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RESEARCH PAPER

Dynamics of a stochastic SEIQR model: stationary distribution and disease extinction with quarantine measures

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Abstract

This paper investigates the dynamics of a stochastic $\mathcal{SEIQR}$ epidemic model, which integrates quarantine measures and a saturated incidence rate to more accurately reflect real-world disease transmission. The model is based on the classical $\mathcal{SEIR}$ framework, with the addition of a quarantined compartment, offering insights into the impact of quarantine on epidemic control. The saturated incidence rate accounts for the diminishing rate of new infections as the susceptible population grows, addressing the limitations of traditional bilinear incidence rates in modeling epidemic spread under high disease prevalence. We first establish the basic reproductive number, $\mathcal{R}_0$, for the deterministic model, which serves as a threshold parameter for disease persistence. Through the stochastic Lyapunov function method, we identify the necessary conditions for the existence of a stationary distribution, focusing on the case where $\mathcal{R}_0^* > 1$, signals the potential long-term persistence of the disease in the population. Furthermore, we derive sufficient conditions for disease extinction, particularly when $\mathcal{R}_S^* < 1$, indicating that the disease will eventually die out despite the inherent randomness in disease transmission. Numerical simulations confirm that environmental noise and quarantine rates shape disease dynamics. Simulations show that more noise or higher quarantine rates speed up disease extinction, offering key policy insights. Our results clarify how quarantine, noise intensity, and disease dynamics interact, aiding epidemic modeling in stochastic settings.

Keywords: Stochastic epidemic model; Lyapunov function; stationary ergodic distribution; extinction

AMS 2020 Classification: 37M05; 37N25; 34D08; 60G10

1 Introduction

Mathematical models have been indispensable in understanding infectious disease dynamics since Daniel Bernoulli's pioneering work in 1766 [1]. These models elucidate disease transmission

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dynamics and analyze the behavior of diseases among populations with varying health statuses. Notably, Kermack and McKendrick's research in 1927 [2] explored infectious disease dynamics using mathematical models, paving the way for numerous subsequent models aimed at understanding epidemic behavior and controlling its spread. The development of stochastic epidemic models, stemming from simple deterministic models, has enabled accurate predictions of disease dissemination and facilitated public health awareness campaigns. Such models are instrumental in preventing disease dissemination and reducing infection rates in society.

Stochastic epidemics trace back to the 1920s, with McKendrick developing the first stochastic SIR model in 1926 [2]. Researchers like Anderson, Roy, Daley, and Gani have since contributed to the field by analyzing infectious disease epidemiology through mathematical models [3, 4]. Their work has focused on determining transmission probabilities for infectious agents and examining the effects of interventions such as vaccination and quarantine, offering theoretical and numerical frameworks for disease prevention and control. There has been a significant increase in the application of mathematical models to investigate mechanisms within infectious diseases such as polio, diphtheria, tuberculosis, HIV, COVID-19, and others [5–11]. Quarantine emerges as a crucial method for preventing disease dissemination, as evidenced by its historical efficacy in reducing the spread of various human and animal diseases. Therefore, studying infectious disease models that incorporate quarantine strategies is essential.

The main objective of this study is to develop a stochastic SEIQR epidemic system incorporating temporary immunity, quarantine strategies, and random perturbations. While providing detailed insights into disease persistence, stochastic models may lack positive equilibrium due to environmental noise interference [12, 13]. Understanding ergodicity theory and stationary distributions is crucial for comprehending epidemic transmission patterns and estimating statistical properties essential for effective disease prevention.

Additionally, several mathematical models investigate infectious disease dynamics under quarantine models [14, 15]. For instance, Dieu et al. [16] have developed the threshold of a stochastic SIQS epidemic model with standard isolation, while Zhou et al. [17] have investigated an SQEIAR stochastic epidemic model with media coverage and asymptomatic infection. Zhang et al. [18] proposed the stationary distribution and extinction of a stochastic SEIQ epidemic model with a general incidence function and temporary immunity. Currently, researchers are actively investigating the SIQR model [19–21].

Therefore, many mathematical biologists consider more realistic factors, such as demographic changes, migration, cross-infections, and other practical elements. Every time infectious diseases affect people, people take precautions to minimize their impact. The quarantine method has been used for controlling contagious diseases, which is one of the most effective ways to prevent epidemic disease outbreaks. Mathematicians and biologists are drawn to this area of research.

2 Mathematical model

Vaccinations and quarantines for disease prevention and disease control have become more crucial in modern medicine developments. In recent years, several researchers have examined the effect of vaccination and quarantines on disease [22–25]. Many infectious diseases incubate within a host for a period of time before becoming infectious, so the duration of infection must also be taken into account. In response to the aforementioned research, the following system describes an epidemic model with imperfect quarantine based on the $\mathcal{SEIQR}$ model:

$$\frac{d\mathcal{S}}{d\mathbf{t}} = \Theta - \frac{\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - \mu\mathcal{S},$$

$$\begin{aligned}
\frac{d\mathcal{E}}{d\mathbf{t}} &= \frac{\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - (\gamma+\mu)\mathcal{E}, \\
\frac{d\mathcal{I}}{d\mathbf{t}} &= \gamma\mathcal{E} - (\xi+\eta+\alpha_1+\mu)\mathcal{I}, \\
\frac{d\mathcal{Q}}{d\mathbf{t}} &= \eta\mathcal{I} - (\delta+\alpha_2+\mu)\mathcal{Q}, \\
\frac{d\mathcal{R}}{d\mathbf{t}} &= \xi\mathcal{I} + \delta\mathcal{Q} - \mu\mathcal{R}.
\end{aligned} \tag{1}$$

In total, the population size is estimated to be $N(\mathbf{t}) = \mathcal{S}(\mathbf{t}) + \mathcal{E}(\mathbf{t}) + \mathcal{I}(\mathbf{t}) + \mathcal{Q}(\mathbf{t}) + \mathcal{R}(\mathbf{t})$. In these epidemic models, the saturated incidence rate is $h(\mathcal{I})\mathcal{S}$ and the model is composed of a $(\mathcal{SEIQR})$ epidemic model. In model (1), the transitions between compartments describe an $\mathcal{SEIQR}$ epidemic model without temporary immunity. The saturation level of $h(\mathcal{I})$ occurs when $\mathcal{I}$ increases in size.

$$h(\mathcal{I})\mathcal{S} = \frac{\beta\mathcal{I}\mathcal{S}}{1+\mathbf{k}\mathcal{I}}.$$

The infection force is represented by $\beta\mathcal{I}$, and when numbers increase, it causes the behavior of susceptible individuals to change. $\frac{1}{1+\mathbf{k}\mathcal{I}}$ represents this inhibition effect. Obviously the region,

$$\mathcal{D} = \{(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})/\mathcal{S}\geq 0,\mathcal{E}\geq 0,\mathcal{I}\geq 0,\mathcal{Q}\geq 0,\mathcal{R}\geq 0,\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}\leq \Theta/\mu\},$$

is a collection of model (1) that is positively invariant. The reproduction number of the model is

$$\mathcal{R}_0 = \frac{\beta\Theta\gamma}{\mu(\gamma+\mu)(\xi+\mu+\alpha_1+\eta)}.$$

Table 1. Model variables and their descriptions

Variables	Descriptions
$\mathcal{S}(\mathbf{t})$	Population of Susceptible
$\mathcal{E}(\mathbf{t})$	Population of Exposed
$\mathcal{I}(\mathbf{t})$	Population of Infected
$\mathcal{Q}(\mathbf{t})$	Population of Quarantined
$\mathcal{R}(\mathbf{t})$	Population of Recovered

Table 2. Model parameters, explanations and units

Parameters	Explanations	Units
Θ	Birth rate	Individual/ day
β	Transmission co-efficient	$(\text{Individual}\times \text{day})^{-1}$
μ	Natural death rate	1/day
γ	Infective individuals of exposed people	1/day
η	Infective individuals of quarantined recovery rate	1/day
ξ	Recovery rate of infective people	1/day
α_1	Rates of disease-induced death among infected individuals	1/day
α_2	Disease induced death rate of quarantined people	1/day
δ	Recovery rate of quarantined individuals	1/day
$\mathbf{k}$	Saturation rate of the inhibition effect rate	1/day

Remark 1 *i. If $\mathcal{R}_0 \leq 1$ holds, then the model (1) will only have an disease-free equilibrium of $\mathbf{E}_0 = (\mathcal{S}_0, 0, 0, 0, 0)$ in which there is global asymptotical stability. Then, only a vulnerable and healthy population remains after the pandemic illnesses have gone away.*
ii. When $\mathcal{R}_0 > 1$ is valid, in positive equilibria, there is an asymptotically stable value $\mathbf{E}^* = (\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Q}^*, \mathcal{R}^*)$ in the area $\mathcal{D}$ for the model (1), indicating that epidemic diseases will continue to exist.

It is impossible to describe the behavior of species using deterministic models in the natural world. Noise from the environment can sometimes cause disturbance to species. Therefore, there should be a fair amount of stochasticity in some parameters [26–28]. The ecosystem is dominated by this phenomenon without a doubt. Therefore, a substantial amount of research has been conducted on the effects of stochastic perturbations on disease [20, 29, 30]. In many branches of applied sciences, including disease dynamics, stochastic differential equations (SDEs) play an important role because they are capable of predicting the future dynamics of their deterministic counterpart. Until now, very few studies have been conducted on the global dynamics of stochastic $\mathcal{SEIQR}$ epidemic models. The model (1) is made more reasonable and realistic by assuming that $\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{Q}(t)$, and $\mathcal{R}(t)$ are directly proportional to environmental noise. Afterward, in accordance with model (1), a stochastic version may be obtained by

$$\begin{aligned}
d\mathcal{S} &= \left[\Theta - \frac{\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - \mu\mathcal{S}\right] d\mathbf{t} + \varrho_1 \mathcal{S} dW_1(\mathbf{t}), \\
d\mathcal{E} &= \left[\frac{\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - (\gamma+\mu)\mathcal{E}\right] d\mathbf{t} + \varrho_2 \mathcal{E} dW_2(\mathbf{t}), \\
d\mathcal{I} &= [\gamma\mathcal{E} - (\xi+\eta+\alpha_1+\mu)\mathcal{I}]\, d\mathbf{t} + \varrho_3 \mathcal{I} dW_3(\mathbf{t}), \\
d\mathcal{Q} &= [\eta\mathcal{I} - (\delta+\alpha_2+\mu)\mathcal{Q}]\, d\mathbf{t} + \varrho_4 \mathcal{Q} dW_4(\mathbf{t}), \\
d\mathcal{R} &= [\xi I + \delta\mathcal{Q} - \mu\mathcal{R}]\, d\mathbf{t} + \varrho_5 \mathcal{R} dW_5(\mathbf{t}),
\end{aligned} \tag{2}$$

where, $W_i = B_i = 1, 2, 3, 4, 5$ are independent standard one-dimensional Brownian motion and $\varrho_i(t)$, are the intensity of the white noise, $i = 1, 2, 3, 4, 5$. All other parameters are similar to those in model (1). As a result, the paper has been organized as follows: In Section 3, this model provides a significant unique global solution to the model (2). In Section 4, we prove that model (2) has an ergodic stationary distribution under certain conditions. In Section 5, we establish what conditions must be met for the disease to be wiped out. In the Section 6, numerical simulations are provided to illustrate the theoretical results. A brief summary of the main findings is presented in Section 7.

