mathcal{T}(\lambda)\Psi = 0 \quad \text{where} \quad \mathcal{T}(\lambda) = \mathcal{R} - \lambda\mathcal{S} \text{ is the operator-pencil }, \tag{3.8}$$

$\Psi = (f(x), \kappa) \in \mathbb{H}$ and κ is defined in (3.1).

Lemma 3.6. Let Ψ_0 be any generalized eigenfunction of the SLP (1.1)-(1.7) which belongs to the eigenvalue λ_0. Then, the eigenpair (λ_0, Ψ_0) satisfies (3.8) in the Hilbert space $\mathbb{H}$.

Lemma 3.7. $\pm 2Re(\int f\overline{\varrho}dz) \geq -\|f\|^2 - \|\varrho\|^2$

Proof. The proof of this lemma is a direct consequence of the polar identity. □

Theorem 3.8. There exists a constant $C_0 > 0$ such that for all $\lambda_0 \in \mathbb{C}$ satisfying $|\lambda_0| > C_0$ the operator-polynomial is positive definite.

PROOF. Taking in view (3.6), we get

$$\begin{aligned} \langle \mathcal{T}(-\lambda_0)\Psi, \Psi\rangle_{\mathbb{H}} &= \langle f(x), f(x)\rangle_{\oplus W^1_{2,p,q}} + \langle \mathcal{T}_0 f(x), f(x)\rangle_{\oplus W^1_{2,p,q}} + \langle \mathcal{T}_2\kappa, f(x)\rangle_{\oplus W^1_{2,p,q}} \\ &+(\mathcal{T}_2^* f(x))\overline{\kappa} - \cot\alpha \sec^2\alpha|\kappa|^2 + \lambda_0\left\{\langle \mathcal{T}_1 f(x), f(x)\rangle_{\oplus W^1_{2,p,q}} + \sec^2\alpha|\kappa|^2\right\} \end{aligned} \tag{3.9}$$

Define the functionals

$$P(f) := \langle pf', f'\rangle_{\oplus W^1_2}, Q(f) := \langle qf, f\rangle_{\oplus W^1_2} \text{ and } R(f) := \langle rf, f\rangle_{\oplus W^1_2} \tag{3.10}$$

Using well-known

$$|f(x_j)|^2 \leq C_{j1}\varepsilon_j P(f) + \frac{C_{j2}}{\varepsilon_j}Q(f), f \in \oplus W^1_2, j \in \{1,2,3,4\} \tag{3.11}$$

holds for sufficiently small positive ε_j, where $x_1 = 0, x_2 = c_1^{\mp}, x_2 = c_2^{\mp}, x_4 = \pi$.

Further, it is convenient to denote

$$P(f) := \left\{\int_0^{c_1^-} + \int_{c_1^+}^{c_2^-} + \int_{c_2^+}^{\pi}\right\} p(x)|f'(x)|^2 dx \; Q(f) := \left\{\int_0^{c_1^-} + \int_{c_1^+}^{c_2^-} + \int_{c_2^+}^{\pi}\right\} q(x)|f(x)|^2 dx$$

Thus,

$$\|f\|^2_{\oplus W^1_{2,p,q}} = P(f) + Q(f) \tag{3.12}$$

Since the functions $q(x)$ and $r(x)$ are positive and bounded, there exist constant $M > 0$ such that

$$\langle \mathcal{T}_1 f(x), f(x)\rangle_{\oplus W^1_{2,p,q}} = \Lambda_1(f,f) \geq MQ(f) \tag{3.13}$$

If we consider Theorem 3.4 and (3.9)-(3.13), we obtain the following inequality.

$$\langle \mathcal{T}(-\lambda_0)\Psi, \Psi\rangle_{\mathbb{H}} \geq \Xi_1 P(f) + \Xi_2(\lambda_0)Q(f) + \Xi_3(\lambda_0)|\kappa|^2$$

where

$$\Xi_1 := 1 - \left(|\cot\alpha| + \frac{1}{\gamma_1|\sin\alpha|}\right) C_{11}\varepsilon_1 - |\frac{b}{a}|C_{21}\varepsilon_2 - |ab|C_{31}\varepsilon_3 - |\cot\beta|C_{41}\varepsilon_4$$

$$\Xi_2(\lambda_0) := 1 - \left(|\cot\alpha| + \frac{1}{\gamma_1|\sin\alpha|}\right)\frac{C_{12}}{\varepsilon_1} - |\frac{b}{a}|\frac{C_{22}}{\varepsilon_2} - |ab|\frac{C_{32}}{\varepsilon_3} - |\cot\beta|\frac{C_{42}}{\varepsilon_4} + \lambda_0 M$$

and

$$\Xi_3(\lambda_0) := -|\cot\alpha|\sec^2\alpha - \frac{\gamma_1}{|\sin\alpha|} + \lambda_0\sec^2\alpha$$

