Some Equity in Tax Collections of the Government in a Commodity Market

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Abstract

The Laffer effect has been discussed before in context of macroeconomic endogenous growth models or in labor market. Discussion have been mainly about whether a tax cut on wages would induce workers to spend more time on work rather than leisure and at the same time leading to an increase in income tax revenues of the government. In this paper, we are interested in providing a general formula for the revenue-maximizing government using an ad valorem tax rate in a single (micro) commodity market such as automobiles, liquor or cigarettes in the case of non-linear demand and supply curves. It turns out that the optimal commodity tax rate depends on the after-tax demand elasticity. Therefore, in practice the government officials should try to project the after-tax elasticity and not rely on the before-tax elasticity, which is commonly assumed in the economics literature. More importantly, if the government imposes an ad valorem tax on a product in a micro market, then the consumers' share of burden of tax does not change no matter what the tax rate is. Hence, in that sense, we find some equity in taxing. Some additional important theoretical results are derived when the demand and supply curves have different positions.

JEL: C02, C65, D01.

Key words: Laffer effect; a commodity market; optimal ad valorem tax rate; non-linear model; consumers' share of burden of tax.

1. Introduction

The Laffer effect has been discussed in the literature in the context of endogenous macroeconomics models to examine the possibility of whether a tax cut on physical capital/bonds ownership would revive the economy to such an extent that the government's tax collections would improve (Ireland (1994), Bruce and Turnovsky (1999), Agell and Perrson (2001)). On the microeconomics side, whether a tax cut on wages would persuade people to work more, resulting in increasing the number of hours worked, and at the same time improving the government's tax collections has been investigated.

However, this paper deals with the maximum tax collection of the government, regarding a single commodity market like, say, the automobile sector with respect to an *ad valorem* tax rather than in a macroeconomic setting. The ad valorem type of tax is very often encountered in practice in the USA, and it is based on the value of the product rather than the units sold. For example there are about 6,400 different ad valorem sales taxes across the United States which can go as high as 8.5 percent (Perloff, 2008).

Özçam and Özçam (2012) discussed a real world example where the Turkish government decreased the special consumption tax (SCT) for automobiles below 1600 cc (making up 85 percent of domestic market sales) temporarily (from March to September 2009) and partially (from 37 percent down to 18 and, then up to 27 percent and finally to 37 percent again) to support the domestic automobile market against the likely negative effects of the global crisis which had started being felt relatively strongly at the beginning of 2009 in Turkey and estimated the demand for automobiles using an econometric model for 2006-2010. Constructing price indexes (Laspeyres, Paasche,etc.) for different segments (such as C2, B2, C1) they evaluated to what extent this partial tax concession was passed on to the consumers by producers in the form of price discounts and the lengths of periods (3-4 months) over which these discounts were offered on a segment basis. As a conclusion, they asserted that, during the global crisis, the portion of SCT decrease given by the Turkish government that passed onto the Turkish auto consumers was about 50 percent and short-lived, pointing to the possibility of the price elasticity of demand being approximately equal to the elasticity of supply.

As a result, in the real world, as we tried to explain above in the Turkish automobile industry example in 2009, the government may try to support a very big micro sector like the automobile industry in certain circumstances like a severe recession and may also target the very same sector for more tax collections in situations like a deteriorating budget deficit. A large percentage of tax collections (about $600 million/month in the case of Turkey) may come from such indirect taxes like sales taxes which are of ad valorem type and therefore, aiming at a large sector for additional tax-revenues makes sense from a point of view of a revenue-maximizing government.

Özçam (2014) discussed whether before or after tax elasticity equilibria points mattered using a mathematical model where the demand and supply curves were linear and the government imposed an *ad valorem* tax on the commodity. He also discussed a special case where the supply curve was perfectly elastic. This specific situation coincided very well with the case where the auto producers would have passed fully the partial and temporary tax decrease given by the Turkish government onto the consumers during the global crisis of 2009.

Mas-Colell, Whinston and Green (2004, p. 331), discussed the tax issue in the context of the welfare authority keeping a balanced budget and returning the tax to consumers by lump-sum transfers and deadweight loss triangle. They also mention a case where in a general equilibrium context, levying a tax on labor in one town leads to a wage rate fall in all other towns (p.538). However, the maximum tax collections of the government does not seem to have been previously discussed in the literature in the context of a product's market.

Section 2 considers a product's market where the demand and supply functions are non-linear and the government imposes an ad valorem tax. The optimal tax rate that the government can charge to maximize tax revenues will be calculated. In Section 3, the linear demand and supply curves will be used to further show our main theorem in the non-linear case given in Section 2 where the optimal tax rate depends on the after-tax equilibrium demand elasticity. In Section 4, some results with respect to various elasticities of the demand and supply curves will follow. Section 5 concludes the paper.