3 Uniqueness of global solution

An epidemic models dynamic behavior can be studied by determining whether or not the solution exits and remains nonnegative. It is well known that with stochastic differential equations, numerical solutions must satisfy both the local Lipschitz condition and the linear growth condition so that they have an exclusive global solution. Model (2) requires linear growth, despite its Lipschitz continuous coefficients, so a finite-time explosion may occur if the linear growth condition is not met. It is necessary to consider the existence and positivity of solutions to the model (2) before studying population system dynamics.
Firstly, we consider stochastic differential equations in d-dimensions

$$d\mathcal{X} = f(\mathcal{X}(\mathbf{t}), \mathbf{t})d\mathbf{t} + g(\mathcal{X}(\mathbf{t}), \mathbf{t})dB(\mathbf{t}), \quad for \quad \mathrm{t} \geq \mathbf{t}_0,$$

with the initial condition for $\mathcal{X}(0) = \mathcal{X}_0 \in \mathbb{R}^{\mathsf{d}}$. The differential operator $\mathcal{L}$ associated with the equation above can be defined as follows:

$$\mathcal{L} = \frac{\partial}{\partial \mathbf{t}} + \sum_{\mathsf{i}=1}^{d} f_{\mathsf{i}}(\mathcal{X}, \mathbf{t}) \frac{\partial}{\partial \mathcal{X}_{\mathsf{i}}} + \frac{1}{2} \sum_{\mathsf{i},\mathsf{j}=1}^{d} \left[g^T(\mathcal{X}, \mathbf{t}) g(\mathcal{X}, \mathbf{t}) \right]_{\mathsf{ij}} \frac{\partial^2}{\partial \mathcal{X}_{\mathsf{i}} \partial \mathcal{X}_{\mathsf{j}}}.$$

If $\mathcal{L}$ acts on a function $\mathcal{V} \in \mathcal{C}^2 \left(\mathbb{R}^d \times [t_0, \infty; \mathbb{R}_+] \right)$, then

$$\mathcal{L}\mathcal{V}(\mathcal{X}, \mathbf{t}) = \mathcal{V}_{\mathbf{t}}(\mathcal{X}, \mathbf{t}) + \mathcal{V}_{\mathcal{X}}(\mathcal{V}\mathcal{X}, \mathbf{t}) f(\mathcal{X}, \mathbf{t}) + \frac{1}{2} trace \left[g^T(\mathcal{X}, \mathbf{t}) \mathcal{V}_{\mathcal{X}\mathcal{X}}(\mathcal{X}, \mathbf{t}) g(\mathcal{X}, \mathbf{t}) \right],$$

where, $\mathcal{V}_{\mathbf{t}} = \frac{\partial \mathcal{V}}{\partial \mathbf{t}}, \mathcal{V}_{\mathcal{X}} = \left(\frac{\partial \mathcal{V}}{\partial \mathcal{X}_1}, \frac{\partial \mathcal{V}}{\partial \mathcal{X}_2}, ..., \frac{\partial \mathcal{V}}{\partial \mathcal{X}_d} \right), \mathcal{V}_{\mathcal{X}\mathcal{X}} = \left(\frac{\partial^2 \mathcal{V}}{\partial \mathcal{X}_i \partial \mathcal{X}_j} \right)_{d \times d}$. Thus, by Ito's formula, if $\mathcal{X}(\mathbf{t}) \in \mathbb{R}^d$, then

$$d\mathcal{V}(\mathcal{X}(\mathbf{t}), \mathbf{t}) = \mathcal{L}\mathcal{V}(\mathcal{X}(\mathbf{t}), \mathbf{t}) d\mathbf{t} + \mathcal{V}_{\mathcal{X}}(\mathcal{X}(\mathbf{t}), \mathbf{t}) g(\mathcal{X}(\mathbf{t}), \mathbf{t}) d\mathcal{B}(\mathbf{t}).$$

Lemma 1 *As a result of the model (2), we get a positive local and unique solution* $(\mathcal{S}(\mathbf{t}), \mathcal{E}(\mathbf{t}), \mathcal{I}(\mathbf{t}), \mathcal{Q}(\mathbf{t}), \mathcal{R}(\mathbf{t}))$ *for* $\mathbf{t} \in [-\varpi, \mathbf{e})$, *where* $\tau_{\mathbf{e}}$ *is the time of the explosion [31], at any starting value* $(\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)) \in \mathbb{R}^5_+$.

Theorem 1 *The model (2) has a unique positive solution* $(\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{Q}(t), \mathcal{R}(t)) \in \mathbb{R}^5_+$ *on* $t \geq 0$ *at any starting value* $(S(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)) \in \mathbb{R}^5_+$.

Proof In the case of model (2), the coefficients are Lipschitz continuous on the region $\mathbb{R}_+$. Two parts are involved in the following proof.
Part − I. According to Lemma 1, In model (2), for any given initial state $(\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)) \in \mathbb{R}^5_+$. there is a positive local solution $(\mathcal{S}(\mathbf{t}), \mathcal{E}(\mathbf{t}), \mathcal{I}(\mathbf{t}), \mathcal{Q}(\mathbf{t}), \mathcal{R}(\mathbf{t}))$.
Part − II. Now, we demonstrate that $\tau_{\mathbf{e}} = +\infty$ a.s, there is only one positive solution, and that is a global solution. If $\mathbf{n}_0 \geq 0$ is sufficiently large, then $\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0)$, and $\mathcal{R}(0)$ will all lie in the range $\left[\frac{1}{\mathbf{n}_0}, \mathbf{n} \right]$. For every integer $\mathbf{n} \geq \mathbf{n}_0$, as a general rule, stopping times can be defined as follows:

$$\begin{aligned} \tau_{\mathbf{n}} \quad = \quad & \inf \left\{ \mathbf{t} \in [\varpi, \tau_{\mathbf{e}}) : \mathcal{S}(\mathbf{t}) \notin \left(\frac{1}{\mathbf{n}}, \mathbf{n} \right), \mathcal{E}(\mathbf{t}) \notin \left(\frac{1}{\mathbf{n}}, \mathbf{n} \right), \mathcal{I}(\mathbf{t}) \notin \left(\frac{1}{\mathbf{n}}, \mathbf{n} \right), \right. \\ & \left. \mathcal{Q}(\mathbf{t}) \notin \left(\frac{1}{\mathbf{n}}, \mathbf{n} \right) \quad or \quad \mathcal{R}(\mathbf{t}) \notin \left(\frac{1}{\mathbf{n}}, \mathbf{n} \right) \right\}. \end{aligned} \tag{3}$$

In the case of an empty set $\varnothing$, we define here $\inf \varnothing = +\infty$. There is no doubt that $\tau_{\mathbf{n}}$ increases strictly when $\mathbf{n} \to \infty$. Assume $\tau_\infty = \lim_{\mathbf{n} \to \infty} \tau_{\mathbf{n}}$; therefore, $\tau_\infty = \tau_{\mathbf{e}}$. The only thing left to do is prove $\tau_\infty = +\infty$ a.s. If $\tau_\infty = +\infty$ is not true, then there exist a both constants $\mathcal{T} > 0$ and $\zeta \in (0, 1)$ such that $\mathbb{P}\tau_\infty \leq \mathcal{T} > \zeta$. Consequently, there exists $\mathbf{n}_1 \geq \mathbf{n}_0 (\mathbf{n}_1 \in \mathbb{N}_+)$.
Define a $\mathcal{C}^{1,2}-$ function $\widehat{\mathcal{V}} : \mathbb{R}^5_+ \to \mathbb{R}_+$

$$\begin{aligned} \widehat{\mathcal{V}}(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \quad = \quad & \mathcal{S} - 1 - \ln \mathcal{S} + \mathcal{E} - 1 - \ln \mathcal{E} + \mathcal{I} - 1 - \ln \mathcal{I} \\ & + \mathcal{Q} - 1 - \ln \mathcal{Q} + \mathcal{R} - 1 - \ln \mathcal{R}. \end{aligned} \tag{4}$$

By applying Definition in view of model (2), we get

$$\begin{aligned} d\widehat{\mathcal{V}} &= p\widehat{\mathcal{V}}\mathbf{dt} + \varrho_1(\mathcal{S}-1)dW_1(\mathbf{t}) + \varrho_2(\mathcal{E}-1)dW_2(\mathbf{t}) + \varrho_1(\mathcal{I}-1)dW_3(\mathbf{t}) \\ &\quad +\varrho_4(\mathcal{Q}-1)dW_4(\mathbf{t}) + \varrho_5(\mathcal{R}-1)dW_5(\mathbf{t}), \end{aligned} \tag{5}$$

where,

$$\begin{aligned} p\widehat{\mathcal{V}} &= \left(1-\frac{1}{\mathcal{S}}\right)\left[\Theta - \frac{\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - \mu\mathcal{S}\right] + \left(1-\frac{1}{\mathcal{E}}\right)\left[\frac{\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - (\gamma+\mu)\mathcal{E}\right] \\ &\quad + \left(1-\frac{1}{\mathcal{I}}\right)[\gamma\mathcal{E} - (\xi+\eta+\alpha_1+\mu)\mathcal{I}] + \left(1-\frac{1}{\mathcal{Q}}\right)[\eta\mathcal{I} - (\delta+\alpha_2+\mu)\mathcal{Q}] \\ &\quad + \left(1-\frac{1}{\mathcal{R}}\right)[\xi\mathcal{I} + \delta\mathcal{Q} - \mu\mathcal{R}] \\ \\ &= \Theta - \frac{\Theta}{\mathcal{S}} - \frac{\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - \frac{\beta\mathcal{S}\mathcal{I}}{\mathcal{E}(1+\mathbf{k}\mathcal{I})} + 5\mu - \mu(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}) \\ &\quad +\gamma+\xi+\eta+\alpha_1+\alpha_2+\delta-\alpha_1\mathcal{I}+\alpha_2\mathcal{Q} - \frac{\gamma\mathcal{E}}{\mathcal{I}} - \frac{\eta\mathcal{I}}{\mathcal{Q}} - \frac{\xi\mathcal{I}}{\mathcal{R}} - \frac{\delta\mathcal{Q}}{\mathcal{R}} \\ &\quad + \frac{\varrho_1^2+\varrho_2^2+\varrho_3^2+\varrho_4^2+\varrho_5^2}{2} \\ &\leq \Theta + 5\mu - \frac{\beta}{\mathbf{k}} + \gamma+\xi+\eta+\delta+\alpha_1+\alpha_2 + \frac{\varrho_1^2+\varrho_2^2+\varrho_3^2+\varrho_4^2+\varrho_5^2}{2} \\ &\leq \mathcal{K}_0 \in R_+. \end{aligned}$$

Hence, $\mathcal{K}_0$ is a positive constant.
Thus,

$$\begin{aligned} d\widehat{\mathcal{V}} &\leq \mathcal{K}_0\mathbf{dt} + \varrho_1(\mathcal{S}-1)dW_1(\mathbf{t}) + \varrho_2(\mathcal{E}-1)dW_2(\mathbf{t}) + \varrho_3(\mathcal{I}-1)dW_3(\mathbf{t}) \\ &\quad +\varrho_4(\mathcal{Q}-1)dW_4(\mathbf{t}) + \varrho_5(\mathcal{R}-1)dW_5(\mathbf{t}). \end{aligned} \tag{6}$$