Since $\theta = \cos^2\alpha > 0$, there are positive numbers $a, b, \gamma_1, \gamma_2, \varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 and the positive parameter λ_0 such that the inequalities $\Xi_1 > 0$, $\Xi_2(\lambda_0) > 0$ and $\Xi_3(\lambda_0) > 0$ holds. Denoting

$$\Xi(\lambda_0) := \min\left(\Xi_1 \, \Xi_2(\lambda_0) \, , \Xi_3(\lambda_0)\right)$$

we have

$$\langle \mathcal{T}(-\lambda_0)\Psi, \Psi\rangle_{\mathbb{H}} \geq \Xi(\lambda_0)\|\Psi\|^2_{\mathbb{H}}$$

for all $\Psi \in \mathbb{H}$. Consequently $\langle \mathcal{T}(-\lambda_0)\Psi, \Psi\rangle_{\mathbb{H}}$ is positive definite quadratic form for sufficiently large $\lambda_0 > 0$. □

Theorem 3.9. The operators $\mathcal{R}$ and $\mathcal{S}$ are completely continuous and compact.

PROOF. The proof is obvious because the operators $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_2^*$ are completely continuous. □

Theorem 3.10. The operator polynomial $\mathcal{T}(-\lambda_0)$ is self-adjoint and completely continuous.

PROOF. Since the linear operators $\mathcal{R}$ and $\mathcal{S}$ are completely continuous and compact, $\mathcal{T}(-\lambda_0) = \mathcal{R}+\lambda_0\mathcal{S}$ is also self-adjoint and completely continuous. □

Corollary 3.11. $\mathrm{T}(-\lambda_0)$ is symmetric operator.

4. Basis Property of the Generalized Eigenfunctions of the SLP (1.1)-(1.7)

After the shift of the spectral parameter $\mu = \lambda + \lambda_0$, where λ_0 is the parameter, from Theorem 3.8, (3.8) takes the form

$$\mathcal{T}(-\lambda_0)\Psi = \mu S\Psi$$

with the new spectral parameter μ. Now the transformation $\Phi = (\mathcal{T}(-\lambda_0))^{\frac{1}{2}}\Psi$ introduced to apply Theorem 2.7. Here $(\mathcal{T}(-\lambda_0))^{\frac{1}{2}}$ is the positive square root, which is invertible of the positive selfadjoint operator $\mathcal{T}(-\lambda_0)$.

Definition 4.1. Given a self-adjoint, bounded, and positive linear operator $\mathcal{T} : \mathbb{H} \to \mathbb{H}$, where $\mathbb{H}$ is a Hilbert space. Then, a linear self-adjoint, bounded and positive operator $\mathbb{B} : \mathbb{H} \to \mathbb{H}$ is called a positive square root of $\mathcal{T}$ and is denoted by $\mathbb{B} = \sqrt{\mathcal{T}}$ or $\mathbb{B} = \mathcal{T}^{\frac{1}{2}}$ (see [26]).

Lemma 4.2. The transformed functions $\Phi = (\mathcal{T}(-\lambda_0))^{\frac{1}{2}}\Psi$ satisfy the operator equation

$$\Phi - \mu\mathcal{L}(\lambda_0)\Phi = 0$$

$$\mathcal{L}(\lambda_0) := (\mathcal{T}(-\lambda_0))^{\frac{-1}{2}} S(\mathcal{T}(-\lambda_0))^{\frac{-1}{2}}$$

which for sufficiently large, fixed λ_0 has a positive compact selfadjoint operator $\mathcal{L}(\lambda_0)$.