2. A commodity market model with non-linear demand and supply curves in the case of an ad valorem tax

In the case of an ad valorem tax, suppose that the non-linear market demand and supply curves are given by

$$D(P) \text{ and } S(P,t) \tag{1}$$

where t is the tax rate (as a percentage) which is inserted into the supply function that the government imposes on the product. In this version of the model, P is the demand price. Setting up a two-equation system and letting Q to be the equilibrium quantity, Q = D(.) = S(.)

$$F_1(P,Q;t) = D(P) - Q = 0$$
$$F_2(P,Q;t) = S(P,t) - Q = 0 \tag{2}$$

We shall invoke the Implicit Function Theorem, since both the demand and supply functions are assumed to possess continuous partial derivatives and the endogenous variables Jacobian is nonzero

(3)

$$|J| = \begin{vmatrix} \frac{\partial F_1}{\partial P} & \frac{\partial F_1}{\partial Q} \\ \frac{\partial F_2}{\partial P} & \frac{\partial F_2}{\partial Q} \end{vmatrix} = \begin{vmatrix} \frac{dD}{dP} & -1 \\ \frac{\partial S}{\partial P} & -1 \end{vmatrix} = \left(\frac{\partial S}{\partial P} - \frac{dD}{dP}\right) > 0$$

Even though we cannot solve explicitly for $\bar{P} = \bar{P}(t)$ and $\bar{Q} = \bar{Q}(t)$, the implicit function theorem insures that the equations in eq. (2) above, hold exactly in some neighborhood of the equilibrium solution, so that we may also write

$$D(\bar{P}) - \bar{Q} \equiv 0 \tag{4}$$
$$S(\bar{P},t) - \bar{Q} \equiv 0$$

Taking the total differential of each identity in turn, and rearranging we obtain a linear system in

$$d\bar{P} \text{ and } d\bar{Q}$$

(5)

$$\frac{dD}{d\bar{P}} d\bar{P} - d\bar{Q} = 0$$
$$\frac{\partial S}{\partial \bar{P}} d\bar{P} - d\bar{Q} = -\frac{\partial S}{\partial t}$$

Dividing through *dt*

$$\begin{bmatrix} \frac{dD}{d\bar{P}} & -1 \\ \frac{\partial S}{\partial \bar{P}} & -1 \end{bmatrix} \begin{bmatrix} \frac{d\bar{P}}{dt} \\ \frac{d\bar{Q}}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-\partial S}{\partial t} \end{bmatrix} \tag{6}$$

By Cramer's rule, we find the solutions for endogenous variables to be

$$\left(\frac{d\bar{P}}{dt}\right) = \begin{vmatrix} 0 & -1 \\ \frac{-\partial S}{\partial t} & -1 \end{vmatrix} / |J| = \frac{-\partial S}{\partial t} \; / \; |J| \; > \; 0 \tag{7}$$

$$\left(\frac{d\bar{Q}}{dt}\right) = \begin{vmatrix} \frac{dD}{d\bar{P}} & 0 \\ \frac{\partial S}{\partial \bar{P}} & \frac{-\partial S}{\partial t} \end{vmatrix} / |J| = \left(\frac{dD}{d\bar{P}}\right) * \left(\frac{-\partial S}{\partial t}\right) \; / \; |J| < \; 0 \tag{8}$$

Generally, there are two types of *ad valorem* tax rates. The government may calculate its tax basis on producers' prices or consumers' prices[36]. This paper investigates the former case.

Then, the tax revenues of the government (*TR*) are given by

$$TR = \bar{P} * \frac{t}{1+t} * \bar{Q} \tag{9}$$

36 *In this paper we investigate the case where the tax rate is calculated on producers' price, which forms a tax basis and a tax rate like a special consumption tax is added to the producers' price. In the alternative situation where the tax rate is levied out of consumers' price, then eq.(12) becomes* $S = -c + d * P * (1 - t)$ *and the formula for the Tax Revenues (eq. (9) above) becomes* $TR = \bar{P} * t * \bar{Q}$.

To see how the tax revenues change as the tax policy of the government changes in the *ad valorem* case, eq. (9) is differentiated with respect to the tax rate, *t*

$$\frac{dTR}{dt} = \frac{\bar{P}}{1+t} + \frac{d\bar{P}}{dt}t + \frac{dD}{d\bar{P}}\frac{\bar{P}}{\bar{Q}}\frac{d\bar{P}}{dt}t = 0 \tag{10}$$

and the Revenue-Maximizing or the optimal tax amount (t_{opt}) from the government's perspective is found to be

$$t_{opt} = \frac{\dfrac{P}{(1+t)}}{\dfrac{d\bar{P}}{dt}(\varepsilon_d - 1)} = \frac{P_s}{\dfrac{d\bar{P}}{dt}(\varepsilon_d - 1)} \tag{11}$$

where $\bar{P}$ is the after-tax demand price, P_s is the after-tax supply price, ε_d is the price elasticity of demand at the after-tax equilibrium (in absolute value).

<u>THEOREM 1:</u> If the government imposes an *ad valorem* tax on a product like automobiles, liquor, cigarettes etc., the optimal tax rate depends on the elasticity of demand at the after-tax equilibrium.

Proof: This is given in eqs. (7), (8) and (11) above. Notice that the elasticity of demand was calculated at $(\bar{P},\bar{Q})$, which is the after tax equilibrium point. Theorem 1 will be discussed further in Section-3 below (eq. (21)) using a numerical example to show that the relevant elasticity is indeed the one that is after-tax equilibrium.

3. The linear market model with an ad valorem tax

Now the linear version of demand and supply curves in eq. (1) above is considered, where all five parameters. a, b, c, d and t are non-negative. Again, the tax variable, *t* appears in the supply function and therefore *P* is the demand price.

$$D = a - b * P \qquad \text{and} \qquad S = -c + d * \frac{P}{1+t} \tag{12}$$

Letting Q to be the equilibrium quantity and rearranging the terms

$$\begin{aligned} -b * P - Q &= -a \\ \frac{d}{(1+t)} * P - Q &= c \end{aligned} \tag{13}$$

where the Jacobian is

$$|J| = \begin{vmatrix} -b & -1 \\ \frac{d}{1+t} & -1 \end{vmatrix} = \frac{(b(1+t)+d)}{1+t} > 0 \tag{14}$$

