Volume 14, Issue 2, Page 1-10, 2025 https://doi.org/10.46810/tdfd.1570735 Research Article

BİNGÖL UNIVERSITY

Türk Doğa ve Fen Dergisi
Turkish Journal of Nature and Science
www.dergipark.gov.tr/tdfd

Two Approaches For Solving Nonlinear Equation Systems: Newton Raphson and Red Fox

Bayram KÖSE [1], Bahar DEMİRTÜRK [2*], Şükran KONCA [3]

[1] Izmir Bakırçay University, Faculty of Engineering and Architecture, Department of Electrical and Electronics Engineering, 35660 Menemen, İzmir, Türkiye
[2,3] Izmir Bakırçay University, Faculty of Engineering and Architecture, Department of Fundamental Sciences, 35660 Menemen, İzmir, Türkiye
Bayram KÖSE ORCID No: 0000-0003-0256-5921
Bahar DEMİRTÜRK ORCID No: 0000-0002-5911-5190
Şükran KONCA ORCID No: 0000-0003-4019-958X

**Corresponding author: bahar.demirturk@bakircay.edu.tr*

(Received: 20.10.2024, Accepted: 27.02.2025, Online Publication: 27.06.2025)

Keywords
Root finding,
Nonlinear systems of equations,
Optimization algorithms,
Newton Raphson algorithm,
Red fox algorithm

Abstract: In this study, the results obtained by using the Red Fox method, a new metaheuristic optimization method, and the Newton Raphson method, which is one of the numerical methods, in finding the solutions of nonlinear systems of equations, are compared and these comparative analyses are evaluated and recommendations are presented.

Doğrusal Olmayan Denklem Sistemlerini Çözmek İçin İki Yaklaşım: Newton Raphson ve Kızıl Tilki

Anahtar Kelimeler
Kök bulma,
Doğrusal olmayan denklem sistemleri,
Optimizasyon algoritmaları,
Newton Raphson algoritması,
Kızıl tilki algoritması

Öz: Bu çalışmada, doğrusal olmayan denklem sistemlerinin çözümlerini bulmak için yeni bir meta sezgisel optimizasyon yöntemi olan kızıl tilki yöntemi ile sayısal yöntemlerden biri olan Newton Raphson yöntemi kullanılarak elde edilen sonuçlar karşılaştırılmış, bu karşılaştırmalı analizler değerlendirilmiş ve öneriler sunulmuştur.

1. INTRODUCTION

Many engineering problems cover a wide range of topics that require modelling and analysis of complex systems. These models involve many variables and the relationships between them. These relationships may not always be linear. Nonlinear systems of equations are quite common in engineering problems and appear in many fields such as experimental design, manufacturing optimization, materials science, heat transfer, fluid mechanics, control systems, etc.

The roots of these equations are critical for solving the problem. For example, to determine the optimum conditions of a chemical reaction, it is necessary to find the roots of a nonlinear system of equations representing the rate of reaction or the design of a heat exchanger requires the solution of a nonlinear system of equations modelling heat transfer [1].

Engineering problems often involve systems of nonlinear equations. In experimental design, production and development phases, the root of these equations gives important information for decision making. The methods used for solving engineering problems should be chosen depending on the specific nature and complexity of the problem.

Solutions of systems of nonlinear equations can be more complex than solutions of linear equations. This is because analytical solutions of nonlinear equations are not always possible and numerical methods or metaheuristic optimization techniques are needed to solve these problems. Analytical methods, graphical methods, numerical methods, simulation and optimization techniques, software and computational tools are widely used to solve such systems. Numerical methods, which are widely used for solving systems of nonlinear equations, are usually iterative techniques and are used to calculate approximate values of the solution when the solution cannot be expressed by an exact formula. The most widely used of these numerical methods is the Newton Raphson method. Metaheuristics are also powerful approaches that can be very effective in solving systems of nonlinear equations. These methods are generally developed for optimization problems and are known for their ability to find global solutions in complex and multidimensional solution spaces. The most widely used metaheuristics are genetic algorithms, particle swarm optimization, simulated annealing, tabu search, differential growth algorithm, etc.

When simulating and analysing aircraft motion in flight, a nonlinear model with appropriate initial conditions is used. Millidere et al. [2] used classical Newton Raphson method to solve a nonlinear system of algebraic equations to find the trim condition, a point at which the aircraft flight conditions should not change abruptly.

Gower et al. [3] proposed a new randomised method for solving systems of nonlinear equations by taking the gradients of the component functions and using Bregman projections onto the solution space of a Newton equation, which can find sparse solutions or solutions under certain simple constraints.

Pourrajabian et al. [4] solved three different nonlinear algebraic equation systems, a single nonlinear equation, a simple set of nonlinear equations and a complex set of nonlinear equations by using genetic algorithms.

Kotsireas et al. [5] gave a synthesis of the literature on the solution of systems of nonlinear equations and they aimed to assist interested readers who wish to identify appropriate solution techniques for solving various systems of nonlinear equations that may be encountered in real-world applications. Moreover, Odan [6] investigates the effectiveness of Genetic Algorithms in solving both linear and nonlinear systems of equations and compares their performance with conventional methods such as Gaussian Elimination, Newton's Method and Levenberg-Marquardt. Verma and Parouha [7] presented an innovative hybrid algorithm of Particle Swarm Optimization Algorithm and Evolutionary Algorithms to find the solution of nonlinear equation systems.

There are many problems raised in Chapra's book that have attracted the attention of researchers from many different disciplines. For example, the torsional buckling problem of open constant bisymmetric cross section constrained thin-walled bars was solved by the Newton Raphson method in [8] and it attracted the attention of Kujawa [9]. Also, a problem given by Chapra and Canale in [8] given by the equation governing the L-C-R circuit in electrical engineering is considered by Sharma et al. in [10].