In order to take the expectation, we integrate both sides of $\tau_{\mathbf{n}} \wedge \mathcal{T}$ to get,

$$\mathbb{E}\left[\widehat{\mathcal{V}}\left(\mathcal{S}(\tau_{\mathbf{n}}\wedge\mathcal{T}), \mathcal{E}(\tau_{\mathbf{n}}\wedge\mathcal{T}), \mathcal{I}(\tau_{\mathbf{n}}\wedge\mathcal{T}), \mathcal{Q}(\tau_{\mathbf{n}}\wedge\mathcal{T}), \mathcal{R}(\tau_{\mathbf{n}}\wedge\mathcal{T})\right)\right]$$

$$\begin{aligned} &\leq \widehat{\mathcal{V}}\left(\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)\right) + \mathbb{E}\left[\int_0^{\tau_{\mathbf{n}}\wedge\mathcal{T}} \mathcal{K}_0\mathbf{dt}\right] \\ &\leq \widehat{\mathcal{V}}\left(\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)\right) + \mathcal{K}_0\mathcal{T}. \end{aligned} \tag{7}$$

Allow it to set $\Omega_{\mathbf{n}} = (\tau_{\mathbf{n}} \leq T)$ for $\mathbf{n} \geq \mathbf{n}_1$, we have $\mathbb{P}(\Omega_{\mathbf{n}}) \geq \varepsilon$ with $\varepsilon \in (0,1)$. Note not for each $\omega \in \Omega_n$, a minimum of one of these exist $\mathcal{S}(\tau_{\mathbf{n}}, \omega)$ or $\mathcal{E}(\tau_{\mathbf{n}}, \omega)$ or $\mathcal{I}(\tau_{\mathbf{n}}, \omega)$ or $\mathcal{Q}(\tau_{\mathbf{n}}, \omega)$ or $\mathcal{R}(\tau_{\mathbf{n}}, \omega)$ either of these is equal $(\mathbf{n} \quad or \quad 1/\mathbf{n})$.

In that case, $\widehat{\mathcal{V}}(\mathcal{S}(\tau_{\mathsf{n}},\varpi),\mathcal{E}(\tau_{\mathsf{n}},\varpi),\mathcal{I}(\tau_{\mathsf{n}},\varpi),\mathcal{Q}(\tau_{\mathsf{n}},\varpi),\mathcal{R}(\tau_{\mathsf{n}},\varpi))$ cannot be less than either

$$\left(\frac{1}{\mathsf{n}}-1-\ln\frac{1}{\mathsf{n}}\right) \quad or \quad (\mathsf{n}-1-\ln\mathsf{n})=\left(\frac{1}{\mathsf{n}}-1+\ln\mathsf{n}\right).$$

The results are as follows:
$\widehat{\mathcal{V}}(\mathcal{S}(0),\mathcal{E}(0),\mathcal{I}(0),\mathcal{Q}(0),\mathcal{R}(0))+\mathcal{K}_0\mathcal{T}$

$$\begin{aligned}
&\geq \mathbb{E}\left[l_{\Omega_{\mathsf{n}}(\varpi)}\widehat{\mathcal{V}}(\mathcal{S}(\tau_{\mathsf{n}},\varpi),\mathcal{E}(\tau_{\mathsf{n}},\varpi),\mathcal{I}(\tau_{\mathsf{n}},\varpi),\mathcal{Q}(\tau_{\mathsf{n}},\varpi),\mathcal{R}(\tau_{\mathsf{n}},\varpi))\right]\\
&\geq \zeta\left[\frac{1}{\mathsf{n}}-1-\ln\frac{1}{\mathsf{n}}\right]\wedge(\mathsf{n}-1-\ln\mathsf{n}). \qquad (8)
\end{aligned}$$

Using a stochastic differential equation, this may be explained if Ω_{n} is denoted by $l_{\Omega_{\mathsf{n}}(\varpi)}$. Suppose $\mathsf{n}\to+\infty$, this implies

$$+\infty>(\mathcal{S}(0),\mathcal{E}(0),\mathcal{I}(0),\mathcal{Q}(0),\mathcal{R}(0))+\mathcal{K}_0\mathcal{T}=+\infty.$$

Therefore, it is contradictory, and therefore, we have $\tau_\infty=+\infty$. The proof is complete.

An equilibrium solution's asymptotic activities in a disease-free system

There is no doubt that $\mathsf{E}^*(\mathcal{S}_0,0,0,0,0)$ satisfies the condition for the model (1), and is associated with disease-free equilibrium. At some points, the disease will be extinct because the solution E^* is global stochastically asymptotically stable. The idea of disease-free equilibrium as a means of containing infectious diseases has thus gained popularity. In this section, the stochastic Lyapunov function is used primarily to achieve the stability of the disease-free equilibrium solution.

Theorem 2 *The stochastic model* (2) *saddles* $\mathcal{D}$ *in the disease-free equilibrium* $\mathsf{E}^*(\mathcal{S}_0,0,0,0,0)$, *when* $\mathcal{R}_0\leq 1$.

Proof Create a Lyapunov function $\mathcal{C}^2-$ on $\mathcal{V}:\mathbb{R}^5_+\to\mathbb{R}_+$ as follows

$$\mathcal{V}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})=\ln(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2+\ln\mathcal{E}+\ln\mathcal{I}.$$

Applying the infinitesimal generator $\mathcal{L}$ is applied to $\mathcal{V}$, we obtain

$$\begin{aligned}\mathcal{LV}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) &= \left(\Theta-\frac{\beta\mathcal{SI}}{1+\mathsf{k}\mathcal{I}}-\mu\mathcal{S}\right)\left(\frac{2}{\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}}\right)\\ &+\left(\frac{\beta\mathcal{SI}}{1+\mathsf{k}\mathcal{I}}-(\gamma+\mu)\mathcal{E}\right)\left(\frac{2}{\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}}+\frac{1}{\mathcal{E}}\right)\\ &+(\gamma\mathcal{E}-(\xi+\eta+\alpha_1+\mu)\mathcal{I})\left(\frac{2}{\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}}+\frac{1}{\mathcal{I}}\right)\\ &+(\eta\mathcal{I}-(\delta+\alpha_2+\mu)\mathcal{Q})\left(\frac{2}{\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}}\right)\\ &+(\xi I+\delta\mathcal{Q}-\mu\mathcal{R})\left(\frac{2}{\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}}\right)\\ &-\left(\frac{2}{2(\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2}\right)\varrho_1^2S^2\\ &+\frac{1}{2}\left(\frac{-2}{(\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2}-\frac{1}{\mathcal{E}^2}\right)\varrho_2^2E^2\\ &+\frac{1}{2}\left(\frac{-2}{(\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2}-\frac{1}{\mathcal{I}^2}\right)\varrho_3^2I^2\\ &+\frac{1}{2}\left(\frac{-2}{(\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2}\right)\varrho_4^2Q^2\\ &-\left(\frac{2}{2(\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2}\right)\varrho_4^2R^2.\end{aligned} \tag{9}$$

As a result of simplifying $\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}\leq 1$, we get

$$\begin{aligned}\mathcal{LV}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) &= \frac{2}{\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}}(\Theta-\mu\mathcal{S}-\mu\mathcal{E}-\mu\mathcal{I}-\mu\mathcal{Q}-\mu\mathcal{R})\\ &+(\beta\mathcal{S}-(\gamma+\mu))+(\gamma-(\xi+\eta+\alpha_1+\mu))\\ &-\frac{\varrho_1^2\mathcal{S}^2+\varrho_2^2\mathcal{E}^2+\varrho_3^2\mathcal{I}^2+\varrho_4^2\mathcal{Q}^2+\varrho_5^2\mathcal{R}^2}{(\mathcal{S}-1+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^2}-\frac{\varrho_2^2}{2}-\frac{\varrho_3^2}{2}.\end{aligned}$$

It follows that $\mathcal{LV}$ will be negative and definite on $\mathcal{D}$ if $\mathcal{R}_0<1$ holds. A disease-free equilibrium solution $\mathsf{E}^*(\mathcal{S}_0,0,0,0,0)$ in $\mathcal{D}$ is global asymptotically stable for the stochastic model (2).

Remark 2 *The overhead Theorem 2 proves that the disease cases exist if $\mathcal{R}_0<1$ holds. The stability condition of the disease $\mu>\beta\gamma\mathcal{S}-((\gamma+\mu)+(\xi+\eta+\alpha_1+\mu))$ disease will be disappear. Taking the reproductive number $\mathcal{R}_0<1$, as a consequence, $\mathsf{E}^*(\mathcal{S}_0,0,0,0,0)$ is stochastically asymptotically stable in the large in the stochastic system (2). According to Theorem 2, the stochastic model (2) will approach disease-free equilibrium if the intensity of white noise is high enough. Since the intensity of white noise ϱ_i (for $i=1,2,3,4,5$) is small, the solutions of stochastic model (2) will generally fluctuate around the disease-less equilibrium of deterministic model (1).*

4 Ergodicity and stationary distribution

It is not only important to study epidemiological dynamics to determine when a disease will eventually become extinct. It is also to determine how long the disease will persist in the population. The endemic equilibrium does not exist for model (2). Therefore, this section examines whether there is a stationary distribution, which indicates the prevalence of a disease, according

to Hasminskii [32].
Then, let $X(t)$ be a time-homogeneous Markov process in $E_n \subset \mathbb{R}_n$. In order to explain this idea, a stochastic differential equation approach can be used.

$$d\mathcal{X}(t) = f(\mathcal{X})dt + \sum_{k=1}^{n} \sigma_k(\mathcal{X})dB_k(t). \tag{10}$$

In this case, E_n represents an n-dimensional Euclidean space. Following is a description of the diffusion matrix:

$$\widehat{\mathcal{A}}(x) = \left(a_{ij}(x)\right), a_{ij}(x) = \sum_{k=1}^{n} \sigma_k^i(x)\sigma_k^j(x). \tag{11}$$

Assumption 1 *The following properties are satisfied by a bounded domain $U \subset E_n$ with a regular boundary Π, such that $\overline{U} \subset E_n$ ($\overline{U}$ is the closure of U):*

i. The diffusion matrix $\widehat{\mathcal{A}}(x)$ is bounded away from zero in the domain U and some nearby neighborhoods.
ii. The mean time it takes for a path leading from 'x' to reach the set U is finite if $x \in E_n \backslash U$, and this holds true for each compact subset of E_n.

Lemma 2 *[32]. If Assumption 1 hold, then the Markov process $\mathcal{X}(t)$ has a stationary distribution $\widehat{\omega}(.)$. Besides, the measure $\widehat{\omega}$ may be integrated when $f(.)$ is a function, then*

$$\mathbb{P}_x\left\{\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f(\mathcal{X}(t))dt = \int_{E_n} f(x)\widehat{\omega}(dx)\right\} = 1,$$

for all $x \in E_n$.

Remark 3 *The demonstration of Assumption 1 (i) [33] involves showing that a bounded domain H has uniform ellipticity F; here is an example.*

$$\mathcal{F}_u = b(x)u_x \frac{1}{2} trace(\mathcal{A}(x)u_{xx}).$$

In particular, there exists a positive number $\mathfrak{Z}$ such that

$$\sum_{i,j=1}^{n} a_{ji}(x)\xi_i\xi_j \geq \mathfrak{Z}\,|\xi|^2, \quad x \in \overline{\mathcal{H}}, \xi \in \mathbb{R}^n.$$

It is possible to prove Assumption 1 (ii) [34] if there is a certain neighborhood U and some non-negative $\mathcal{C}^{2,1}$-function $\mathcal{V}$ such that for all $x \in E_n \backslash U$,
$\mathcal{L}\mathcal{V}(x) < 0$.
The following main results can be obtained by using Lemma 2.