PROOF. The proof of this lemma is obvious because the operator $\mathcal{T}(-\lambda_0)$ is positive compact selfadjoint operator. □

Lemma 4.3. If (μ_n, Ψ_n) is any eigenpair of the Sturm-Liouville problem (1.1)-(1.7), Then, μ_n is the eigenvalue of the operator $\mathcal{L}(\lambda_0)$ with corresponding eigenelement $\Phi_n = (\mathcal{T}(-\lambda_0))^{\frac{1}{2}}\Psi_n$, i.e., the functions (Φ_n) satisfy the operator equation

$$\Phi_n - \mu_n\mathcal{L}(\lambda_0)\Phi_n = 0$$

PROOF. From the identity$\mathcal{T}(-\lambda_0)\Psi_n - \mu S\Psi_n = 0$ we have

$$(\mathcal{T}(-\lambda_0))^{\frac{1}{2}}\Psi_n = \mu_n(\mathcal{T}(-\lambda_0))^{\frac{-1}{2}}\mathcal{S}\Psi_n$$

Denoting $\Phi_n = (\mathcal{T}(-\lambda_0))^{\frac{1}{2}}\Psi_n$, we get the needed identity

$$\Phi_n = \mu_n\mathcal{L}(\lambda_0)\Phi_n$$

□

Theorem 4.4. The SLP (1.1)-(1.7) has infinitely many real eigenvalues $\{\lambda_n\}$ with accumulation point at $+\infty$ and the system of corresponding generalized (weak) eigenfunctions forms a Riesz basis of the Hilbert space $\mathbb{H}$.

PROOF. Since $\mathcal{T}(-\lambda_0)$ is bounded positive operator and $(\mathcal{T}(-\lambda_0))^{\frac{1}{2}}$ is invertible, the proof follows immediately from Lemma 4.2 and Theorem 2.7. $\square$

Corollary 4.5. The system of the generalized (weak) eigenfunctions of the SLP (1.1)-(1.7) is complete system in $\oplus L_2$.

5. Conclusion

This study investigates a new type of SLP, which differs from the regular SLPs in that the equation has discontinuities at two interior points, and four additional conditions are specified at these points, the so-called jump conditions. Naturally, analyzing such types of discontinuous SLPs is much more complicated than classical SLPs since it is unclear how to adapt Sturmian theory's known methods to such discontinuous problems. To establish some important spectral properties of the problem under consideration, we construct new self-adjoint polynomial operators in the appropriate Hilbert space. In particular, we prove that this polynomial operator is positive definite, the spectrum is discrete, and the generalized (weak) eigenfunctions system forms a Riesz basis. Many problems in physics, engineering, and other branches of natural science lead to discontinuous Sturm-Liouville problems with parameter-dependent jump conditions posed at the points of discontinuity. Similar or modified methods can investigate the case where both the boundary and jump conditions depend on the spectral parameter.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] B. P. Belinskiy, J. W. Hiestand, J. V. Matthews, *Piecewise uniform optimal design of a bar with an attached mass*, Electronic Journal of Differential Equations 2015 (2015) Article Number 206 17 pages.

[2] N. J. Guliyev, *Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter*, Journal of Mathematical Physics 60 (6) (2019) 063501.

[3] D. B. Hinton, *An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition*, Quarterly Journal of Mathematics (30) (1979) 33–42.

[4] A. Kawano, A. Morassi and R. Zaera, *Detecting a prey in a spider orb-web from in-plane vibration*, SIAM Journal on Applied Mathematics 81 (6) (2021) 2297-2322.

[5] J. Walter, *Regular eigenvalue problems with eigenvalue parameter in the boundary conditions*, Mathematische Zeitschrift 133 (1973) 301–312.

[6] A. Yakar, Z. Akdoğan, *On the fundamental solutions of a discontinuous fractional boundary value problem*, Advances in Difference Equations, 2017 (2017) Article ID 378 15 pages.

[7] Z. Akdoğan, A. Yakar, M. Demirci, *Discontinuous fractional Sturm-Liouville problems with transmission conditions*, Applied Mathematics and Computation 350 (2019) 1–10.

[8] B. P. Allahverdiev, H. Tuna, *Eigenfunction expansion for singular Sturm-Liouville problems with transmission conditions*, Electronic Journal of Differential Equations 3 (2019) 1–10.