Chapra and Canale [8] found approximate solutions to a system of nonlinear equations using the fixed-point iteration method, a derivative-free root-finding method and the Newton Raphson method, a derivative root-finding method. From this point of view, in this study, the Red Fox method, a metaheuristic optimization method, is used to find the solutions of the given system of nonlinear equations and the results are compared with the Newton Raphson method. The original aspect of this study is the use of a new metaheuristic method, the Red Fox search algorithm, in solving a nonlinear system of equations and comparing it with a numerical method, the Newton Raphson method, and presenting the evaluated results and recommendations.

2. SYSTEMS OF NONLINEAR EQUATIONS

A linear equation, for $1 \le i \le n$ and with a_i, b constants, can be defined in general form $f(x) = a_1x_1 + a_2x_2 + ... + a_nx_n - b = 0$. The solution of this linear equation is $x = (x_1, x_2, ..., x_n)$. The solution of the system of equations,

$$\begin{aligned} f_1(x_1, x_2, ..., x_n) &= 0 \\ f_2(x_1, x_2, ..., x_n) &= 0 \\ &. \\ &. \\ &. \\ f_n(x_1, x_2, ..., x_n) &= 0 \end{aligned} \tag{1}$$

consisting of n equations and n unknowns can be defined as finding the value $x = (x_1, x_2, ..., x_n)$ of all equations at the same time, which results in all equations being equal to zero [8].

Equations that do not obey the form $f(x) = a_1x_1 + a_2x_2 + ... + a_nx_n - b = 0$ are called nonlinear equations.

Example. The equations $x^2 + xy = 10$ and $y + 3xy^2 = 57$ are nonlinear equations with two unknowns. These equations can also be expressed in the form

$$\begin{aligned} u(x, y) &= x^2 + xy - 10 = 0 \\ v(x, y) &= y + 3xy^2 - 57 = 0. \end{aligned} \tag{2}$$

Therefore, the solution will be the values of x and y that set the functions $u(x, y)$ and $v(x, y)$ equal to zero. Most of the approaches used to find such solutions are extensions of explicit methods for solving single equations. In [8], Chapra and Canale considered two of these methods, fixed point iteration and Newton Raphson methods, to find the solution of the system of equations given in (1).

In this study, the solutions of the system of equations given by (1) are found by using the Newton Raphson method from numerical methods and the Red Fox optimization method from metaheuristics methods and the results obtained are compared.

3. METHODS

In this section, the Newton Raphson method, the most widely used numerical method, and the Red Fox Optimization algorithm, one of the metaheuristics, will be discussed.

Numerical methods are highly effective tools that are frequently used in solving systems of nonlinear equations. These methods combine mathematical and computer science tools to provide solutions to complex problems for which analytical solutions cannot be obtained. Numerical methods, which usually follow an iterative approach, aim to obtain approximate values by approaching the solution step by step. Powerful approaches such as metaheuristics are used to solve engineering problems that cannot be solved by analytical methods. Metaheuristic algorithms used in solving optimization problems aim to reach the best solution with heuristic approaches. These algorithms are used with mathematical optimization algorithms. While mathematical optimization algorithms aim to reach a solution by scanning the entire solution set, metaheuristic optimization algorithms aim to reach the best solution or a near-optimal solution through heuristic approaches. The performance of these algorithms can vary depending on the problem type and test functions. They usually consist of models inspired by nature and mimicking the behavior of biological systems.

3.1. Newton Raphson Algorithm

The Newton Raphson method, which has a very fast convergence rate, is one of the most popular numerical methods that iteratively approaches the solution using derivative information and provides an efficient solution. This method uses the values of the function and the derivative function to generate a new point closer to the solution at each iteration. Although the method has some limitations, it provides a practical and reliable solution to many problems in terms of computational efficiency.

Select an arbitrary value x_n and calculate the value of $f(x_n)$ corresponding to this point. After this point, a tangent is drawn to the function at the point $(x_n, f(x_n))$. The point where the tangent crosses the x-axis is taken as the new value x_{n+1} and the same procedure is applied for x_{n+1}.

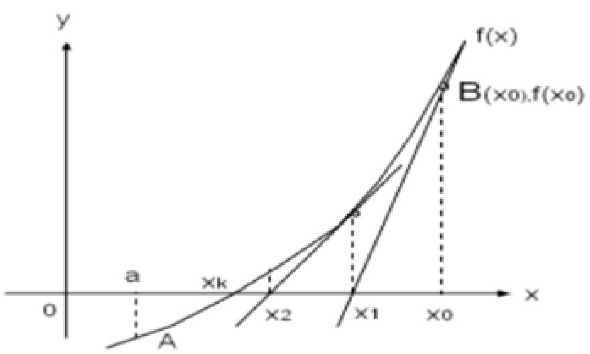


Figure 1. Newton Raphson method

The Newton Raphson method is based on the following iterative formula for algebraic equations in one variable:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The formula for systems of equations in two variables is given as follows:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1}_{(x_0,y_0)} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_{(x_0,y_0)}$$

The convergence rate of the Newton Raphson method shows a quadratic convergence rate. This means that with each iteration the distance to the solution decreases quadratically. This property makes the method very fast and efficient. This is also the reason why Newton Raphson is used in calculators that can perform solutions. Only one starting point needs to be chosen instead of an interval as in the bisection method. The success of the Newton Raphson method depends on the choice of this starting point. If the starting point is not chosen close enough to the solution, the method may not converge or may reach an incorrect solution. Therefore, it is important to determine a suitable starting point before applying the method.

