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$(\Delta_v^m)_u$-Statistical Boundedness and Convergence of Order α

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ABSTRACT

In this paper, we defined the concepts of $(\Delta_v^m)_u$-statistical convergence and $(\Delta_v^m)_u$-statistical boundedness for sequences u and v with nonzero terms. Then, we extend these concepts to the concepts of $\left(\Delta_{\lambda,v}^m\right)_u$-statistical convergence and $\left(\Delta_{\lambda,v}^m\right)_u$-statistical boundedness using the sequences (λ_n) satisfying the conditions $\lambda_1 = 1$, $\lambda_{n+1} \le \lambda_n + 1$ and $\lambda_n \to \infty$ $(n \to \infty)$. Then, using the concepts of $\left(\Delta_{\lambda,v}^m\right)_u$-statistical convergence and $\left(\Delta_{\lambda,v}^m\right)_u$-statistical boundedness, we defined the sequence spaces $\left(\Delta_{\lambda,v}^m\right)_u(S_c^\alpha)$ and $\left(\Delta_{\lambda,v}^m\right)_u(S_b^\alpha)$ with the help of numbers α satisfying the condition $0 < \alpha \le 1$. We also investigated the inclusion relations between these sequence spaces and between the sequence spaces obtained in some special cases.

Keywords: Difference Sequences, Statistical Boundedness, Statistical Convergence

α. Mertebeden $(\Delta_v^m)_u$-İstatistiksel Sınırlılık ve Yakınsaklık

ÖZ

Bu makalede, terimleri sıfırdan farklı u ve v sayı dizileri için $(\Delta_v^m)_u$-istatistiksel yakınsaklık ve $(\Delta_v^m)_u$-istatistiksel sınırlılık kavramlarını tanımladık. Daha sonra bu kavramları $\lambda_1 = 1$, $\lambda_{n+1} \le \lambda_n + 1$ ve $\lambda_n \to \infty$ $(n \to \infty)$ şartını sağlayan (λ_n) dizilerini kullanarak $\left(\Delta_{\lambda,v}^m\right)_u$-istatistiksel yakınsaklık ve $\left(\Delta_{\lambda,v}^m\right)_u$-istatistiksel sınırlılık kavramlarına genişlettik. Daha sonra $\left(\Delta_{\lambda,v}^m\right)_u$-istatistiksel yakınsaklık ve $\left(\Delta_{\lambda,v}^m\right)_u$-istatistiksel sınırlılık kavramlarını kullanarak $0 < \alpha \le 1$ şartını sağlayan α sayıları yardımıyla $\left(\Delta_{\lambda,v}^m\right)_u(S_c^\alpha)$ ve $\left(\Delta_{\lambda,v}^m\right)_u(S_b^\alpha)$ dizi uzaylarını tanımladık. Ayrıca bu dizi uzayları arasındaki ve bazı özel durumlarında elde edilen dizi uzayları arasındaki kapsama bağıntılarını inceledik.

Anahtar Kelimeler: Fark Dizileri, İstatistiksel Sınırlılık, İstatistiksel Yakınsaklık

INTRODUCTION

Statistical convergence was studied independently by Fast and Steinhaus in 1951 [1,2]. Then, in 1953, Buck introduced the concept of statistical convergence for real and complex number sequences and its relationship with the theory of summability [3]. The same subject was introduced by Schoenberg as a method of summability [4].

The natural density of a set K is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|$, where $|\{k \le n : k \in K\}|$, denotes the number of elements of the set K not greater than n.

A sequence $x = (x_k)$ if satisfies that: $\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - l| \ge \varepsilon\}| = 0$, for all $\varepsilon > 0$ and some l, we say x is statistically convergent to l and denote it by $S - lim x_k = l$ or $x_k \to l(S)$. The space of all statistically convergent sequences is denoted by Sc. If $l = 0$ then x is called statistically null sequence and the space of all statistically null sequences is denoted by Sc_0. The space of statistically convergent sequences is a sequence algebra so; if $x = (x_k), y = (y_k) \in Sc$ then $xy = (x_k y_k) \in Sc$. If $S - lim x_k = l_x$ and $S - lim y_k = l_y$, then $S - lim x_k y_k = l_x l_y$. It is known that classical convergence implies statistical convergence, namely $c \subset Sc$.

In 2000, Mursaleen introduced the concept of λ-statistical converge, which is a more general form of statistical convergence, using the concept of $\lambda = (\lambda_n)$ sequence [5]. Such that if $\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - l| \ge \varepsilon\}| = 0$ is satisfied for all $\varepsilon > 0$ and some l, then t he sequence x is said to be λ-statistically convergent or S_λ-convergent to l, and the space of all λ-statistically convergent sequences is denoted by S_λ. It can easily be seen that the λ-statistical convergence is same as the statistical convergence for $\lambda_n = n$, where all sequence $\lambda = (\lambda_n) \in \Gamma$ satisfies $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ conditions and diverges to infinity, and $I_n = [n - \lambda_n + 1, n]$. In this study, we will take the same.