Theorem 3 *If*

$$\mathcal{R}_0^* = \frac{\beta b\gamma}{(\mu + \frac{\varrho_1^2}{2})(\gamma + \mu + \frac{\varrho_2^2}{2})(\xi + \eta + \mu + \alpha_1 + \frac{\varrho_3^2}{2})} > 1,$$

and for the any initial value given as $(\mathcal{S}(0),\mathcal{E}(0),\mathcal{I}(0),\mathcal{Q}(0),\mathcal{R}(0)) \in \mathbb{R}^5_+$, *then the system has a uniquely stationary distribution* $\widehat{\omega}(.)$ *and it is has a ergodic property.*

Proof Define a function $\mathcal{C}^2$ such that,

$$\widehat{\mathcal{V}} : \mathbb{R}^5_+ \to \mathbb{R}_+,$$

$$\begin{aligned}\widehat{\mathcal{V}}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) &= \Gamma\left[(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R}) - c_1\ln\mathcal{S} - c_2\ln\mathcal{E} - c_3\ln\mathcal{I}\right]\\ &\quad + \frac{1}{p+1}(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa+1} - \ln\mathcal{S} - \ln\mathcal{E} - \ln\mathcal{Q}\\ &\quad - \ln\mathcal{R} + (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})\\ &= \Gamma\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 + \mathcal{V}_5 + \mathcal{V}_6 + \mathcal{V}_7, \end{aligned} \tag{12}$$

here κ and c_i, $(\mathbf{i} = 1,2,3)$ the positive constant satisfying the condition

$$0 < \kappa < 2\mu\left(\frac{1}{\varrho_1^2+\varrho_2^2+\varrho_3^2+\varrho_4^2+\varrho_5^2}\right),$$

$$c_1 = \frac{\Theta}{\mu+\frac{\varrho_1^2}{2}}, \quad c_2 = \frac{\Theta}{\gamma+\mu+\frac{\varrho_2^2}{2}}, \quad c_3 = \frac{\Theta}{\xi+\mu+\eta+\alpha_1+\frac{\varrho_3^2}{2}},$$

we can consider $\Gamma > 0$ and make it large enough, such that

$$\Gamma\phi + \mathcal{M} \leq -2,$$

obviously,

$$\liminf_{\pi\to(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathbb{R}^5_+\setminus\mathbb{U}_\pi} \overline{\mathcal{V}}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) = +\infty, \tag{13}$$

and here, $\mathbb{U}_\pi = (\frac{1}{\pi},\pi)\times(\frac{1}{\pi},\pi)\times(\frac{1}{\pi},\pi)\times(\frac{1}{\pi},\pi)\times(\frac{1}{\pi},\pi)$.
There exists a point $(\mathcal{S}^*,\mathcal{E}^*,\mathcal{I}^*,\mathcal{Q}^*,\mathcal{R}^*)$ in $\mathbb{R}^5_+$ that is the minimum point of $\widehat{\mathcal{V}}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})$ because $\widetilde{\mathcal{V}}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})$ is a continuous function. The positive define $\mathcal{C}^2-$ function $\mathcal{V} : \mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$\mathcal{V}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) = \widehat{\mathcal{V}}(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}) - \widehat{\mathcal{V}}(\mathcal{S}^*,\mathcal{E}^*,\mathcal{I}^*,\mathcal{Q}^*,\mathcal{R}^*), \tag{14}$$

from Ito's formula,

$$\begin{aligned}\mathcal{L}\mathcal{V}_1 &= -\mu - \frac{c_1\Theta}{\mathcal{S}} - \frac{c_2\beta\mathcal{S}\mathcal{I}}{\mathcal{E}(1+\mathbf{k}\mathcal{I})} - \frac{c_3\gamma\mathcal{E}}{\mathcal{I}} + \frac{c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} + \Theta - \alpha_1\mathcal{I} - \alpha_2\mathcal{Q}\\ &\quad + c_1\left(\mu+\frac{\varrho_1^2}{2}\right) + c_2\left(\gamma+\mu+\frac{\varrho_2^2}{2}\right) + c_3\left(\xi+\mu+\eta+\alpha_1+\frac{\varrho_3^2}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\mu - \frac{c_1\Theta}{\mathcal{S}} - \frac{c_2\beta\mathcal{S}\mathcal{I}}{\mathcal{E}(1+\mathbf{k}\mathcal{I})} - \frac{c_3\gamma\mathcal{E}}{\mathcal{I}} + \frac{c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} + \Theta + c_1\left(\mu + \frac{\varrho_1^2}{2}\right) \\
&\quad + c_2\left(\gamma + \mu + \frac{\varrho_2^2}{2}\right) + c_3\left(\xi + \mu + \eta + \alpha_1 + \frac{\varrho_3^2}{2}\right) \\
&\leq -4\left(\frac{\mu c_1 c_2 c_3 b\beta\gamma}{1+\mathbf{k}\mathcal{I}}\right)^{\frac{1}{4}} + \frac{c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} + \Theta + c_1\left(\mu + \frac{\varrho_1^2}{2}\right) + c_2\left(\gamma + \mu + \frac{\varrho_2^2}{2}\right) \\
&\quad + c_3\left(\xi + \mu + \eta + \alpha_1 + \frac{\varrho_3^2}{2}\right) \\
&\leq -4\left(\frac{\mu c_1 c_2 c_3 \Theta\beta\gamma}{1+\mathbf{k}\mathcal{I}}\right)^{\frac{1}{4}} + \frac{c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - 4\mu \\
&\leq -4\mu\left[\left(\frac{\Theta\beta\gamma}{(1+\mathbf{k}\mathcal{I})\left(\mu + \frac{\varrho_1^2}{2}\right)\left(\gamma + \mu + \frac{\varrho_2^2}{2}\right)\left(\xi + \mu + \eta + \alpha_1 + \frac{\varrho_3^2}{2}\right)}\right)^{\frac{1}{4}} - 1\right] \\
&\quad + \frac{c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} \\
&\leq \frac{-4\mu}{1+\mathbf{k}\mathcal{I}}\left[(\mathcal{R}_0^*)^{\frac{1}{4}} - 1\right] + \frac{c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}}, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\mathcal{V}_2 &= (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa}\left[\Theta - (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})\mu - \alpha_1\mathcal{I} - \alpha_2\mathcal{Q}\right] \\
&\quad + \frac{\kappa(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa+1}}{2}\left(\varrho_1^2\mathcal{S}^2 + \varrho_2^2\mathcal{E}^2 + \varrho_3^2\mathcal{I}^2 + \varrho_4^2\mathcal{Q}^2 + \varrho_5^2\mathcal{R}^2\right) \\
&\leq \Theta(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa} - \left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \\
&\quad \times (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa+1} \\
&\leq \omega - \frac{1}{2}\left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right](\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa+1} \\
&< \omega - \frac{1}{2}\left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \\
&\quad \times (\mathcal{S}^{\kappa+1} + \mathcal{E}^{\kappa+1} + \mathcal{I}^{\kappa+1} + \mathcal{Q}^{\kappa+1} + \mathcal{R}^{\kappa+1}), \qquad (16)
\end{aligned}
$$

where,

$$
\begin{aligned}
\omega &= \sup_{(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathbb{R}_+^5}\{\Theta(\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa} \\
&\quad - \left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \times (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})^{\kappa+1}\Big\} \\
&< \infty.
\end{aligned}
$$

Similarly, we get

$$\mathcal{L}\mathcal{V}_3 = -\frac{\Theta}{\mathcal{S}} + \frac{\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} + \mu + \frac{\varrho_1^2}{2}, \tag{17}$$

$$\mathcal{L}\mathcal{V}_4 = -\frac{\beta\mathcal{S}\mathcal{I}}{\mathcal{E}(1+\mathbf{k}\mathcal{I})} + (\mu+\gamma) + \frac{\varrho_2^2}{2}, \tag{18}$$

$$\mathcal{L}\mathcal{V}_5 = -\frac{\eta\mathcal{I}}{\mathcal{Q}} + (\xi_\mu + \alpha_2) + \frac{\varrho_4^2}{2}, \tag{19}$$

$$\mathcal{L}\mathcal{V}_6 = -\frac{\xi\mathcal{I}+\delta\mathcal{Q}}{\mathcal{R}} + \mu + \frac{\varrho_5^2}{2}, \tag{20}$$

$$\mathcal{L}\mathcal{V}_7 = \Theta - (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})\mu. \tag{21}$$

Therefore,

$$\begin{aligned}\mathcal{L}\mathcal{V} = {} & -\varphi\phi + \frac{\varphi c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}} - \frac{1}{2}\left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \\ & (\mathcal{S}^{\kappa+1} + \mathcal{E}^{\kappa+1} + \mathcal{I}^{\kappa+1} + \mathcal{Q}^{\kappa+1} + \mathcal{R}^{\kappa+1}) \\ & -\frac{\Theta}{\mathcal{S}} - \frac{\beta\mathcal{S}\mathcal{I}}{\mathcal{E}(1+\mathbf{k}\mathcal{I})} - \frac{\eta\mathcal{I}}{\mathcal{Q}} - \frac{\xi\mathcal{I}+\delta\mathcal{Q}}{\mathcal{R}} - (\mathcal{S}+\mathcal{E}+\mathcal{I}+\mathcal{Q}+\mathcal{R})\mu \\ & +\Theta + \pi + \xi + \gamma + 4\mu + \alpha_2 + \frac{\varrho_1^2 \vee \varrho_2^2 \vee \varrho_4^2 \vee \varrho_5^2}{2}.\end{aligned} \tag{22}$$

As a next step, let us examine compact subset $\mathcal{D}$

$$\mathcal{D} = \left\{\varepsilon \le \mathcal{S} \le \frac{1}{\epsilon}, \epsilon_4^2 \le \mathcal{E} \le \frac{1}{\epsilon_4^2}, \epsilon_2^2 \le \mathcal{I} \le \frac{1}{\epsilon_2^2}, \epsilon_3^2 \le \mathcal{Q} \le \frac{1}{\epsilon_3^2}, \epsilon_4^2 \le \mathcal{R} \le \frac{1}{\epsilon_4^2}\right\}.$$

The following conditions must be satisfied if ϵ is a sufficiently small constant:

$$\begin{aligned}
& - \quad \frac{\Theta}{\mathcal{S}} + \frac{\Upsilon c_1\beta}{\mathbf{k}} + F \le -1 \\
& - \quad \Upsilon\phi + \Upsilon c_1\beta\epsilon \le -1 \\
& - \quad 2\left(\frac{\mu\beta}{1+\mathbf{k}\mathcal{I}}\right)^{\frac{1}{2}} + \Upsilon\epsilon_1\beta + \mathcal{M} \le -1 \\
& - \quad \frac{\eta}{\epsilon} + \frac{\Upsilon c_1\beta}{1+\mathbf{k}\mathcal{I}} + \mathcal{M} \le -1 \\
& - \quad \frac{\epsilon}{\epsilon^2} - \frac{\delta}{\epsilon} + \Upsilon c_1\beta + \mathcal{M} \le -1 \\
& - \quad \frac{1}{2}\left[\mu - \frac{\kappa}{2}\left(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2\right)\right]\frac{1}{\epsilon^{i(\kappa+1)}} + \Upsilon c_1\beta\mathcal{I} \\
& \quad +\mathcal{M} \le -1.
\end{aligned}$$