The Newton Raphson method may not give the desired result in some cases, because of some limitations of the method. The function and derivative function must be continuous and differentiable. From the quotient expression in the denominator, division by zero can be a problem when the derivative is zero. This is since the tangent is parallel to the x-axis and does not intersect the x-axis at some point and cannot determine the new x-value. The starting point should be chosen close enough to the solution. If the solution has more than one root, the method may not always converge to the desired root or may only converge to a single root. In addition, the function may have problems such as oscillating around the root and accidentally missing the root [11]. The flowchart of Newton Raphson method can be given as follows:

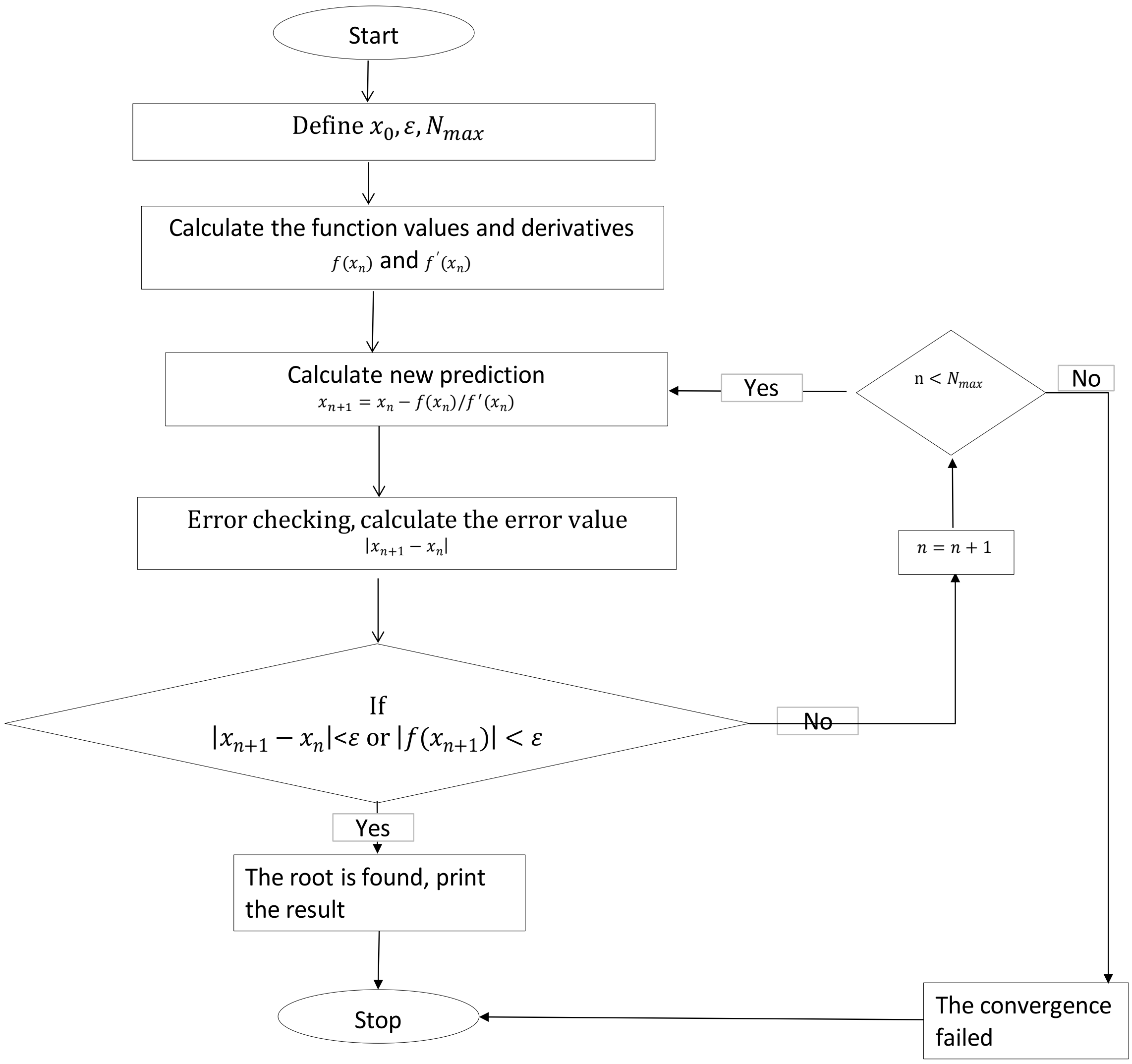


Figure 2. The flowchart of Newton Raphson method

3.2. Red Fox Optimization Algorithm

The advantages of metaheuristics include their applicability to complex and multidimensional problems where analytical methods are inadequate; their potential to find the global solution without getting stuck in the local optimum; and their stochastic nature, which allows them to produce different solutions in different runs. This can help to explore different aspects of the problem and increase the chances of finding the best solution. The Red Fox Algorithm is inspired by the hunting and herding behaviour of red foxes. The algorithm aims to find solutions to optimization problems by using concepts such as predator-prey relationship and within-pack hierarchy. In this section, the Red Fox Optimization Algorithm developed by Połap and Woźniak [12] based on the mathematical model of red fox habits, foraging, hunting and population is described in detail.

The Red Fox Algorithm is inspired by complex behaviours in nature. This gives the algorithm robustness and flexibility. The Red Fox Algorithm has the potential to find the global solution without getting stuck in the local optimum.

A population of red foxes consists of individuals living in specific territories and leading a nomadic life [13]. Each pack shares a single territory under the hierarchy of the alpha pair. When youngsters grow up, they can leave the pack and start their own pack if they have a good chance of taking over another territory. Otherwise, they stay in the family and one day take over the hunting grounds from their parents. The red fox is an effective predator of small animals, both domestic and wild animals. Foxes look for food at every opportunity as they roam the territory, sneaking up on their prey and hiding until they get close enough for an effective attack. This algorithm models foraging as global search, where foxes explore the area while their prey is away. In the second stage, roaming the habitat to get as close as possible before hunting is modelled as local search [14]. The conceptual model of this hunting behavior of the fox is shown in Figure 3.