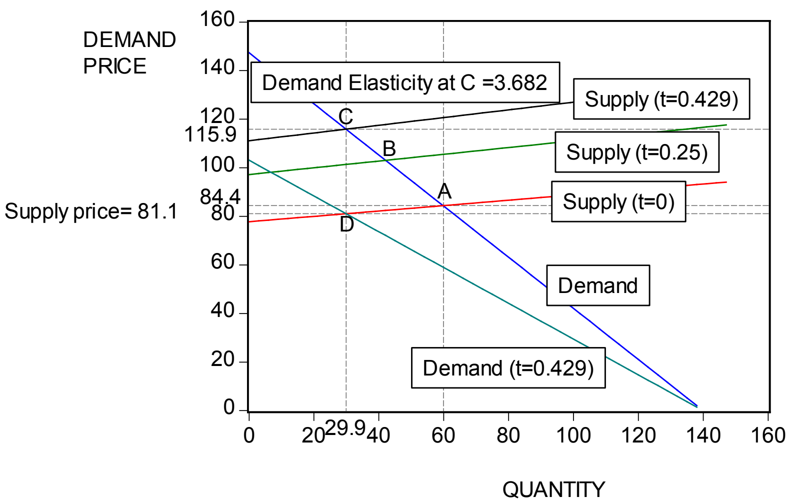
and the equilibria values of price and quantity are,

$$\bar{P} = \begin{vmatrix} -a & -1 \\ c & -1 \end{vmatrix} \, / \, |J| = \frac{(a+c)*(1+t)}{b(1+t)+d}$$

$$\bar{Q} = \begin{vmatrix} -b & -a \\ \frac{d}{1+t} & c+dt \end{vmatrix} \, / \, |J| = (ad - bc(1+t))/(b(1+t)+d) \tag{15}$$

This issue may be clearer with the help of a numerical example. Inserting some hypothetical numbers, one can calculate the equilibria points as shown in Figure 1 below. The initial equilibrium (t=0%) is exhibited as point A, with the initial quantity of 59.8 units and the initial price of $84.4 using a=140, b=0.95, c=700 and d=9. As the tax rate t increases from 0 to 0.25 and to 0.429, the supply curve keeps shifting to the left but it also tilts upward. The equilibrium quantity decreases down to 42.09 and 29.9 units while the equilibrium price (demand price) increases up to \$103 and \$115.9 respectively.

Figure 1. The Demand and the Tax-shifted and Tilted Supply Curves: The Linear Model (a=140, b=0.95, c=700, d=9, and t=0,0.25 and 0.429)



The left-shifted and upward tilted supply function becomes

$$P_d = (c/d)*(1+t) + S*(1+t)/d = 7\ .778*(1+t) + S*(1+t)/9 \qquad (16)$$

where P_d is the demand price.

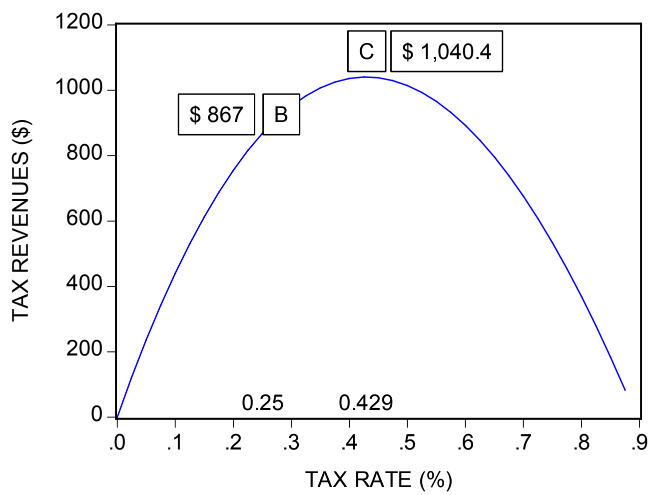
To check the consistency, one can also consider the other version of the model (eq. (12)) where the tax variable (t) would have been inserted in the demand function, $D = a - b * P_s * (1+t)$. Figure 1 additionally shows this case where the inward tilted demand curve is

$$P_s = a/(b*(1+t)) - D/(b*(1+t)) = 147.37/(1+t) - D/(0.95*(1+t)) \qquad (17)$$

With t = 0.429 and D = 29.9 units, P_s in eq. (17) above is equal to \$81,1 which is precisely the supply price, P_s = 115.9/1.429 = \$81.1 obtained from the version of the model where the supply curve shifts and tilts. This is confirmed further geometrically in Figure 1 above, where the tilted-demand curve intersects the initial Supply Curve (t=0) at point D (at \$81.1) which is directly below point C, precisely at the equilibrium quantity of 29.9 units.

In Figure 2 below, using the same values for parameters a, b, c and d, it can be observed that the tax revenues, $TR = \bar{P} * \frac{t}{1+t} * \bar{Q}$ (eq. (9) above) are maximized at t_{opt} = 0.429 and are equal to \$1,040.4 at point C using eq. (9) numerically (Grid approach). Points B and C correspond to those in Figure 1 above.