[9] M. Yücel, O. Mukhtarov, K. Aydemir, *Computation of eigenfunctions of nonlinear boundary-value-transmission problems by developing some approximate techniques*, Boletim da Sociedade Paranaense de Matemática 3 (41) (2023) 1–12.

[10] P. A. Binding, P. J. Browne, B. A. Watson, *Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter II*, Journal of Computational and Applied Mathematics (148) (2002) 147–168.

[11] M. Kandemir, O. Sh. Mukhtarov, *Nonlocal Sturm-Liouville problems with integral terms in the boundary conditions*, Electronic Journal of Differential Equations 2017 (11) (2017) 1–12.

[12] Y. A. Küçükevcilioğlu, E. Bayram , G. G. Özbey, *On the spectral and scattering properties of eigenparameter dependent discrete impulsive Sturm-Liouville equations*, Turkish Journal of Mathematics 45(2) (2021) 988–1000.

[13] O. Sh. Mukhtarov, M. Yücel, K. Aydemir, *A new generalization of the differential transform method for solving boundary value problems*, Journal of New Results in Science 10 (2) (2021) 49–58.

[14] O.Sh. Mukhtarov, H. Olğar, K. Aydemir, *Resolvent operator and spectrum of new type boundary value problems*, Filomat 29 (7) (2015) 1671–1680.

[15] O. Sh. Mukhtarov, M. Yücel, K. Aydemir, *Treatment a new approximation method and its justification for Sturm-Liouville problems*, Complexity 2020 (2020) Article ID 8019460 8 pages

[16] H. Olğar, *Self-adjointness and positiveness of the differential operators generated by new type Sturm-Liouville problems*, Cumhuriyet Science Journal 40 (1) (2019) 24–34.

[17] E. Şen, M. Açıkgöz, S. Aracı, *Spectral problem for Sturm-Liouville operator with retarded argument which contains a spectral parameter in the boundary condition*, Ukrainian Mathematical Journal 68(8) (2017) 1263–1277.

[18] E. Uğurlu, K. Taş, *A new method for dissipative dynamic operator with transmission conditions*, Complex Analysis and Operator Theory 12(4) (2018) 1027–1055.

[19] A. Wang, J. Sun, X. Hao, S. Yao, *Completeness of eigenfunctions of Sturm-Liouville problems with transmission conditions*, Methods and Application of Analysis 16 (3) (2009) 299–312.

[20] N. B. Kerimov, R. G. Poladov, *Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions*, Doklady Mathematics (2012) 85 (1) 8–13.

[21] O. Sh. Mukhtarov, K. Aydemir, *Basis properties of the eigenfunctions of two-interval SturmLiouville problems*, Analysis and Mathematical Physics 9 (2019) 1363–1382.

[22] H. Olğar, O. Sh. Mukhtarov, *Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions*, Journal of Mathematical Physics 58 (2017) Article ID 042201.

[23] H. Olğar, O. Sh. Mukhtarov, K. Aydemir, *Some properties of eigenvalues and generalized eigenvectors of one boundary value problem*, Filomat 32 (3) (2018) 911–920.

[24] A. M. Sarsenbi, A. A. Tengaeva, *On the basis properties of root functions of two generalized eigenvalue problems*, Differential Equations 48 (2) (2012) 306–308.

[25] M. V. Keldysh, *On the characteristics values and characteristics functions of a certain class of nonselfadjoint equations (in Russian)*, Doklady Akademii Nauk (77) (1951) 11–14.

[26] I. Gohberg, S. Goldberg, Basic operator theory, Birkhauser, Boston-Basel-Stuttgart, 1981.

[27] O. A. Ladyzhenskaia, The boundary value problems of mathematical physics, Springer-Verlag, New York 1985.

[28] A. S. Markus, Introduction to the spectral theory of polynomial pencils, Translation of Mathematical Monographs, American Mathematical Society, Providence, Rhode Island, 1988.

[29] L. Rodman, An introduction to operator polynomials, Birkhaüser Verlag, Boston, Massachusetts, 1989.

[30] B. P. Belinskiy, J. P. Dauer, *Eigenoscillations of mechanical systems with boundary conditions containing the frequency*, Quarterly of Applied Mathematics (56) (1998) 521–541.

[31] E. Kreyszig, Introductory functional analysis with application, New-York, 1978.