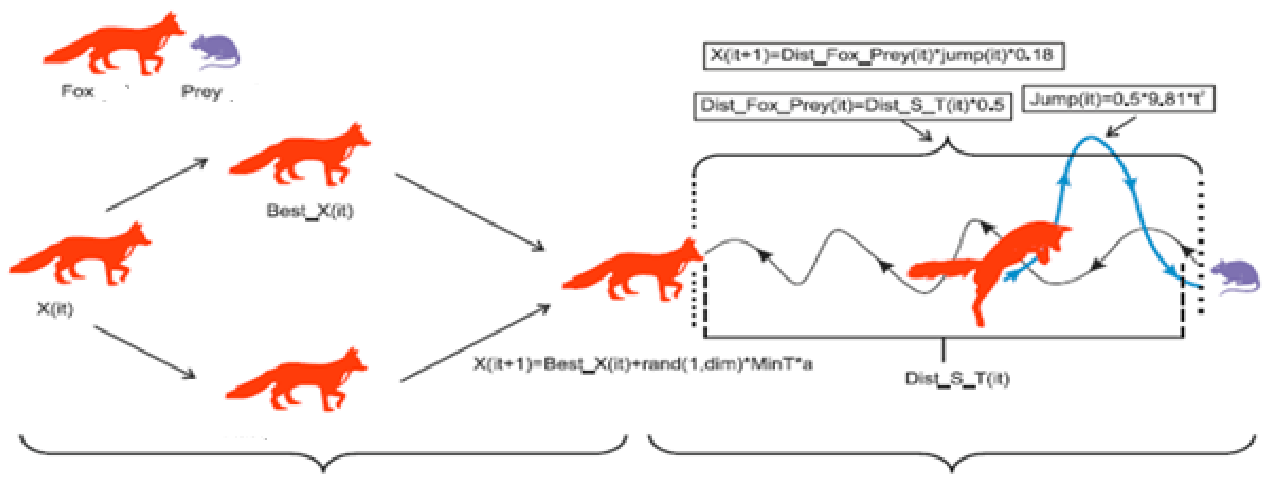


Figure 3. Red fox predation impact: reproduction and exploitation [14]

The red fox algorithm mimics the hunting behavior of the red fox as it dives under the snow to catch its prey. The basic techniques are based on the red fox trying to catch the best prey. The steps of red fox behavior are described below:

1. When snow covers the ground, the snow blocks the sight of the prey. The red fox tries to search for prey randomly.
2. The red fox finds the prey by hearing the ultrasound sound of the prey. However, it takes time for the fox to get close to the prey.
3. By listening to the sound of the prey and calculating the time difference, the red fox can determine the distance between itself and the prey.
4. After determining the distance, the red fox estimates the jumping distance needed to catch prey.
5. The walking process is randomized according to the minimum time and the best position.

Initially, the red fox moves randomly in the search space to discover the prey. This random movement is used to provide exploratory behavior. While searching, the red fox can hear the prey. Upon hearing the sound, the red fox enters the exploitation phase. It takes time for the sound of prey to reach the fox. The distance the sound travel can be calculated by multiplying the speed of sound in air by 343 m/s. The red fox tries to move forward to catch the prey according to the reception of the sound and decides to jump on the prey and tries to jump depending on the time it takes for the sound to be received. Červený et al. [15] showed that the fox prefers to jump in the northeast direction based on the magnetic alignment effect. In the study, it was found that the fox had an 82% chance of catching prey when jumping in the northeast direction, while it had an 18% chance of catching prey when jumping in the opposite direction.

The basic algorithm proposition assumes that at each iteration, individuals consist of a fixed number of foxes. Each is represented as a point with n coordinates $x = (x_0, x_1, \ldots, x_{n-1})$. Each fox is distinguished by the notation $((x_i^j)_t)$ at iteration t, where i is the number of foxes in the population and j is the coordinate with respect to the dimensions of the solution space. Foxes are assumed to search for solutions in the criterion function for optimal values.

In Search of Food - Global Search Phase: In a pack, each fox needs to play an important role for the survival of all family members. When food is not available in local habitats or to explore other territories, pack members travel to distant destinations. They share the knowledge they gain from this exploration with the family.

Roaming Local Habitat - Local Search Phase: The red fox roams its territory looking for potential prey. When it sees a possible prey, the fox begins to sneak up on it to get as close as possible without being noticed. Therefore, it circles and deceives the prey to trick it and convince it that the prey is not interested in it.

Reproduction and Pack Separation: In the wild, the red fox faces many dangers. There may be no food in the local habitat area, so it may be necessary to move elsewhere. Another danger comes from humans. If there are large losses in the pet population, people may hunt the fox.
To model these behaviors, in each iteration we select the worst 5% of the population according to the criterion function value. This is assumed to be foxes that leave the pack or are shot by hunters. To keep the number of individuals in the population constant, we replace them with new individuals using the habitat territory established by the alpha pair.

In this work, we define the number of foxes numFoxes = 40, maximum iterations numIterations = 100, lower limits lb = [0, 0], upper limits ub = [15, 15], size nVar = length(ub), to identify the worst 5% foxes alpha_fraction = 0.05, Alpha pair reproduction threshold is 0.44, $c_1 = 0.18$ and $c_2 = 0.82$, p variable is selected by the interval [0,1] randomly, and a random variable r is used to balance the exploration and exploitation phases.

The following parameters are taken and the steps below are followed in the flowchart of Red Fox Algorithm given by Figure 4.

$X_{(it+1)}$: The new position of the red fox after jumping,
$BestX_{it}$: The best solution ever found,
$rand(1, \text{dimension})$: Value used to allow the fox to discover prey by random walking,

$MinT$: Minimum time variable,
a : A value to reduce the search performance based on $BestX_{it}$ value,
$Dist\_Fox\_Prey_{it}$: Distance of the red fox from the prey,
$Jump_{it}$: The jumping value of red fox.

Th following parameters are taken and the steps below are followed in the flowchart of Red Fox Algorithm, given in Figure 4.