Figure 2. Tax Revenues: The Linear Model (a=140, b=0.95, c=700, d=9)



One can also calculate the optimal rate of an ad valorem tax from the government's perspective analytically in the linear model using eqs. (10) and (15), as follows:

$$\frac{dTR}{dt}=\frac{(a+c)}{b(1+t)+d}+\frac{(a+c)}{(b(1+t)+d)^{2}}-bdt\frac{(a+c)^{2}*(1+t)}{(ad-bc(1+t))*(b(1+t)+d)}=0 \qquad (18)$$

multiplying by $b(1+t)+d$ and dividing by $(a+c)$ throughout

$$\frac{dTR}{dt}=(b+d)*ad-(b+d)*bc-t*(b^{2}c+2bcd+abd)=0 \qquad (19)$$

$$t_{opt}=\frac{(b+d)*(ad-bc)}{b*(bc+2cd+ad)}=\frac{(9.95)*(1260-665)}{0.95*(665+12600+1260)}=0.429 \qquad (20)$$

Our calculation of the optimal tax rate in eq. (20) is confirmed by both the Grid approach (numerical checking) and the geometrical display in Figure 2 above where the maximum of the tax revenues occurs indeed at t = 0.429. The analytical derivation in eqs. (18), (19) and (20) above support further this finding.

Table 1 below shows the calculations/analytical results of Figures 1 and 2. The tax revenues (4th column) are indeed maximized at t=0.429 and equal to $1,040.4 shown as point C in Figures 1 and 2 above. Moreover, as the tax rate increases in the first column, the equilibrium quantity decreases and the equilibrium demand price rises as expected. Next, it can be observed that the after-tax elasticity in column 5 increases as t increases. This shows the very well-known result that the price elasticity increases up along the demand curve. Indeed, we will show that the correct elasticity calculation (column 5 and row 7) that enters the formula for the optimal tax rate must be carried out at the intersection of demand and the after-tax supply curve which occurs at the final equilibria point (Point C). Moreover, this is the after-tax demand elasticity and it must be calculated along the fixed demand curve.

To show our claim, one can calculate the after-tax elasticity of demand ε_d^C (in absolute value) at point C (Figure-1) and then check whether t_{opt} is equal to 0,429 or not, using the formula in eq. (11) for the general functions model, when our claimed value of 3.682 is inserted for ε_d^C.

$$\varepsilon_d^C = b * \frac{\bar{P}}{\bar{Q}} = \frac{(0.95 * 115.9)}{29.9} = 3.682$$

$$t_{opt} = \frac{\frac{\bar{P}}{(1+t)}}{\frac{d\bar{P}}{dt}(\varepsilon_d^C - 1)} = \frac{P_s}{\frac{d\bar{P}}{dt}(\varepsilon_d^C - 1)} = \frac{115.9/1.429}{70.47*(3.682-1)} = 0.429 \tag{21}$$

Table 1. Tax Revenues: The Linear Model (a=140, b=0.95, c=700, d=9)

Tax Rates (%)	Eq. Quantity	Eq. Demand Price ($)	Tax Revenues ($)	AFTER-TAX Demand Elasticity	$\left(\frac{d\bar{P}}{dt}\right)$	RHS of Eq. (11) at various tax amounts	Consumers' Burden of Tax (CB) (%)
0	59.8	84.4	0	1.34	76.4	3.24	0
0.05	56.19	88.2	236	1.49	75.6	2.56	0.9045
0.1	52.6	91.9	440	1.66	75	1.69	0.9045
0.25	42.09	103	867	2.33	72.8	0.85	0.9045
0.42	30.5	115.3	1039	3.59	70.58	0.44	0.9045
0.429*	**29.9**	**115.9**	**1,040.4**	**3.682**	**70.47**	**0.429***	0.9045
0.6	18.63	127.8	892	6.51	68.3	0.21	0.9045
0.875	1.22	146.1	83	11.4	65	0.01	0.9045

Notes: * Optimal tax rate.

The optimal tax amount calculated in the linear case, eq. (20), coincides nicely with that calculated in the general functions case, eq. (21), and thus confirming the validity of eq. (11) given in the Theorem-1 above. The 7th column shows the right hand side (RHS) of eq. (11). It is exactly equal to the optimal tax rate, $t_{opt} = 0.429$ when $t = 0.429$ in the first column. In other words, at all equilibria points other than point C, eq. (11) is not satisfied. The calculations in the linear version of demand and supply curves point out to the correctness of our derivation of eq. (11) above, in case of general non-linear functions. Hence, we see that the appropriate demand elasticity that enters the optimal tax formula is the one calculated at after-tax level as stated in Theorem 1 above.

Furthermore, a result similar to the case of a specific tax (Özçam, 2015) is obtained and is given in the following Theorem. One can observe that Consumers' Burden of Tax (CB) remains the same (0.9045) in the 8th column of Table 1.

THEOREM 2: If the government imposes an ad valorem tax rate on a product in a micro market such as automobiles, liquor, and cigarettes, then the Consumers' Share of Burden of Tax (and therefore the Producers' Burden of Tax also) does not change no matter what the tax rate is for given linear demand and supply curves. Therefore, some equity is preserved when the government changes the tax rate it levies on the public.

Proof:

$$\text{CB} = \frac{(\bar{P} - P_1)}{(\bar{P} - P_s)} = \frac{\dfrac{(a+c)*(1+t)}{b(1+t)+d} - \dfrac{(a+c)}{(b+d)}}{\dfrac{(a+c)*t}{b(1+t)+d}} = \frac{(1+t)*(b+d) - b - bt - d}{(b+d)*t} = \frac{d}{b+d}$$

using eqs. (13) and (15) above. Therefore, the consumers' share of burden of tax (CB) depends on fixed parameter values (b and d) and hence is constant. In particular, CB does not depend on the tax rate, t, each of which corresponds to a different position of the after-tax supply curve (shifted and tilted) as in Figure 1 above (points B, C …).