Here ϵ is a sufficiently small constant, where $i = 1, 2, 3, 4, 5$. In that case, $\mathbb{R}^5_+ \backslash \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup, ..., \cup \mathcal{D}_{10}$ with,

$$\begin{aligned}
\mathcal{D}_1 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / 0 < \mathcal{S} < \epsilon\right\}, \\
\mathcal{D}_2 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{S} \geq \epsilon, \mathcal{I} \geq \epsilon_2^2, 0 < \epsilon < \epsilon_4^2\right\}, \\
\mathcal{D}_3 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{S} \geq \epsilon, 0 < \mathcal{I} < \epsilon_3^2\right\}, \\
\mathcal{D}_4 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{I} \geq \epsilon_2^2, 0 < \mathcal{Q} < \epsilon_3^2\right\}, \\
\mathcal{D}_5 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{S} \geq \epsilon, \mathcal{I} \geq \epsilon_2^2, \mathcal{Q} \geq \epsilon_3^2, 0 < \mathcal{R} < \epsilon_4^2\right\}, \\
\mathcal{D}_6 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{S} > \frac{1}{\epsilon}\right\}, \\
\mathcal{D}_7 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{E} > \frac{1}{\epsilon_4^2}\right\}, \\
\mathcal{D}_8 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{I} > \frac{1}{\epsilon_3^2}\right\}, \\
\mathcal{D}_9 &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{Q} > \frac{1}{\epsilon_3^2}\right\}, \\
\mathcal{D}_{10} &= \left\{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ / \mathcal{R} > \frac{1}{\epsilon_4^2}\right\}.
\end{aligned}$$

As a result, we can now calculate the negative $\mathcal{LV}$ value for every $(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ \backslash \mathcal{D}$.
Case-1: Suppose $(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathcal{D}_1$

$$\begin{aligned}
\mathcal{LV} &\leq -\frac{\Theta}{\mathcal{S}} + \frac{\Upsilon c_1 \beta \mathcal{I}}{1 + \mathbf{k}\mathcal{I}} - \frac{1}{2}\left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \\
&\quad \times (\mathcal{S}^{\kappa+1} + \mathcal{E}^{\kappa+1} + \mathcal{I}^{\kappa+1} + \mathcal{Q}^{\kappa+1} + \mathcal{R}^{\kappa+1}) + \Theta + \pi + \xi + \gamma + 4\mu + \alpha_2 \\
&\quad + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2 + \varrho_5^2}{2} \\
&\leq -\frac{\Theta}{\mathcal{S}} + \frac{\Upsilon c_1 \beta}{\mathbf{k}} + F \leq -1.
\end{aligned} \tag{23}$$

Case-2: In this case, $(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathcal{D}_2$

$$\begin{aligned}
\mathcal{LV} &\leq -\Upsilon\phi + \frac{\Upsilon c_1 \beta \mathcal{S}\mathcal{I}}{1 + \mathbf{k}\mathcal{I}} + \frac{\beta \mathcal{S}\mathcal{I}}{\mathcal{E}(1 + \mathbf{k}\mathcal{I})} - \frac{1}{2}\left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \\
&\quad (\mathcal{S}^{\kappa+1} + \mathcal{E}^{\kappa+1} + \mathcal{I}^{\kappa+1} + \mathcal{Q}^{\kappa+1} + \mathcal{R}^{\kappa+1}) + \pi + \xi + \gamma + 4\mu + \alpha_2 \\
&\quad + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2 + \varrho_5^2}{2} \\
&\leq -\Upsilon\phi + \frac{\Upsilon c_1 \beta}{1 + \mathbf{k}\mathcal{I}} \frac{1}{2}\left[\mu - \frac{\kappa}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2)\right] \\
&\quad \times (\mathcal{S}^{\kappa+1} + \mathcal{E}^{\kappa+1} + \mathcal{I}^{\kappa+1} + \mathcal{Q}^{\kappa+1} + \mathcal{R}^{\kappa+1}) + \pi + \xi + \gamma + 4\mu + \alpha_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varrho_1^2+\varrho_2^2+\varrho_3^2+\varrho_4^2+\varrho_5^2}{2}\\
&\leq -\Upsilon\phi+\frac{\Upsilon c_1\beta}{\mathbf{k}}+G\\
&\leq -\Upsilon\phi+\Upsilon c_1\beta\epsilon\leq -1.
\end{aligned}
\tag{24}
$$

Case-3: In case of $(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathcal{D}_3$

$$
\begin{aligned}
\mathcal{LV} &= -2\left(\frac{\mu\beta\mathcal{S}\mathcal{I}}{\mathcal{E}(1+\mathbf{k}\mathcal{I})}\right)^{\frac{1}{2}}+\frac{\Upsilon c_1\beta\mathcal{S}\mathcal{I}}{1+\mathbf{k}\mathcal{I}}+\pi+\xi+\gamma+4\mu+\alpha_2\\
&\quad+\frac{\varrho_1^2+\varrho_2^2+\varrho_3^2+\varrho_4^2+\varrho_5^2}{2}\\
&\leq -2\left(\frac{\mu\beta}{1+\mathbf{k}}\right)^{\frac{1}{2}}+\frac{\Upsilon\epsilon_1\beta}{1+\mathbf{k}\mathcal{I}}+M\\
&\leq -2\left(\frac{\mu\beta}{1+\mathbf{k}}\right)^{\frac{1}{2}}+\Upsilon\epsilon_1\beta+M\leq -1.
\end{aligned}
\tag{25}
$$

Case-4: In case $(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathcal{D}_4$

$$
\begin{aligned}
\mathcal{LV} &\leq -\frac{\eta\mathcal{I}}{\mathcal{Q}}+\frac{\Upsilon c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}}+b+\pi+\xi+\gamma+4\mu+\alpha_2\\
&\quad+\frac{\varrho_1^2+\varrho_2^2+\varrho_3^2+\varrho_4^2+\varrho_5^2}{2}\\
&\leq -\frac{\eta}{\epsilon}+\frac{\Upsilon c_1\beta}{1+\mathbf{k}\mathcal{I}}+M\leq -1.
\end{aligned}
\tag{26}
$$

Case-5: In this case, $(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathcal{D}_5$

$$
\begin{aligned}
\mathcal{LV} &\leq -\frac{\xi\mathcal{I}-\delta\mathcal{Q}}{\mathcal{R}}+\frac{\Upsilon c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}}+M\\
&\leq -\frac{\epsilon}{\epsilon^2}-\frac{\delta}{\epsilon}+\Upsilon c_1\beta+M\leq -1.
\end{aligned}
\tag{27}
$$

Case-6: Suppose $(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathcal{D}_6$

$$
\begin{aligned}
\mathcal{LV} &\leq -\frac{1}{2}\left[\mu-\frac{\kappa}{2}\left(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2\vee\varrho_5^2\right)\right]\mathcal{S}^{\kappa+1}+\frac{\gamma c_1\beta\mathcal{I}}{1+\mathbf{k}\mathcal{I}}+M\\
&\leq -\frac{1}{2}\left[\mu-\frac{\kappa}{2}\left(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2\vee\varrho_5^2\right)\right]\frac{1}{\epsilon^{\kappa+1}}+\gamma c_1\beta\mathcal{I}+M\\
&\leq -1.
\end{aligned}
\tag{28}
$$

Subsequently, providing under requirement (4.19) for $i=1$. We get $\mathcal{LV}\leq -1$ on $\mathcal{D}_6$. Similarly, it follows from the equation (4.19) for $i=2,\ldots,4$, the same procedure can deduced for the compartments $\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R}$ on $\mathcal{LV}\leq -1$ and for $\mathcal{D}_i, i=7,\ldots,10$.
Based on the 10 cases mentioned above, it can be concluded that According to the discussion of the above ten cases, it becomes clear that, for a sufficiently small ε,

$$\mathcal{LV} \leq -1 \quad \text{for all} \quad (\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Q}, \mathcal{R}) \in \mathbb{R}^5_+ \setminus \mathcal{D}. \tag{29}$$

Therefore, Assumption 1 (ii) is satisfied. A diffusion matrix is also presented for model (2) as follows:

$$A = \begin{pmatrix} \varrho_1^2\mathcal{S}^2 & 0 & 0 & 0 & 0 \\ 0 & \varrho_2^2\mathcal{E}^2 & 0 & 0 & 0 \\ 0 & 0 & \varrho_3^2\mathcal{I}^2 & 0 & 0 \\ 0 & 0 & 0 & \varrho_4^2\mathcal{Q}^2 & 0 \\ 0 & 0 & 0 & 0 & \varrho_5^2\mathcal{R}^2 \end{pmatrix}.$$

A positive number exists,

$$\mathcal{X} = \min_{(\mathcal{S},\mathcal{E},\mathcal{I},\mathcal{Q},\mathcal{R})\in\mathbb{D}} \left\{ \varrho_1^2\mathcal{S}^2, \varrho_2^2\mathcal{E}^2, \varrho_3^2\mathcal{I}^2, \varrho_4^2\mathcal{Q}^2, \varrho_5^2\mathcal{R}^2 \right\},$$

such that,

$$\begin{aligned} \sum_{i,j=1}^{5} a_{ij}\xi_i\xi_j &= \varrho_1^2\mathcal{S}^2\xi_1^2 + \varrho_2^2\mathcal{E}^2\xi_2^2 + \varrho_3^2\mathcal{I}^2\xi_3^2 + \varrho_4^2\mathcal{Q}^2\xi_4^2 + \varrho_5^2\mathcal{R}^2\xi_5^2 \\ &\geq \mathcal{X}\,|\xi|^2. \end{aligned} \tag{30}$$

In this case, Assumption 1 (i) is satisfied. Due to this, the model (2) has an ergodic distribution $\widehat{\omega}(.)$ with a uniquely stationary distribution. Proof of the Theorem is complete.

Remark 4 *Theorem 3 indicates that model (2) has an uniquely ergodic stationary distribution if $\mathcal{R}_0^* > 1$. If $\varrho_i = 0 (i = 1, 2, 3, 4, 5)$, the expression of $\mathcal{R}_0^*$ corresponds to the reproduction number $\mathcal{R}_0$ of the model (1). It is clear from this that the results of model (1) can be generalized. Furthermore, this theorem shows that the disease can be resistant to small levels of environmental noise in order to remain persistent.*

5 Extinction

The purpose of this section is to discuss the parameter conditions for the extinction of diseases in the model (2). First, we give a useful lemma before proving our main conclusions:

Lemma 3 *If $(\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{Q}(t), \mathcal{R}(t))$ be a solution of the system for any given initial value $(\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{Q}(t), \mathcal{R}(t)) \in \mathbb{R}^5_+$ has the following properties,*

$$\lim_{t\to\infty} \frac{1}{t} (\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{Q}(t), \mathcal{R}(t)) = 0, \qquad a.s.$$

In addition, when $\mu > \frac{1}{2}\left(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2\right)$ *holds.*

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{S}(x)dB_1(x) &= 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{E}(x)dB_2(x) = 0,\\
\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{I}(x)dB_2(x) &= 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{Q}(x)dB_4(x) = 0,\\
\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{R}(x)dB_5(x) &= 0, \quad a.s.
\end{aligned} \tag{31}$$

Proof The proof of Lemma 3 follows the same way as [35, 36]; therefore, we omit it.