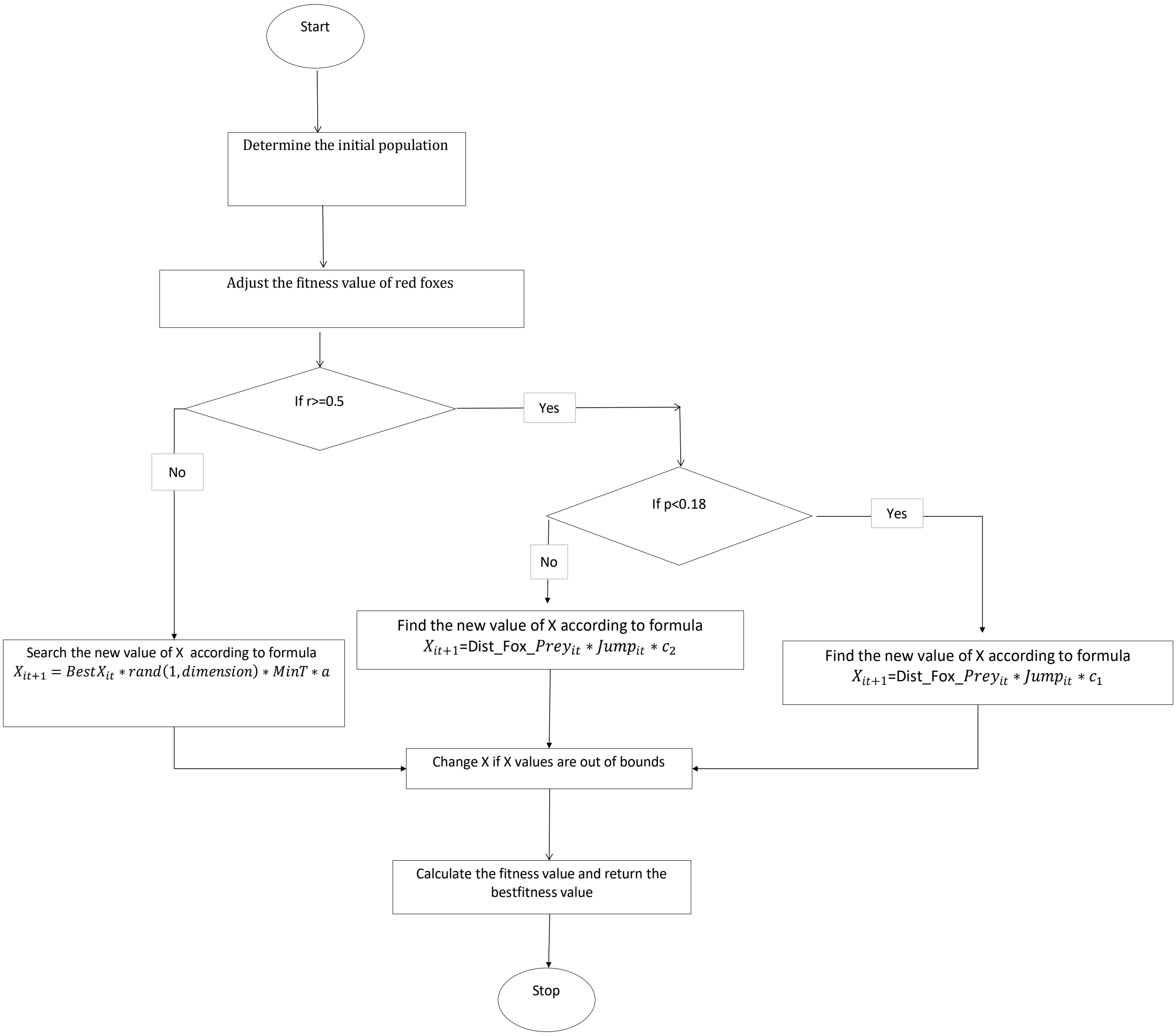


Figure 4. The flowchart of Red Fox Algorithm [16]

3.3. Optimization Approach for Finding the Roots

When the optimization process is used to find the roots of algebraic equations, the problem of finding the unknown values in each equation becomes an optimization problem to be solved by numerical methods. Optimization is the process of obtaining the best value of an objective function according to specified criteria. Since the numerical approach to finding roots in algebraic equations usually involves an iterative process, similarly, in finding roots with an optimization algorithm, starting from a given starting point, candidate root values are iteratively updated and reach a minimum or maximum value when the objective function is sufficiently close or a certain tolerance value is reached.

As an explicit consequence of Intermediate Value, Bolzano and Extreme Value Theorems which are known from classical Mathematical Analysis, we can give Theorem 3.3.1. Moreover, we will give Theorem 3.3.2 as a generalization of Theorem 3.3.1 for equations in one variable for finding roots in algebraic equations.

Theorem 3.3.1. (Root Search in Optimization Algorithm)

Let $I=[a,b]$ and $I \subset \mathbb{R}$. If the function $f{:}I \rightarrow R$ is continuous, then it has at least one minimum on this interval and if $|f(x_i)|=0$ then there exists at least one $x_i \in I,\ (i \in N)$ satisfying this equality (Köse et al., [17]).

Theorem 3.3.2. (Root Finding Algorithm for Nonlinear Equation Systems)

Let $I=[a,b]$ and $I \subset \mathbb{R}$. If the functions $f_i : I^n \rightarrow \mathbb{R}$ are continuous, then for each $1 \leq i \leq n$ the functions f_i have at least one minimum value in this interval and have at least one point $x=\left(x_1,x_2,...,x_n\right) \in I$ that satisfies the equality $\sum_{i=1}^{n}\left|f_i(x_i)\right|=0$ (Köse et al., [17]).

4. PERFORMANCE EVALUATION AND ANALYSIS

In this section, the analysis of performance of two methods for approaching the roots is given. Firstly, we can examine the approximation graphics of Newton Raphson and Red Fox methods in Figure 5 for the first case.