Using our numerical values (b=0.95 and d=9), CB $= \frac{9}{0.95+9} = 0.9045$ we can check that this is indeed the case by calculating CB for example

at point B, CB = $\frac{(\bar{P}-P_1)}{(\bar{P}-P_s)} = \frac{(103-84.4)}{(103-82.4)} = 0.9045$, or at point C,

CB = $\frac{(\bar{P}-P_1)}{(\bar{P}-P_s)} = \frac{(115.9-84.4)}{(115.9-81.1)} = 0.9045$ where P_1 is the initial

equilibrium price ($84.4 in our case) and $P_s = \frac{\bar{P}}{(1+t)}$ is the supply price.

4. Some Special Cases

Now, we consider three special cases with respect to the positions of demand and supply curves:

a) The supply curve becomes more inelastic starting from a perfectly elastic case:

If the initial equilibrium point is fixed at point A (59.8 units, $84.4) as in Figure 1 above starting with a perfectly elastic supply curve where the parameter c is infinitely large and the demand curve staying the same (a=140, b=0.95), the parameter d must adjust according to eq. (12) to keep point A fixed (with t =0) as the steepness of the supply curve increases.

Table 2. Various Supply Curves and Optimal Tax Rates (a=140, b=0.95)

c	d	Optimal Tax rate (%)	Eq. Quantity	Eq. Demand Price ($)	Elasticity of Demand at optimal tax rates	Tax Revenue ($)	$\left(\frac{d\bar{P}}{dt}\right)$	Consumers' Burden of Tax (CB) (%)
∞	∞	0.3728	29.9	115.9	3.682	941.03	84.42	1
90,000	1,066	0.3732	29.9	115.9	3.682	941.86	84.29	0.99
7,000	83.6	0.378	29.9	115.9	3.682	952	82.8	0.988
5,000	60	0.38	29.9	115.9	3.682	955	82.1	0.98
1,000	12.6	0.41	29.9	115.9	3.682	1,012	74.1	0.93
700	**9**	**0.429**	**29.9**	**115.9**	**3.682**	**1,040.4**	**70.47**	**0.9045**
100	1.9	0.69	29.9	115.9	3.682	1,413	37.1	0.67
-20	0.5	4.5	29.9	115.9	3.682	2,837	1.7	0.33
-29.9	0.35	-	29.9	115.9	3.682	3,473	0	0.27

As the parameter *c* decreases in the first column of Table 2 above, the supply curve becomes more and more inelastic (more vertical). Notice that the 7th row of Table-2 above (in bold) coincides precisely with that of Table-1 above, since a=140, b=0.95, c=700 and d=9 was the base example.

Moreover, observe that the equilibrium quantity and demand price (29.9 units and $115.9) in columns 4 and 5 do not change and therefore stay fixed while the steepness of the supply curve increases. These fixed values of equilibrium quantity and demand price correspond exactly to point C in Figure 1 above. Consequently, when the government levies an optimal ad valorem tax rate, the after-tax demand elasticity (3.682) does not change either (column 6), since it is calculated at point C every time.Furthermore, the 3rd column gives the optimal tax rates at each position (elasticity) of the supply curve. It is well known in the literature that the tax revenues are higher (as shown in column 7) and the Consumers' Burden (CB) is lower (as shown in column 9) for a given tax rate, as the supply curve becomes more inelastic.

However, here we are discussing a different issue. We are comparing the equilibria at various optimal tax rates depending on the steepness's of the supply curves. We additionally find numerically that then the necessary revenue-maximizing tax rate (or the optimal tax rate) is higher (column 3).

COROLLARY 1: For a given demand curve, the steeper (less elastic) the supply curve, the greater the tax rate in order for the government to maximize its tax collections.

One can also draw some theoretical results regarding the limiting case where the supply curve is perfectly elastic (Row 2 in Table 2) as given in the following Corollary.

COROLLARY 2: When the supply curve is perfectly elastic, then at the optimal tax rate it becomes, $\varepsilon_D = \dfrac{(1+t_{opt})}{t_{opt}}$. Moreover, since the function $\dfrac{(1+t)}{t}$ is asymptotic to 1 from above, the Laffer effect can never occur if the demand curve at after-tax equilibrium is inelastic (less than 1).

Proof: Using eq. (11) above in Section 2 for the general demand and supply functions,

$$t_{opt} = \frac{\dfrac{\bar{P}}{(1+t)}}{\dfrac{d\bar{P}}{dt}(\varepsilon_d^C - 1)} = \frac{\dfrac{115.9}{1.3728}}{84.42*(\varepsilon_d^C - 1)} = \frac{1}{(\varepsilon_d^C - 1)} = \frac{1}{(3.682-1)} = 0.3728$$

$$\varepsilon_d^C = \frac{(1+t_{opt})}{t_{opt}} = \frac{(1+0.3728)}{0.3728} = 3.682 \qquad (22)$$

Since (1+t)/t declines as t increases, eq. (22) suggests that for a given level of demand elasticity (in absolute value), the tax rate must be high enough for the Laffer effect to occur. Alternatively, for a given level of tax rate, the elasticity of demand must be high enough. For example, if the tax rate is 100 percent, then the demand elasticity must be greater than 2. Even if the tax rate increases to 120 percent, as in the Turkish auto industry real world example for relatively luxurious autos above 2000cc, the demand elasticity still needs to be greater than 1.83. Since the demand elasticity at point C was equal to 3.682 (much greater than1) in our numerical example, the Laffer effect did in fact occur at a much smaller tax rate, 42.9 percent (Figure 2 above). The tax revenues had the concave shape allowing for the possibility of the Laffer effect.