Theorem 4 *Let* $\mu > \frac{1}{2}\left(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2\right)$ *at any given initial value* $(\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{Q}(t), \mathcal{R}(t)) \in \mathbb{R}_+^5$, *if*

$$\mathcal{R}_S^* = \frac{2\gamma\beta(\gamma+\mu)}{\left[(\xi+\mu+\alpha_1+\eta+\frac{\varrho_3^2}{2})(\gamma+\mu)^2\right] \wedge \left(\frac{\gamma^2\varrho_2^2}{2}\right)} < 1.$$

Then

$$\lim_{t\to\infty}\mathcal{E}(t) = \lim_{t\to\infty}\mathcal{I}(t) = \lim_{t\to\infty}\mathcal{Q}(t) = \lim_{t\to\infty}\mathcal{R}(t) = 0, \quad a.s.$$

Furthermore,

$$\lim_{t\to\infty}\langle \mathcal{S}\rangle = \frac{\Theta}{\mu} = \mathcal{S}_0, \quad a.s.$$

Proof In the following equation, define a function V_0 that is differentiable

$$\begin{aligned}
dV_0 &= \left\{\frac{\Upsilon(\beta\mathcal{S}\mathcal{I}/1+\mathbf{k}\mathcal{I}) - (\gamma+\mu)(\xi+\mu+\alpha_1+\eta)\mathcal{I}}{\Upsilon\mathcal{E}+(\gamma+\mu)\mathcal{I}} - \frac{\gamma^2\varrho_2^2\mathcal{E}^2 + (\gamma+\mu)^2\varrho_3^2\mathcal{I}^2}{2\left[\gamma\mathcal{E}+(\gamma+\mu)\mathcal{I}\right]^2}\right\}dt\\
&\quad + \frac{\Upsilon\varrho_2\mathcal{E}}{\Upsilon\mathcal{E}+(\gamma+\mu)\mathcal{I}}dB_2(t) + \frac{(\gamma+\mu)\varrho_3\mathcal{I}}{\Upsilon\mathcal{E}+(\gamma+\mu)}dB_3(t).\\
&\leq \left\{\frac{\Upsilon\beta}{(\gamma+\mu)} - \frac{(\gamma+\mu)^2\mathcal{I}^2(\xi+\mu+\alpha_1+\eta+\frac{\varrho_3^2}{2}) + (\frac{\gamma^2\varrho_2^2}{2})\mathcal{E}^2}{\left[\gamma\mathcal{E}+(\gamma+\mu)\mathcal{I}\right]^2}\right\}\\
&\quad + \frac{\Upsilon\varrho_2\mathcal{E}}{\Upsilon\mathcal{E}+(\gamma+\mu)\mathcal{I}}dB_2(t) + \frac{(\gamma+\mu)\varrho_3\mathcal{I}}{\Upsilon\mathcal{E}+(\gamma+\mu)}dB_3(t)\\
&\leq \left\{\frac{\Upsilon\beta}{\gamma+\mu} - \frac{(\xi+\eta+\alpha_1+\mu+\frac{\varrho_3^2}{2})(\gamma+\mu)^2 \wedge \left(\frac{\gamma^2\varrho_2^2}{2}\right)}{2(\gamma+\mu)^2}\right\}dt\\
&\quad + \frac{\Upsilon\varrho_2\mathcal{E}}{\Upsilon\mathcal{E}+(\gamma+\mu)\mathcal{I}}dB_2(t) + \frac{(\gamma+\mu)\varrho_3\mathcal{I}}{\Upsilon\mathcal{E}+(\gamma+\mu)}dB_3(t).
\end{aligned} \tag{32}$$

The integration from 0 to t and the division by t on both sides of (32) are done as follows:

$$\begin{aligned}\frac{\ln[\gamma\mathcal{E}+\mathcal{I}(\mathbf{t})(\gamma+\mu)]}{\mathbf{t}} \leq\ & \frac{\Upsilon\beta}{\gamma+\mu}-\frac{(\xi+\eta+\alpha_1+\mu+\frac{\varrho_3^2}{2})(\gamma+\mu)^2\wedge\left(\frac{\gamma^2\varrho_2^2}{2}\right)}{2(\gamma+\mu)^2}\\ &+\frac{\ln[\gamma\mathcal{E}(0)+\mathcal{I}(0)(\gamma+\mu)]}{\mathbf{t}}\\ &+\frac{\gamma\varrho_2}{\mathbf{t}}\int_0^{\mathbf{t}}\frac{\mathcal{E}(x)}{\gamma\mathcal{E}(x)+(\gamma+\mu)\mathcal{I}(x)}dB_2(x)\\ &+\frac{(\gamma+\mu)\varrho_3}{\mathbf{t}}\int_0^{\mathbf{t}}\frac{\mathcal{I}(x)}{\gamma\mathcal{E}(x)+(\gamma+\mu)\mathcal{I}(x)}dB_3(x).\end{aligned}$$

By applying Lemma 3, we need to

$$\begin{aligned}\limsup_{\mathbf{t}\to\infty}\frac{\ln[\gamma\mathcal{E}(\mathbf{t})+\mathcal{I}(\mathbf{t})(\gamma+\mu)]}{\mathbf{t}} \leq\ & \frac{\Upsilon\beta}{\gamma+\mu}\\ &-\frac{(\xi+\eta+\alpha_1+\mu+\frac{\varrho_3^2}{2})(\gamma+\mu)^2\wedge\left(\frac{\gamma^2\varrho_2^2}{2}\right)}{2(\gamma+\mu)^2}\\ <\ & 0\quad a.s.\end{aligned}$$

The result of which is

$$\lim_{\mathbf{t}\to\infty}\mathcal{E}(\mathbf{t})=0,\lim_{\mathbf{t}\to\infty}\mathcal{I}(\mathbf{t})=0\quad a.s.$$

The model (2) is easily understood by taking the fourth equation as an example

$$\lim_{\mathbf{t}\to\infty}\mathcal{Q}(\mathrm{t})=0\quad a.s.$$

Furthermore, on both sides of the first equation of model (2), integrating from 0 to t and dividing by t results in the following.

$$\frac{\mathcal{S}(\mathbf{t})-\mathcal{S}(0)}{\mathbf{t}}=\Theta-\frac{\beta}{1+\mathbf{k}\mathcal{I}}\langle\mathcal{S}\mathcal{I}\rangle+\frac{\varrho_1}{\mathbf{t}}\int_0^{\mathbf{t}}\mathcal{S}(x)dB(x),\tag{33}$$

$$\lim_{\mathrm{t}\to\infty}\langle\mathcal{S}\rangle=\frac{\Theta}{\mu}=\mathcal{S}_0\quad a.s.$$

A similar result can be obtained

$$\lim_{\mathbf{t}\to\infty}\langle\mathcal{R}\rangle=0\quad a.s.$$

The proof is validated.

6 Numerical simulations

A numerical simulation was conducted using Matlab software in order to illustrate the results of the above theorems. In order to determine the discretization transformation of the model (2), we use the Milstein method mentioned in [37].

$$\begin{aligned}
\mathcal{S}_{i+1} &= \mathcal{S}_i + \left[\Theta - \frac{\beta \mathcal{S}_i \mathcal{I}_i}{1+\mathbf{k}\mathcal{I}_i} - \mu \mathcal{S}_i\right]\Delta t + \varrho_1 \mathcal{S}_i \sqrt{\Delta t}\zeta_{1,i} + \frac{\varrho_1^2}{2}\mathcal{S}_i(\zeta_{1,i}^2 - 1)\Delta t,\\
\mathcal{E}_{i+1} &= \mathcal{E}_i + \left[\frac{\beta \mathcal{S}_i \mathcal{I}_i}{1+\mathbf{k}\mathcal{I}_i} - (\gamma+\mu)\mathcal{E}_i\right]\Delta t + \varrho_2 \mathcal{E}_i \sqrt{\Delta t}\zeta_{2,i} + \frac{\varrho_2^2}{2}\mathcal{E}_i(\zeta_{2,i}^2 - 1)\Delta t,\\
\mathcal{I}_{i+1} &= \mathcal{I}_i + [\gamma \mathcal{E}_i - (\xi+\eta+\alpha_i+\mu)\mathcal{I}_i]\Delta t + \varrho_3 \mathcal{I}_i \sqrt{\Delta t}\zeta_{3,i} + \frac{\varrho_3^2}{2}\mathcal{I}_i(\zeta_{3,i}^2 - 1)\Delta t,\\
\mathcal{Q}_{i+1} &= \mathcal{Q}_i + [\eta \mathcal{I}_i - (\delta+\alpha_2+\mu)\mathcal{Q}_i]\Delta t + \varrho_4 \mathcal{Q}_i \sqrt{\Delta t}\zeta_{4,i} + \frac{\varrho_4^2}{2}\mathcal{Q}_i(\zeta_{4,i}^2 - 1)\Delta t,\\
\mathcal{R}_{i+1} &= \mathcal{R}_i + [\xi \mathcal{I}_i + \delta \mathcal{Q}_i - \mu \mathcal{R}_i]\Delta t + \varrho_5 \mathcal{R}_i \sqrt{\Delta t}\zeta_{5,i} + \frac{\varrho_5^2}{2}\mathcal{R}_i(\zeta_{5,i}^2 - 1)\Delta t. \qquad (34)
\end{aligned}$$

Suppose $\zeta_{j,i}(j = 1,2,3,4,5; i = 1,2,..,n)$ represent $\mathcal{N}(0,1)$ is independent distributed random variables and Δt is greater than zero.

Example 1 *Assume that model (2) has the following parameters are considered;* $\Theta = 2$, $\mu = 0.4$, $\beta = 1.25, \mathbf{k} = 0.5$, $\gamma = 0.75$, $\alpha_1 = 0.01$, $\alpha_2 = 0.01$, $\eta = 0.5$, $\varrho_i = 0.3, \quad \forall i = 1 to 5$, *as well as the initial condition values* $(\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)) = 0.25$, *and* $\Delta \mathbf{t} = 0.1$. *Then*

$$\begin{aligned}
\mathcal{R}_0^* &= \frac{\beta\Theta\gamma}{(\mu+\frac{\varrho_1^2}{2})(\gamma+\mu+\frac{\varrho_2^2}{2})(\xi+\eta+\mu+\alpha_1+\frac{\varrho_3^2}{2})}\\
&= 2.1968 > 1,
\end{aligned}$$