As seen from Figure 5 (a), with Newton Raphson Method, in the initial iterations, large oscillations towards the solution point are observed. By the 15th iteration, the solution is quite close to the root, but the errors have not completely vanished. This case demonstrates that the Newton Raphson method is sensitive to the initial point and may not converge quickly if the starting value is far from the true root. Regardless of the number of iterations, it always finds the true root in the given nonlinear equation system. However, a poor initial guess can significantly increase the number of iterations required.

Unlike Newton Raphson Method, the Red Fox algorithm does not rely on the initial point and explores a wide search space. In the early iterations, a random search process is observed, while by the 85th iteration, it has approached the root significantly, as seen from Figure 5 (b). Also, it exhibits a more stable convergence pattern, which allows it to find a solution even in cases where the Newton Raphson method struggles. However, it requires more iterations and its convergence is irregular, which can be a disadvantage.

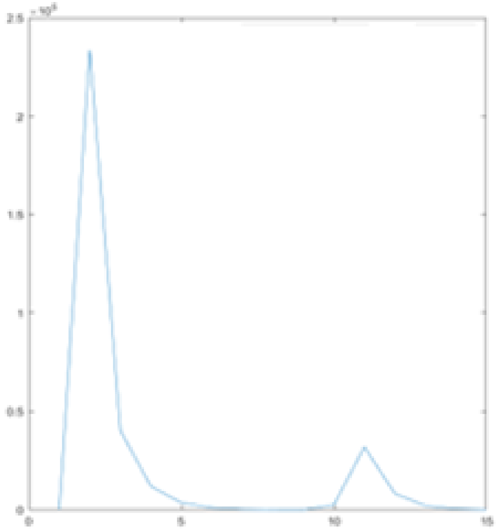


Figure 5. (a) The approximation graphic of Newton Raphson method for the initial values (x,y)=(8,0) case

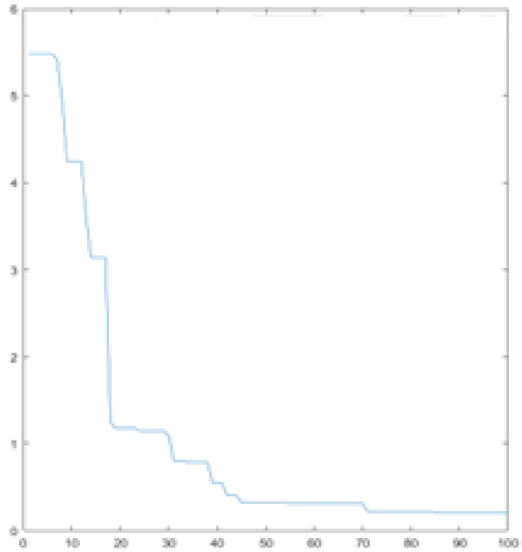
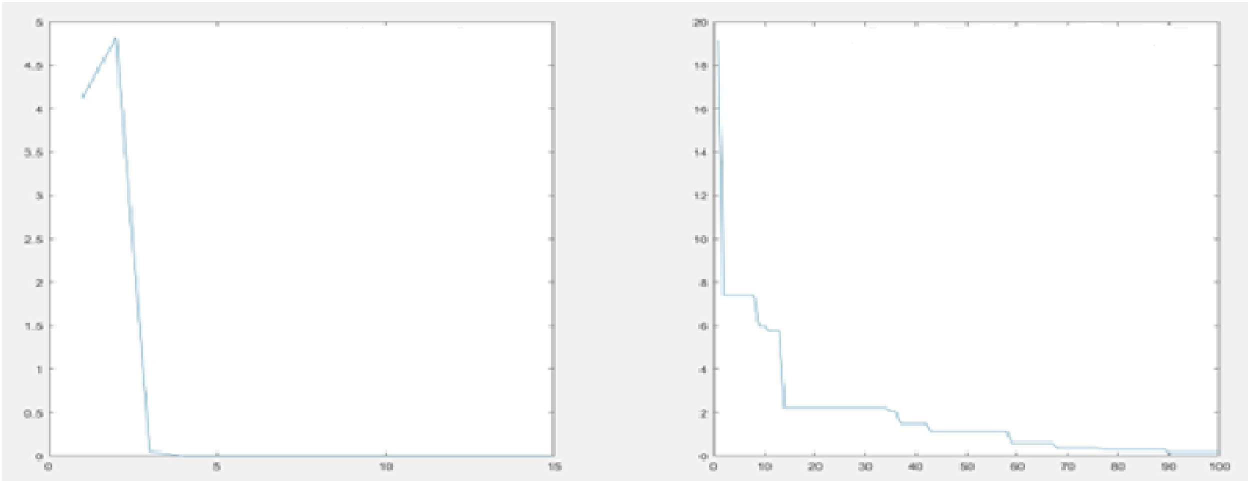


Figure 5. (b) The approximation graphics of Red Fox methods for (x-best,y-best)=(2.6963,2.6200) location case

Now we can give the approximation graphics of Newton Raphson and Red Fox methods in Figure 6 for the second case.