Exactly the same result was derived in Proposition 1 and Corollary 2 in Özçam (2014), where in a micro model (commodity market), an *ad valorem* tax was inserted in the demand curve rather than in the supply curve (as in this paper) and where the supply curve was assumed to be perfectly elastic all along in model equations. A similar derivation to that, but with respect to a perfectly elastic demand curve will be presented in part (b) below.

Furthermore, the fact that the Consumers' share of tax burden (CB) decreases as the supply curve becomes more inelastic for a given tax rate and a given demand curve is a familiar result in the economics literature. However, our situation here is different since we are discussing the consumers' shares of tax burden at various optimal ad valorem tax rates.

<u>THEOREM 3</u>: Given a demand curve, as the supply curve becomes more inelastic, the consumers' share of tax burden decreases for optimal tax policies applied by the government at each predetermined position (elasticity) of the supply curve.

Proof: Using Theorem-2 in Section 3 above,

$$\text{CB} = \frac{(\bar{P} - P_1)}{(\bar{P} - P_s)} = \frac{d}{b+d} \tag{23}$$

In other words, for given demand and supply curves, the consumers' share of tax burden is independent of the levied tax rate by the government. Moreover, as observed in eq. (23), as *d* goes from zero to infinity, the consumers' share of tax (CB) ranges over (0,1) as in Table 2 above.

b) The demand curve is perfectly elastic

Then, the demand price is fixed at the level of $\bar{P} = \$84.4$, and after the imposition of the tax the supply function shifts to the left and also tilts upward,

$$S(P_s) = -c + d * \frac{\bar{P}}{1+t} \tag{24}$$

where $\frac{\bar{P}}{1+t} = P_s$ is the supply price. In Figure 3 below, it is clear that the after-tax price to producers, P_s decreases thus discouraging production (Varian, 1999). The Laffer effect occurs when the government's tax revenues increase as the tax amount/rate decreases. This paper deals with the Laffer effect in the case of an ad valorem tax, *t*, which is expressed in percentage (as rate),

$$\frac{dTR}{dt} = \frac{d(\bar{P} * t/(1+t) * S(P_s))}{dt} = \bar{P} * \left[\frac{t}{1+t} * \frac{dS(P_s)}{dt} + S(P_s) * \frac{1}{(1+t)^2} \right] < 0$$

or $$t * \frac{dS(P_S)}{dP_S} * \frac{-\bar{P}}{(1+t)^2} + S(P_s) * \frac{1}{1+t} < 0 \tag{25}$$

Dividing by $S(P_S)$ yields

$$\varepsilon_s > \frac{1}{t} \tag{26}$$

$$t_{opt} = \frac{1}{\varepsilon_s} \tag{27}$$

Turning to the case of a linear supply curve,

$$\frac{dTR}{dt} = \frac{d(\bar{P} * t/(1+t) * S(P_s))}{dt} = \bar{P}\left[\frac{t}{1+t} * \frac{-d\bar{P}}{(1+t)^2} + (-c + d\frac{\bar{P}}{1+t}) * \frac{1}{(1+t)}\right] = 0$$

$$-td\bar{P} - ct = c - d\bar{P}$$

or

$$t_{opt} = \frac{c - d\bar{P}}{-c - d\bar{P}} = \frac{700 - (9 * 84.4)}{-700 - (9 * 84.4)} = 0.04096 \tag{28}$$

In Figure-3 below, the optimal shifting of the supply curve is shown as Supply Curve (t=0.04096). An interesting question is at which point the supply elasticity, given in eq. (27), ought to be calculated. It turns out that it is neither at the initial equilibrium point (A) nor at the final equilibrium point (G). It is at the point (F) where one can interpret it as an after-tax supply elasticity calculated at the extension of a zero tax amount (t=0) Supply Curve. Using the parameter values,

$$P_s = \frac{\bar{P}}{1+t_{opt}} = \frac{84.4}{1.04096} = 81.1$$

$$\varepsilon^F_{\sup ply} = \frac{d * P_s}{\bar{Q}} = \frac{9*81.1}{29.9} = 24.4117 \quad (29)$$

Finally using eq. (27) above one can see that for this value of after-tax supply elasticity (24.4117):

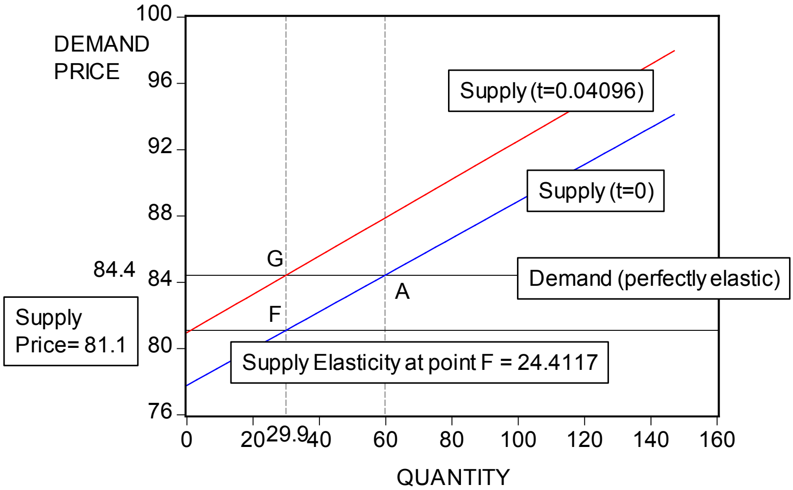
$$t_{opt} = \frac{1}{\varepsilon^F_{\sup ply}} = \frac{1}{24.4117} = 0.04096 \quad (30)$$

which coincides with the result given in eq. (28) above in the case of a linear supply curve.