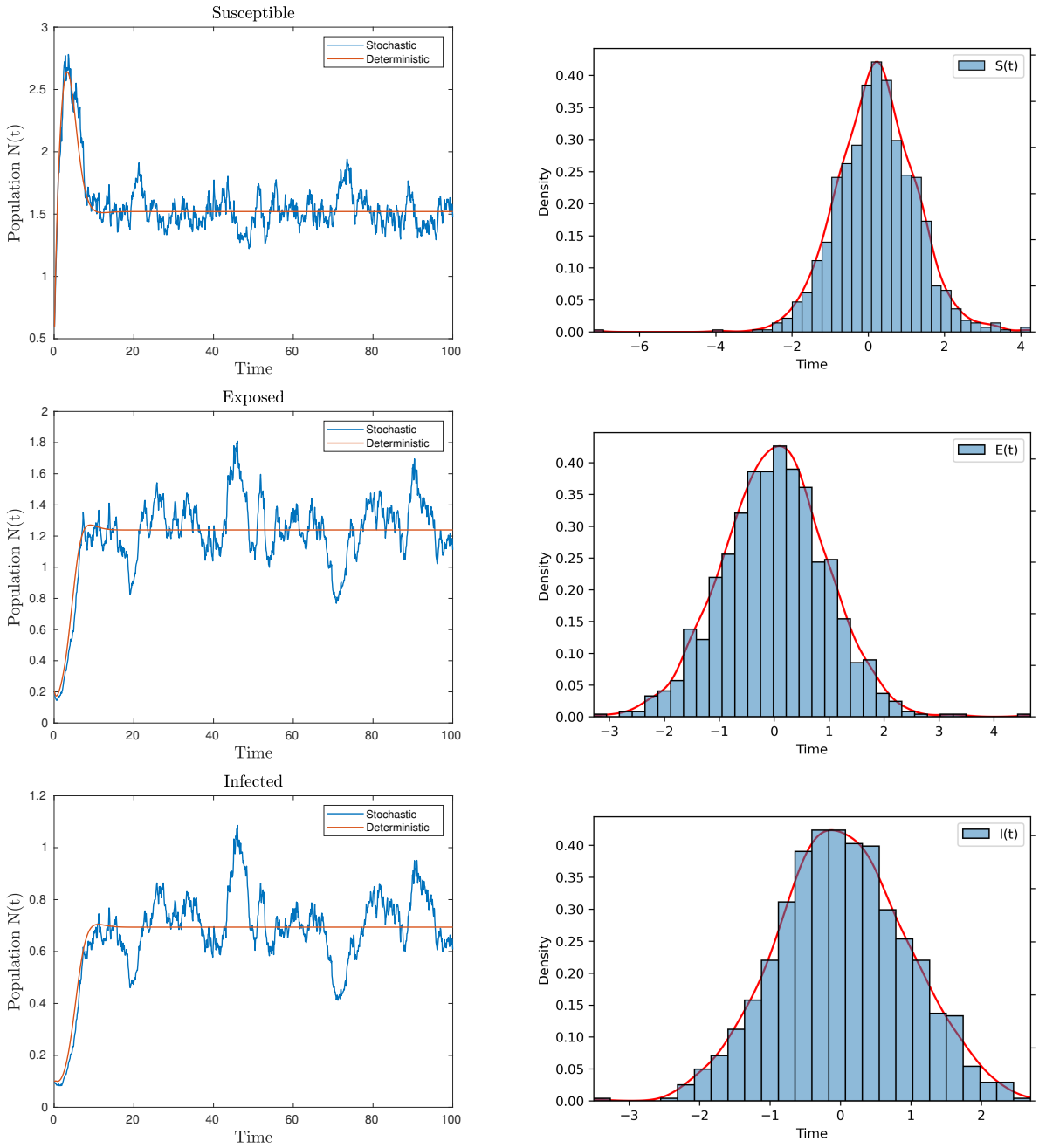
as a result of Theorem 3, model (2) has an ergodic property and a unique stationary distribution $\widehat{\omega}(.)$. This small neighborhood is shown in Figure 1, which shows the ups and downs of the solution of the model (2), as a result, we can see that there is a stationary distribution. In accordance with Theorem 3, Figure 3 and Figure 5a confirms our results.

Example 2 *According to the model (2), the following parameters are considered:* $\Theta = 1.5$, $\mu = 0.5$, $\beta = 1.75$, $\mathbf{k} = 0.25$, $\gamma = 1.5$, $\alpha_1 = 0.2$, $\alpha_2 = 0.2$, $\eta = 0.3$, $\varrho_1 = 0.15$, $\varrho_2 = 1$, $\varrho_3 = 1$, $\varrho_4 = 0.5$, $\varrho_5 = 0.25$, *as well as the initial condition values* $(\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0), \mathcal{R}(0)) = 0.25$, *and* $\Delta \mathbf{t} = 0.5$. *Then*

$$\begin{aligned}
\mathcal{R}_S^* &= \frac{2\gamma\beta(\gamma+\mu)}{\left[(\xi+\mu+\alpha_1+\eta+\frac{\varrho_3^2}{2})(\gamma+\mu)^2\right] \wedge \left(\frac{\gamma^2\varrho_2^2}{2}\right)}\\
&= 0.9315 < 1,
\end{aligned}$$

Figure 2, Figure 4, and Figure 5b satisfy the Theorem 4 conditions; the chances of individuals who will become extinct are almost certainly high when they are exposed, infected, and quarantined. In model (2), we see that the permanent disease can die out through stochastic effects after exposed, infected, and quarantined individuals are sent to extinction. It follows that stochastic disturbances are conducive to controlling epidemic diseases. In Figure 6, we represent that Theorem 2 confirms the stability of the disease-free equilibrium E^* when $R_0 \leq 1$. No negative population values were

observed, ensuring that the model remains biologically valid. This demonstrates the effectiveness of the stochastic Lyapunov function in guaranteeing global asymptotic stability.



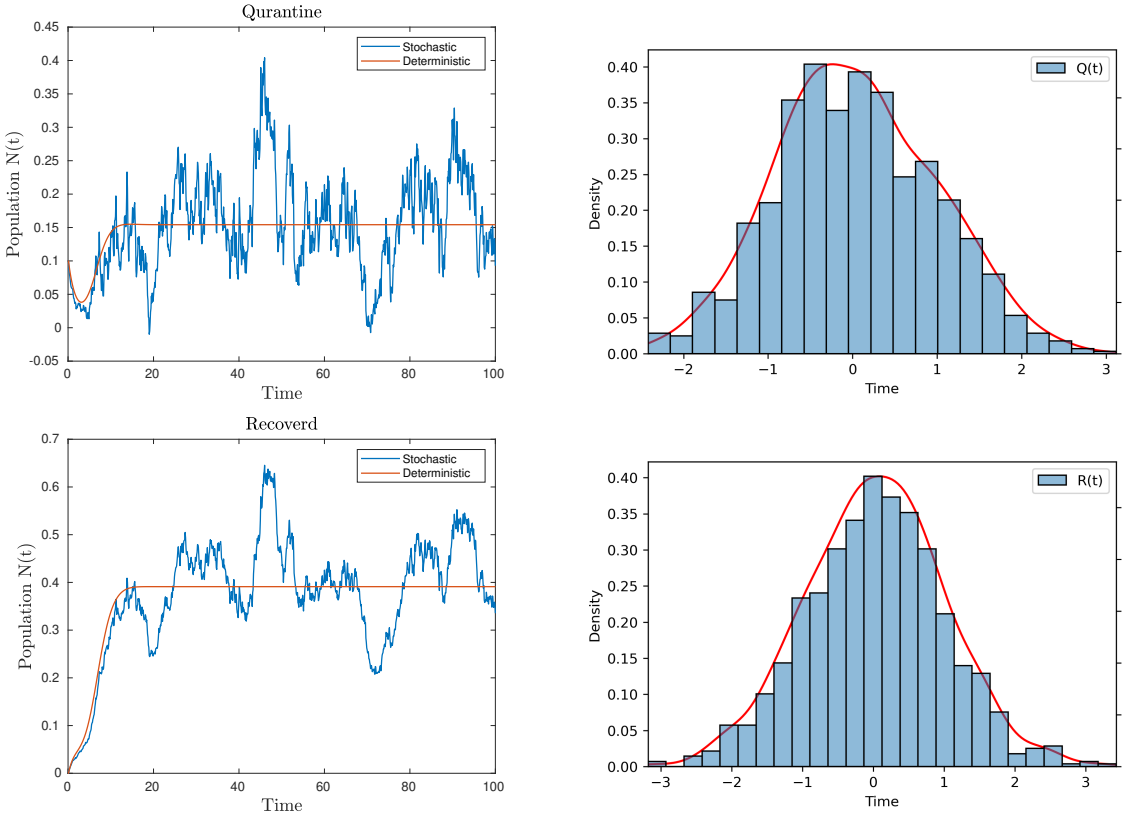


Figure 1. This diagram consists of a time sequence of stochastic persistence and stationary distribution of diseases based on model (2). In the right column, the histogram is represented by the probability density function for $S(\mathbf{t}), E(\mathbf{t}), I(\mathbf{t}), Q(\mathbf{t})$, and $R(\mathbf{t})$

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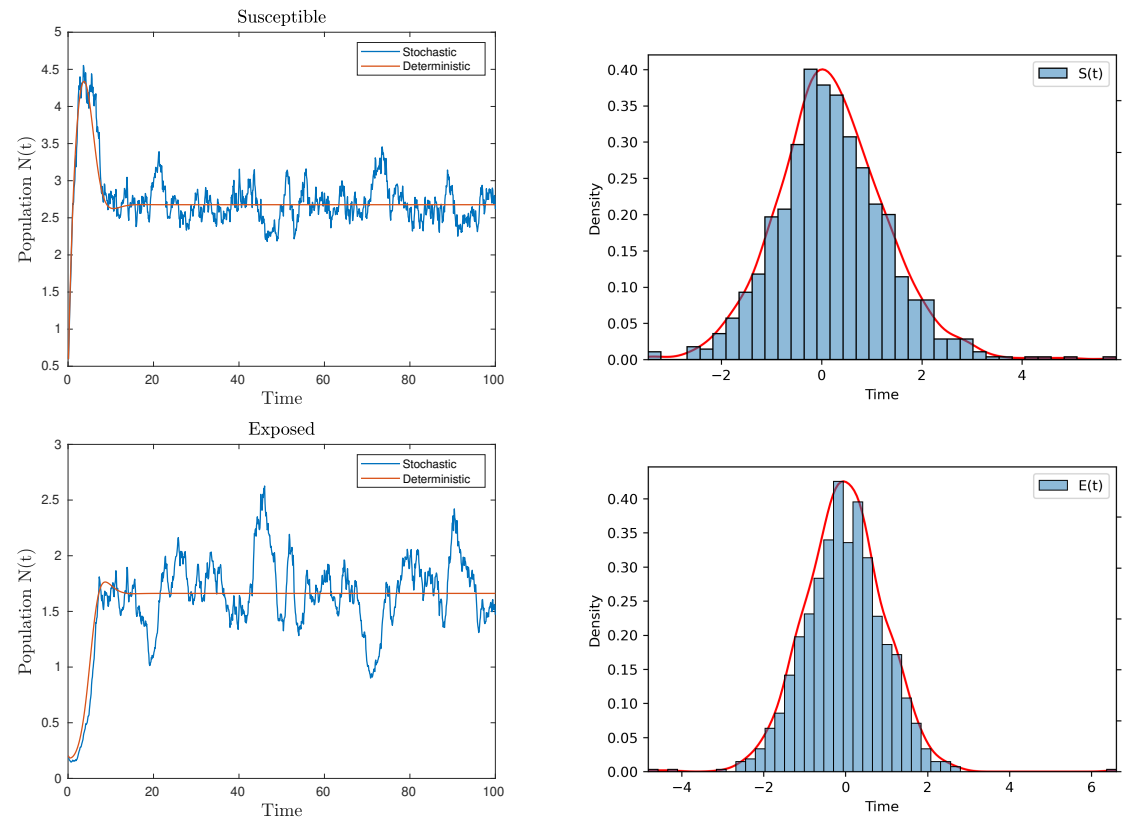