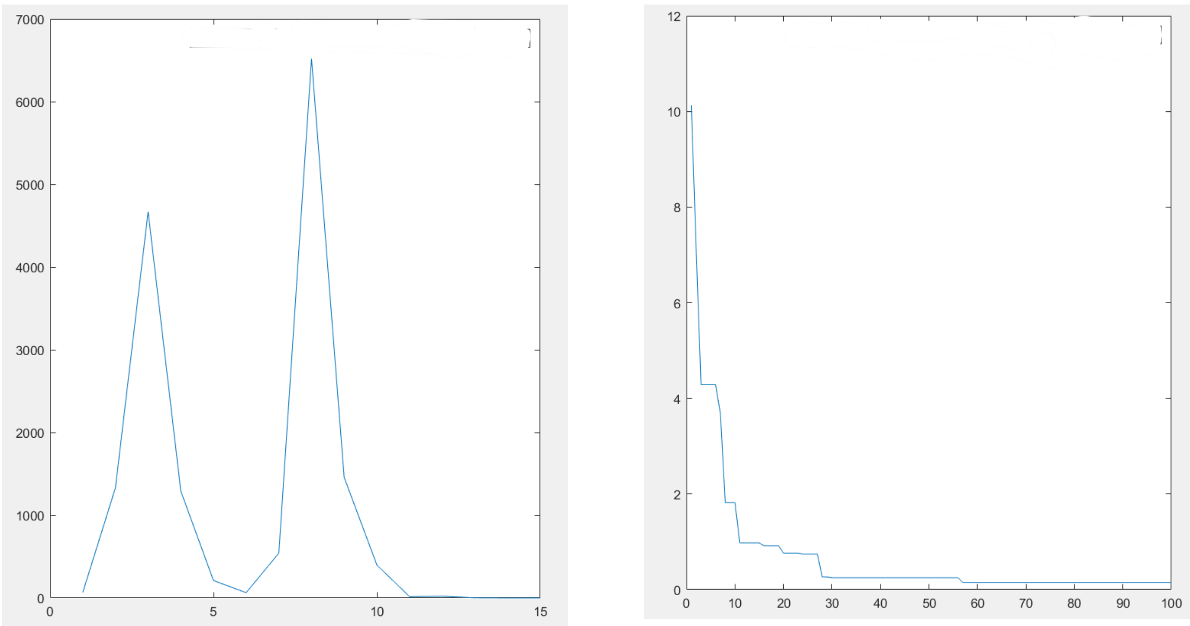
(a) (b)

Figure 6. (a) The approximation graphic of Newton Raphson method for the initial values *(x,y)=(1.5,3.5)* case
(b) The approximation graphic of Red Fox methods for (x-best,y-best)= (0.5,6.2403) location case.

In Case 2, given in Figure 6 (a), with the Newton Raphson Method, a more stable convergence is observed from the 3rd iteration. The errors are significantly reduced and in the 4th iteration, the root is found exactly. Since the initial point is close to the true root, the Newton Raphson method quickly reaches the solution. This case highlights the efficiency of Newton Raphson when a good initial guess is chosen, as it converges much faster than in Case 1.

As seen from Figure 6 (b), with the Red Fox Algorithm, a broad search process is observed in the initial iterations, but the solution is not fully reached until the 86th iteration. Although it eventually gets very close to the root, the convergence is much slower compared to Newton Raphson. In this case, Newton Raphson performs significantly better, as it reaches the solution in just four iterations. While the Red Fox algorithm has the advantage of not depending on the initial point, it requires more time to converge in this scenario.

Finally, we can give the approximation graphics of Newton Raphson and Red Fox methods in Figure 7 for the third case.



(a) (b)

Figure 7. (a) The approximation graphic of Newton Raphson method for the initial values (x,y)=(0.5,0.5) case
(b) The approximation graphic of and Red Fox methods for (x-best,y-best)= (0.5291, 5.3180) case.

In Case 3 given by Figure 7 (a), with the Newton Raphson Method, large errors and an unstable convergence process are observed in the first iterations. The true root is not fully reached until the 13th iteration, and only in the 15th iteration does it fully converge. Since the initial point is far from the root, the method requires more iterations to converge. The error rate is very high at the beginning, but as the iterations progress, it stabilizes and approaches the correct root.

In Case 3, given by Figure 7 (b), with the Red Fox Algorithm, a broad search process continues in the initial iterations, but by the 63rd iteration, the solution is significantly approached. The Red Fox method performs better in cases where Newton Raphson struggles. However, it again requires more iterations than Newton Raphson and shows a slower convergence rate. In this case, Red Fox has an advantage because it does not require derivative information, but when Newton Raphson is applied correctly, it converges in fewer iterations and is more efficient.

Moreover, the roots and errors obtained by Newton Raphson and red Fox methods can be seen in Table 1, for three cases.

Table 1. The roots and errors of Newton Raphson and Red Fox Methods

Newton Raphson Method						Red Fox Method				
Initial Values (x, y)	**Iteration Number**	**x**	**y**	**Error of x**	**Error of y**	**(x-best, y-best) location**	**x-best**	**y-best**	**Error of x**	**Error of y**
(8,0) case	1	8.000	0.000	6.000	3.000	(2.6963, 2.6200)	2.6963	2.6200	0,6963	0,38
	5	3.500	4.200	2.500	1.800		2.5000	3.5000	0,5	0,5
	10	2.000	5.500	1.000	1.500		2.1000	3.9000	0,1	0,9
	15	**1.334**	**6.636**	**0.666**	**3.636**		2.0100	3.7500	0,01	0,75
	50	1.334	6.636	0.666	3.636		2.0000	3.2000	0	0,2
	85	1.334	6.636	0.666	3.636		**1.9639**	**3.0269**	**0,0361**	**0,0269**
(1.5,3.5) case	1	1.500	3.500	0.500	0.500	(0.5, 6.2403)	0.5000	6.2403	1,5	3,2403
	2	1.800	3.200	0.200	0.300		0.8000	5.5000	1,2	2,5
	3	1.9987	3.0023	0.0013	0.0023		1.2000	4.7000	0,8	1,7
	4	**2.000**	**3.000**	**0.000**	**0.000**		1.7500	2.7800	0,25	0,22
	50	2.000	3.000	0.000	0.000		1.9990	3.0010	0,001	0,001
	86	2.000	3.000	0.000	0.000		**1.9994**	**3.0003**	**0,0006**	**0,0003**
(0.5,0.5) case	1	0.500	0.500	1.500	2.500	(0.5291, 5.3180)	0.5291	5.3180	1,9999	2,318
	3	1.000	1.500	1.000	1.500		1.1000	4.0000	0,9	1
	6	1.400	0.500	0.600	2.500		1.5000	3.5000	0,5	0,5
	10	1.800	2.100	0.200	0.900		1.9000	3.2000	0,1	0,2
	13	2.000	3.100	0.000	0.100		1.9600	3.1250	0,04	0,125
	15	**2.000**	**3.000**	**0.000**	**0.000**		1.9100	3.100	0,09	2,69
	63	2.000	3.000	0.000	0.000		**2.0044**	**2.9967**	**0,0044**	**0,0033**

When these three graphs are analyzed, it is seen that in the Newton Raphson method, one of the numerical methods, when the starting point is far from the true root, the graphs oscillate a lot and almost find the true root around the 15th iteration. This supports the assertion that “the starting point should be chosen close enough to the solution”, which is one of the constraints in the Newton Raphson method mentioned earlier. No matter how many iterations it takes, it always finds the true root in the nonlinear equation system we considered.