<u>COROLLARY 3</u>: In the case of a perfectly elastic demand curve, in calculating the optimal ad valorem tax rate, one must consider the after-tax supply elasticity along the initial supply curve (t=0) and not the before-tax supply elasticity at the initial equilibrium point nor the after-tax elasticity at the final equilibrium.

Proof: Proof is given in eqs. (25) - (30) above.

Figure 3: The Infinitely Elastic Demand Curve and the Optimal Tax Rate ($\bar{P} = \$84.4$, c=700, d=9 and $t_{opt} = 0.04096$)



Combining the results of eq. (11) in Section 2 above and eq. (27), the results are shown in the following corollary.

<u>COROLLARY 4</u>: When the demand curve becomes completely elastic, then the optimal tax rate can be calculated either in terms of the supply elasticity or the demand elasticity as follows (but at different points: F versus G),

$$t_{opt} = \frac{1}{\varepsilon^{F}_{supply}} = 0.04096 \quad \underline{or} \quad t_{opt} = \frac{\dfrac{\bar{P}}{1+t}}{\dfrac{d\bar{P}}{dt}(\varepsilon^{G}_{demand} - 1)} = 0.04096 \qquad (31)$$

Proof: t_{opt} being equal to 0.04096 was already given in eq. (30) above in terms of the supply elasticity. Regarding the demand elasticity, very large values for a and b are taken to make the demand elasticity very large, then

$$t_{opt} = \frac{\dfrac{\bar{P}}{1+t}}{\dfrac{d\bar{P}}{dt}(\varepsilon_d^G - 1)} = \frac{\dfrac{84.4}{1.04096}}{(5.92E-24)*((3.34E+26)-1)} = 0.04096 \tag{32}$$

The extreme value of the elasticity of demand (3.34E+26) in the denominator of eq. (32) and those of *a*, *b* are given in the second row of Table 3 below. Even though the supply elasticity must be calculated at point F, the elasticity of demand must be calculated at point G (Figure 3 above).

c) The demand curve becomes more inelastic starting from a perfectly elastic case

One can keep the initial equilibrium point fixed at point A (59.8 units, \$84.4) again starting with a perfectly elastic demand curve where the parameter *a* is infinitely large and the supply curve stays the same (c=700, d=9) as in Figure-4 below. As the parameter *a* decreases in the first column of Table 3, the demand curve becomes more and more inelastic (more vertical). The parameter *b* must adjust accordingly to keep point A fixed,

$$b = \frac{(a-59.8)}{84.4} \tag{33}$$

The optimal tax amount at each combination of (a,b) is obtained by (except b=0)

$$t_{opt} = \frac{(b+d)*(ad-bc)}{b*(bc+2cd+ad)} \quad \text{or} \quad t_{opt} = \frac{\dfrac{\bar{P}}{1+t}}{\dfrac{d\bar{P}}{dt}(\varepsilon_d^{C'} - 1)} \tag{34}$$

and the equilibria quantity and demand price in terms of parameters, and elasticity of demand (at each optimal tax amount) are,

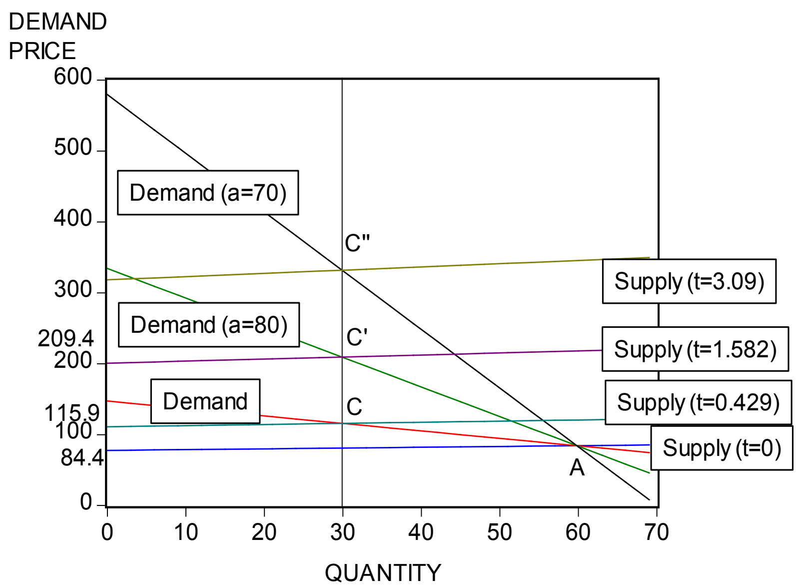
$$\bar{Q} = \frac{d \;\; - b \;\; * (1 + t_{opt})}{b * (1 + t_{opt}) + d} \qquad \bar{P} = \frac{(a + c) * (1 + t_{opt})}{b * (1 + t_{opt}) + d}$$

$$\varepsilon_d^{C'} = \frac{b * \bar{P}}{\bar{Q}} \tag{35}$$

The notation of C' in eqs. (34)and (35) refers to the constantly vertically upward shifting of point C in Figure 4 below (C' and C''), where the equilibrium demand price increases as the optimal tax amount is applied and the equilibrium quantity is always kept constant at 29.9 units (4th column of Table 3).

Firstly, notice that the 7th row of Table 3 coincides with those of Table 1 and Table 2, as a=140 and b=0.95 was the base example. Secondly, as it is observed, the lower the price elasticity of demand (6th column) is, given a supply curve, the higher are the tax rate (3rd column) and the tax revenues (7th column) of the government. Moreover, similar to the case where the supply curve becomes more inelastic, given a demand curve as in Theorem 2 above, one obtains the following figure.