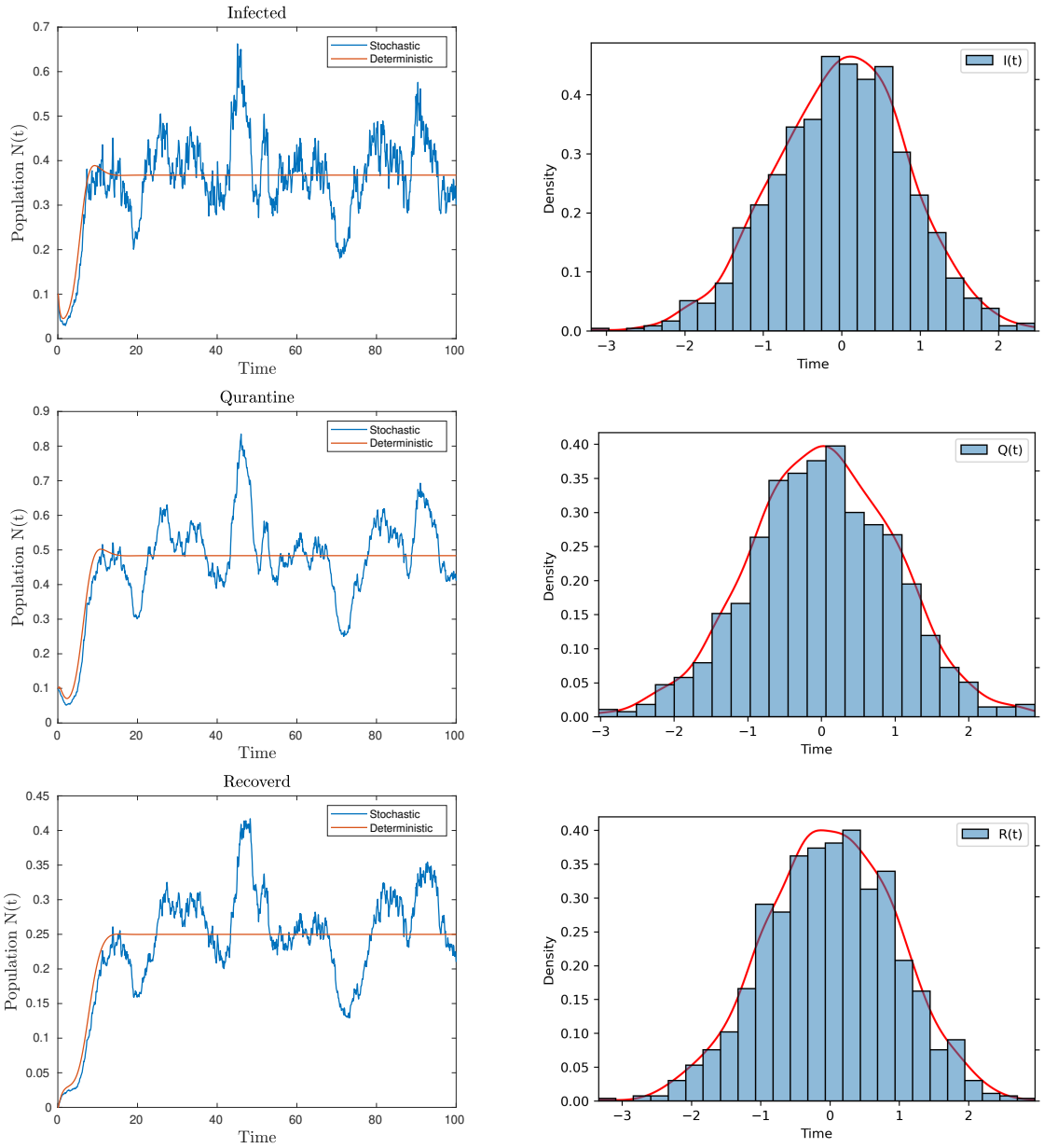
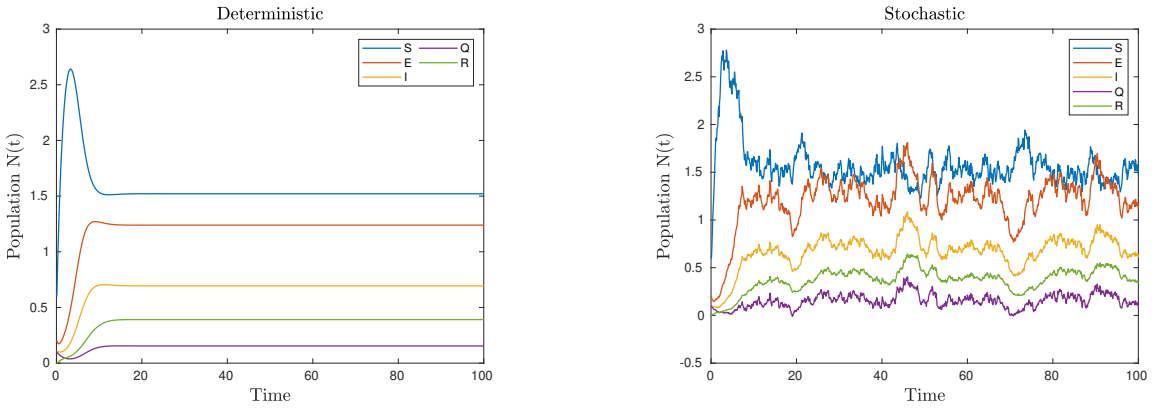


Figure 2. This time sequence diagram illustrates how disease extinction occurs in model (2). In the right column, the histogram is represented by the probability density function for $S(\mathbf{t}), E(\mathbf{t}), I(\mathbf{t}), Q(\mathbf{t})$, and $R(\mathbf{t})$

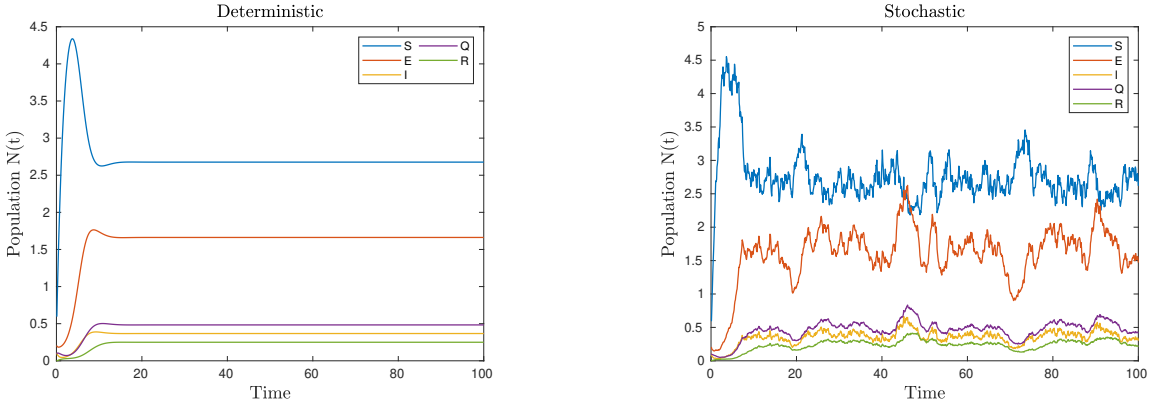
7 Conclusion

An epidemic model with saturated incidence rates is developed in this paper using a stochastic $\mathcal{SEIQR}$ model. As a result of building a suitable stochastic Lyapunov function, we have found that the positive solutions to the model (2) have a stationary distribution when $\mathcal{R}_0^* > 1$. Furthermore, we established sufficient conditions for disease extinction as well, which means, $\mu > \dfrac{\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee \varrho_5^2}{2}$ and $\mathcal{R}_S^* < 1$. Besides, we found that saturated incidence significantly affected disease spread within a population. Furthermore, we find that when white noise intensity is high, the disease disappears, whereas when white noise intensity is low, the disease persists. During stochastic environmental disturbances, white noise prevents disease. This manifests itself at the exact time epidemics arise, affecting epidemic dynamics. The findings show that virus dynamics-based stochastic epidemic models outperform deterministic models in epidemic prediction. Finally, our findings are validated using numerical simulation.



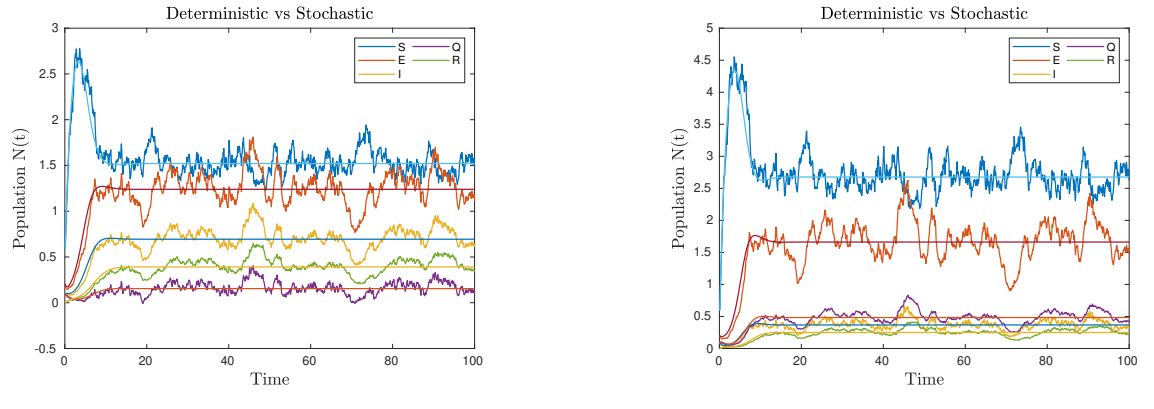
(a) **(b)**

Figure 3. The solution for all class of Deterministic and Stochastic model with $\mathcal{R}_0^*$ greater than 1



(a) **(b)**

Figure 4. The solution for all class of Deterministic and Stochastic model with $\mathcal{R}_S^*$ less than 1



(a) **(b)**

Figure 5. Comparison of solutions on $\mathcal{S}(0), \mathcal{E}(0), \mathcal{I}(0), \mathcal{Q}(0)$, and $\mathcal{R}(0)$: for all class in Deterministic vs Stochastic model with (a) $\mathcal{R}_0^*$ greater than 1 and (b) $\mathcal{R}_S^*$ less than 1

In the future, we will be able to propose practical and complex models, such as models that consider the effects of regime switching on SEIQR epidemics [38] or consider the dynamical characteristics of a stochastic SIR, SEIQR, SEIVQR epidemic model with various incidence and time-based delays [20, 39]. We will continue to investigate these issues in the future.

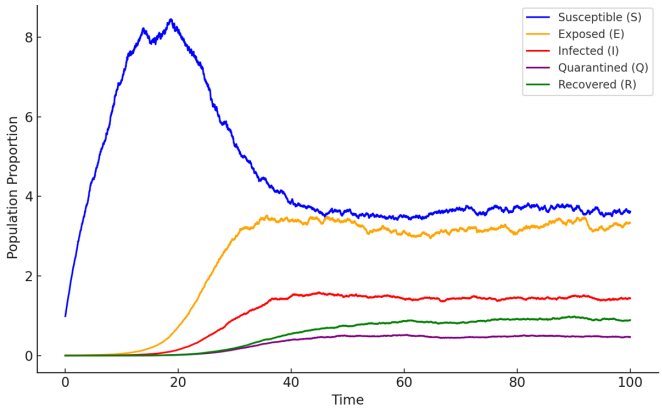


Figure 6. Stochastic SEIQR model simulation in $\mathcal{R}_0 \leq 1$

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

Data availability is not applicable to this article as no new data were created or analyzed in this study.

Ethical approval (optional)

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest regarding the publication of this manuscript.

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Author's contributions

S.S.: Conceptualization, Methodology, Software, Validation, Data Curation, Writing - Original Draft. M.C.: Writing - Review & Editing, Supervision. All authors have read and agreed to the published version of the manuscript.

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