Then Gadjiev and Orhan [6] first introduced the idea of order α of statistical convergence. This concept has attracted more attention after Çolak's works [7, 8]. After these Çolak and Bektaş [9] combined the concepts λ-statistical convergence and statistical convergence of order α together and defined λ-statistical convergence of order α, a more general form of both ideas, as follows: $x = (x_k)$ is λ-statistically convergent to l of order α if satisfies that: $\lim_{n\to\infty} \frac{1}{(\lambda_n)^\alpha} |\{k \in I_n : |x_k - l| \geq \varepsilon\}| = 0,$ for $\lambda = (\lambda_n) \in \Gamma$ and all $\varepsilon > 0$, where $0 < \alpha \leq 1$.

The difference sequence concept defined by Kızmaz [10], was generalized to difference of order m by Et and Çolak [11]. Upon this Et and Nuray [12] generalized the statistical convergence concept to Δ^m-statistical convergence. According to this. $(\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ so $\Delta^0 x = x$ and $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for the sequence $x = (x_k)$ and $\Delta^m(X) = \{x = (x_k) \colon \Delta^m x \in X\}$, for any sequence space X. Here, $m \in \mathbb{N}$ is a finite number and we will use m as a finite natural number in this paper.

Çolak [13] defined generalized sequence space $\Delta_v(X)$ for any sequence spaces X and $v = (v_k)$ sequences consisting of nonzero complex numbers as follows.

$$\Delta_v(X) = \{x = (x_k) : \Delta_v(x) \in X\}$$

and examined the topological properties of these spaces, where $\Delta_v(x) = (\Delta_v(x_k)) = (v_k x_k - v_{k+1} x_{k+1})$. Then these sequence spaces was generalized to Δ_v^m by Et and Esi [14]. According to this,

$$\Delta_v^m(X) = \{x = (x_k) \colon \Delta_v^m(x) \in X\},$$

for $X = l_\infty, c, c_0$, where $\Delta_v^0(x) = (v_k x_k)$, $\Delta_v^m(x) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and such that

$$\Delta_v^m(x_k) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.$$

Statistical boundedness is a much newer and less studied compared to statistical convergence. A real or complex statistically bounded sequence $x = (x_k)$ defined by Fridy and Orhan [15] for some $B \geq 0$ satisfies that; $\delta(\{k \colon |x_k| > B\}) = 0$. Space of all statistically bounded sequences is denoted by S_b. Then Bhardwaj and Gupta [16] generalized the concept of statistical boundedness as follows. If a sequence $x = (x_k)$ is statistically bounded of order α, then there is some $B \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n \colon |x_k| > B\}| = 0.$$

The space of all statistically bounded of order α sequences is denoted by S_b^α. Note that $S_b^\alpha = S_b$ for $\alpha = 1$ and $S_b^\alpha = w$ for $\alpha > 1$. So, throughout this article we will take α less than or equal to 1.

Later Temizsu and Et [17] defined statistical convergence in terms of Δ^m-difference sequences and $\lambda = (\lambda_n) \in \Gamma$ sequences as follows and gave some inclusion theorems related to these concepts. If a sequence $x = (x_k)$ is Δ^m-statistically bounded of order α, then there is some $B \geq 0$ such that $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n \colon |\Delta^m x_k| > B\}| = 0.$ The space of all such sequences $x = (x_k)$ is denoted by $\Delta^m(S_b^\alpha)$ and it is obvious that $x \in w$ for $\alpha > 1$. A sequence $x = (x_k)$ is Δ_λ^m-statistically bounded of order α if there is some $B \geq 0$ such that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n \colon |\Delta^m x_k| > B\}| = 0$ for $\lambda = (\lambda_n) \in \Gamma$. The space of all Δ_λ^m-statistically bounded sequences of order α is denoted by $\Delta_\lambda^m(S_b^\alpha)$. For α=1, it turns into Δ_λ^m-statistically bounded and the space of such these sequences is $\Delta_\lambda^m(S_b)$ [17].

MAIN RESULTS

In this section we will define $(\Delta_v^m)_u$-statistically bounded and $(\Delta_v^m)_u$-statistically convergent of order α and examine some inclusion relations.

Definition 1. Let $X = l_\infty, c, c_0$ and $u = (u_k)$, $v = (v_k)$ any fixed nonzero complex numbers' sequences. We define:

$$(\Delta_v^m)_u(X) = \{x = (x_k) \colon (\Delta_v^m)_u(x) \in X\},$$

where $(\Delta_v^m)_u(x) = (u_k \Delta_v^m x_k)$.

Throughout this paper we consider the sequences $u = (u_k)$, $v = (v_k)$ as sequences of nonzero complex numbers.

Definition 2. If there exists some $B \geq 0$ such that $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n \colon |u_k \Delta_v^m x_k| > B\}| = 0,$ then the sequence $x = (x_k)$ is $(\Delta_v^m)_u$-statistically bounded of order α and we denote the space of such sequences with $(\Delta_v^m)_u(S_b^\alpha)$.