In the Red Fox method, it is observed that the approach to the true root occurs at different iteration numbers regardless of the initial positions. No regular relationship was observed between the initial position and the number of iterations to approach the true root. However, this metaheuristic shows that it is a preferable method as it produces results very close to the true root where the Newton Raphson method gets stuck. The problem was obtained by running the MATLAB program with the other parameters given above up to a maximum of 100 iterations with 40 foxes. It was observed that as the parameters such as the number of populations, maximum number of iterations, initial positions, etc. were changed, the number of steps we approached the true root varied.

5. CONCLUSIONS AND RECOMMENDATIONS

From the performance results obtained above, it can be said that metaheuristic methods as well as the Newton Raphson method, which is a mathematical optimization method, are powerful approaches that can be very effective in solving systems of nonlinear equations.

Since the Newton Raphson method relies on derivative information, it can provide rapid convergence. However, when the initial point is not chosen appropriately, significant oscillations may occur, leading to delays in convergence. Particularly, when the initial point is far from the true root, the convergence process slows down, and the number of iterations increases. Nevertheless, when an appropriate initial point is selected, the Newton Raphson method quickly finds the solution.

On the other hand, the Red Fox algorithm offers a global search capability as it does not depend on the initial point. This algorithm can find solutions in cases where Newton Raphson fails, particularly when derivative information is zero or when dealing with complex functions. However, it generally requires more iterations, and its convergence pattern is irregular.

Although there are studies similar to Chapra and Canale's work in the literature, the number of comparative studies focusing on different methods is noteworthy. We believe that these comparative studies will provide new perspectives for both practitioners and theoretical scientists. Depending on problem's nature, selecting the appropriate method is crucial for achieving efficient and reliable solutions. For problems requiring rapid convergence, where the initial point can be accurately chosen, the Newton Raphson method may be more effective. For problems where the effect of the initial point is uncertain or where derivative information is ambiguous, the Red Fox algorithm can be considered as an alternative approach. Hybrid approaches combining the advantages of both Newton Raphson and Red Fox algorithms can be developed for complex systems. Systematic optimization of the Red Fox algorithm's parameters such as population size and maximum number of iterations can improve convergence speed and stability.

REFERENCES

[1] Victor YP. Root-Finding and Root-Refining for a Polynomial Equation. Computer Science Technical Reports: City University of New York (CUNY); 2012.

[2] Millidere M, Karaman U, Uslu S, Kasnakoğlu C, Çimen T. Newton Raphson methods in aircraft trim: A comparative study. In: AIAA AVIATION 2020 Forum. American Institute of Aeronautics and Astronautics; 2020. p. 3198.

[3] Gower R, Lorenz DA, Winkler M. A Bregman–Kaczmarz method for nonlinear systems of equations. Comput Optim Appl. 2023;87(3):1059–1098.

[4] Pourrajabian A, Ebrahimi R, Mirzaei M, Shams M. Applying genetic algorithms for solving nonlinear algebraic equations. Appl Math Comput. 2013;219(24):11483–11494.

[5] Kotsireas IS, Pardalos PM, Semenov A, Trevena WT, Vrahatis MN. Survey of methods for solving systems of nonlinear equations, Part II: Optimization based approaches. arXiv. 2022. Available from: https://arxiv.org/abs/2208.08532

[6] Odan S. Metaheuristic method for solving systems of equations. arXiv. 2024. Available from: https://arxiv.org/abs/2409.16958

[7] Verma P, Parouha RP. Solving systems of nonlinear equations using an innovative hybrid algorithm. Iranian J Sci Technol Trans Electr Eng. 2022; 46:1005–1027.

[8] Chapra RP, Canale SC. Numerical Methods for Engineers. New York: McGraw-Hill Higher Education; 2002.

[9] Kujawa MA. Torsional buckling of restrained thin-walled bars of open bisymmetric cross-section. TASK Quarterly. Scientific Bulletin of Academic Computer Centre in Gdansk. 2012;16(3-4).

[10] Sharma JR, Kumar S, Cesarano C. An efficient derivative free one-point method with memory for solving nonlinear equations. Mathematics. 2019;7(7):604.

[11] Burden RL, Faires JD, Burden AM. Numerical Analysis. 10th ed. Canada: Cengage Learning; 2015.

[12] Połap D, Woźniak M. Red fox optimization algorithm. Expert Syst Appl. 2021; 166:1-21.

[13] Larivière S, Pasitschniak AM. Vulpes vulpes. Mammal Species. 1996; 537:1–11.

[14] Mohammed H, Rashid T. FOX: a FOX-inspired optimization algorithm. Appl Intell. 2023;53(1):1030–1050.

[15] Červený J, Begall S, Koubek P, Nováková P, Burda H. Directional preference may enhance hunting accuracy in foraging foxes. Biol Lett. 2011;7(3):355–357.

[16] Bektaş Y. Kızıl Tilki Optimizasyon Algoritması. Teknobilim 2023: Optimizasyon Modelleme ve Yapay Zeka Optimizasyon Algoritmaları. İstanbul: Efe Akademi; 2023.

[17] Köse B, Demirtürk B, Konca Ş. Application of Search Algorithms to Root-Finding Problems in Algebraic and Transcendental Contexts. 2025. (accepted in European Journal of Pure and Applied Mathematics)