Figure 4. Various Price Elasticities of Demand (c=700, d=9) and the Optimal Tax Rates with Corresponding Shifts and Tilts in the Supply Curves



<u>THEOREM 4</u>: Given a supply curve, as the demand curve becomes more inelastic, the consumers' share of tax burden increases for a given tax rate or for an optimal tax policy by the government for each value of after-tax demand elasticity. However, the consumers' shares of burden of tax are exactly the same for a given tax rate or for an optimal tax rate given a supply curve.

Proof: Using Theorem 2 in Section 3 above,

$$\text{CB} = \frac{(\bar{P} - P_1)}{(\bar{P} - P_s)} = \frac{d}{b+d} \qquad (36)$$

In other words, for given demand and supply curves, the consumers' share of tax burden is independent of the tax rate levied by the government: optimal or not. One observes in eq. (36) that as *b* goes from infinity to zero, the consumers' share of tax (CB) ranges over (0,1) as given in Table 3 below.

Table 3. Various Price Elasticities of Demand (c=700, d=9)

a	*b*	Optimal Tax amount (%)	Eq. Quantity	Eq. Demand Price ($)	Elasticity of Demand	Tax Revenue ($)	Consumers' Burden of Tax (CB) (%)	Producers' Nominal Tax amount ($)
1E+28	1.2E+26	0.0409	29.9	84.4	3.34E+26	99.33	0	99.33
90,000	1,065	0.0413	29.9	84.45	3,009	100	0.008	99.33
7,000	82.2	0.045	29.9	84.8	233	110	0.098	99.33
1,000	11.14	0.074	29.9	87.1	32	179	0.45	99.33
500	5.21	0.11	29.9	90.2	15	270	0.63	99.33
140	**0.95**	**0.429**	**29.9**	**115.9**	**3.682**	**1,040.4**	**0.9045**	99.33
80	0.24	1.582	29.9	209.4	1.7	3,835	0.97	99.33
59.8	0	∞	-	∞	-	∞	1	99.33

Finally, for a given supply curve, even though the share of the producers' burden of tax (1 –CB) decreases in percentage terms as the demand curve becomes more inelastic, the nominal amount ($99.33) the producers pay is constant (last column of Table 3).

5. Conclusion

This paper tried to tackle the issue of the maximum *ad valorem* tax collection of the government in the context of a single commodity market where the model was non-linear. The consistency of our calculations were checked by comparing the results from both the linear and non-linear models. Our efforts tried to emphasize the fact that the government must calculate the after-tax elasticity of demand (forwardlooking approach) rather than perhaps the unjustifiably common notion of pre-tax (initial equilibrium) elasticity which is unfortunately pretty-well established in the economics literature. Perhaps more importantly, if the government imposes an ad valorem tax on a product in a micro market, then the consumers' share of burden of tax does not change no matter what the tax rate is. Hence, in that sense, we have found some equity in taxation.

We summarize the results of the present study as follows:

A-1) If the government imposes an *ad valorem* tax on a product like, automobiles, liquor, cigarettes etc., the optimal tax rate depends on the elasticity of demand at the after-tax equilibrium. (Theorem-1).

A-2) If the government imposes an ad valorem tax rate on a product in a micro market such as automobiles, liquor, and cigarettes, then the consumers' share of burden of tax (and therefore the producers' burden of tax as well) does not change no matter what the tax rate is for given linear demand and supply curves (Theorem-2). Therefore, some equity is preserved when the government levies a tax on the public.

Some secondary results are as follows:

B-1) For a given demand curve, the steeper (less elastic) the supply curve, the greater the tax rate in order for the government to maximize its tax collections (Corollary-1).

B-2) When the supply curve is perfectly elastic, the Laffer effect can never occur if the demand curve at after-tax equilibrium is inelastic (Corollary-2).

B-3) Given a demand curve, as the supply curve becomes more inelastic, the consumers' share of tax burden decreases for optimal tax policies applied by the government for each predetermined position (elasticity) of the supply curve (Theorem-3).

B-4) In the case of a perfectly elastic demand curve, in calculating the optimal ad valorem tax rate, one must consider the after-tax supply elasticity along the initial supply curve (t=0) and not the before-tax supply elasticity at the initial equilibrium point or the after-tax elasticity at the final equilibrium (Corollary-3).

B-5) When the demand curve becomes completely elastic, then the optimal tax rate can be calculated either in terms of the supply elasticity or the demand elasticity (Corollary-4).

B-6) Given a supply curve, as the demand curve becomes more inelastic, the consumers' share of tax burden increases for a given tax rate or for an optimal tax policy by the government for each value of after-tax demand elasticity. However, the consumers' shares of burden of tax are exactly the same for a given tax rate or for an optimal tax rate given a supply curve (Theorem-4).

Moreover, there are two important topics of further research:

a) This paper considered the case where the government imposed an *ad valorem* tax starting from a tax rate of zero, which is the usual case considered in the literature. Therefore, the problem is not exposed in its entirety since in practice the government usually starts increasing the rate from a non-zero level. This important extension can be researched further.

b) This paper also investigated the case where the tax rate was calculated on producers' price which formed the tax basis to be added to the producers' price. The alternative situation where the tax rate is levied in proportion to consumers' price and eq.(12) above becomes $S = -c + d * P * (1 - t)$ and the formula for the Tax Revenues becomes

$TR = \bar{P} * t * \bar{Q}$ which can be analyzed along the lines outlined in this paper and the results can be compared. An important question is then whether the supply elasticity rather than the demand elasticity enters the formula for the revenue-maximizing tax rate.

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