It can easily be seen that if we take $\alpha > 1$ then the $(\Delta_v^m)_u\, u(S_b^\alpha)$ turns into w. If we take $\alpha = 1$, then the $x = (x_k)$ become $(\Delta_v^m)_u$-statistically bounded and we will denote this sequences' space by $(\Delta_v^m)_u(S_b)$.

Theorem 1. Let $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. Then $(\Delta_v^m)_u(S_b^\alpha) \subset (\Delta_v^m)_u(S_b^\beta)$ and this inclusion is strict.

Proof: Let $x \in (\Delta_v^m)_u(S_b^\alpha)$. Therefore, we know $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n \colon |u_k \Delta_v^m x_k| > B\}| = 0$. Since $\alpha < \beta$ implies $n^\alpha < n^\beta$ and thus $\frac{1}{n^\alpha} > \frac{1}{n^\beta}$,

$$\frac{1}{n^\alpha} |\{k \leq n \colon |u_k \Delta_v^m x_k| > B\}| > \frac{1}{n^\beta} |\{k \leq n \colon |u_k \Delta_v^m x_k| > B\}|.$$

We take the limit of both sides of the above inequality as $n \to \infty$ and find that $x \in (\Delta_v^m)_u(S_b^\beta)$. To prove the strictness of the inclusion we can choose as $x = (x_k) = (-k)$ and $v = (v_k) = (k-1)$. At that rate for $m = 1$, $\Delta_v^m x = \Delta_v x = (\Delta_v x_k) = (x_k v_k - x_{k+1} v_{k+1}) = (2k)$ is found. If we define u by

$$u_k = \begin{cases} \frac{1}{2k}, & k \neq n^3 \\ \frac{1}{2}, & k = n^3 \end{cases} \quad (n = 0,1,2,3,\dots),$$

then we write

$$u_k \Delta_v x_k = \begin{cases} 1, & k \neq n^3 \\ k, & k = n^3 \end{cases} \quad (n = 0,1,2,3,\dots).$$

For $B = 1$, we can easily see $|\{k \leq n: |u_k \Delta_v x_k| > 1\}| = [\![\sqrt[3]{n}]\!] - 1 \leq \sqrt[3]{n} - 1 < \sqrt[3]{n}$. Where $[\![\ ]\!]$ denotes greatest integer function. Therefore, we have:

$$\frac{1}{n^\beta}|\{k \leq n: |u_k \Delta_v x_k| > 1\}| < \frac{\sqrt[3]{n}}{n^\beta}.$$

Let $0 < \alpha < \frac{1}{3} < \beta \leq 1$. By taking the limit as $n \to \infty$, we obtain $x \in (\Delta_v^m)_u(S_b^\beta)$. From the property greatest integer function, we know $\sqrt[3]{n} - 1 < [\![\sqrt[3]{n}]\!] \leq \sqrt[3]{n}$ which implies:

$$\frac{\sqrt[3]{n} - 2}{n^\alpha} < \frac{1}{n^\alpha}|\{k \leq n: |u_k \Delta_v^m x_k| > 1\}|.$$

By taking the limit, we obtain $x \notin (\Delta_v^m)_u(S_b^\alpha)$.

From Theorem 1 by setting $\beta = 1$, we obtain $(\Delta_v^m)_u(S_b^\alpha) \subset (\Delta_v^m)_u(S_b)$. Note that this inclusion is strict for $\alpha \in (0,1)$.

Proposition 1. $(\Delta_v^m)_u(l_\infty) \subset (\Delta_v^m)_u(S_b^\alpha)$, and this inclusion is strict.

Proof: If $x \in (\Delta_v^m)_u(l_\infty)$, then there exists some $B \geq 0$ such that $|(\Delta_v^m)_u x_k| = |u_k \Delta_v^m x_k| \leq B$ for all $k \in \mathbb{N}$. Thus $\{k \leq n: |u_k \Delta_v^m x_k| > B\} = \emptyset$ and hence

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n: |u_k \Delta_v^m x_k| > B\}| = 0.$$

This implies that $x \in (\Delta_v^m)_u(S_b^\alpha)$. To show that the converse inclusion does not always hold, we can take $u_k = v_k = 1$ for all $k \in \mathbb{N}$ and sequence x like this,

$$x = (x_k) = \begin{cases} k^3, & k \text{ is prime number}, \\ k, & \text{otherwise}. \end{cases}$$

For $m = 2$, we have:

$$\Delta_v^2 x = \begin{cases} 6k + 6, & k \text{ is prime number}, \\ 0, & \text{otherwise}. \end{cases}$$

Since the density of prime numbers is zero, we conclude that $(\Delta_v^2)_u x$ is statistically bounded. However, it is clear that this sequence is not bounded.

Corollary 1. $(\Delta_v^m)_u(l_\infty) \subset (\Delta_v^m)_u(S_b)$ and this inclusion is strict.

Theorem 2.
$(\boldsymbol{i})$ If $u = (u_k) \in S_b$, then $\Delta_v^m(S_b) \subset (\Delta_v^m)_u(S_b)$,
$(\boldsymbol{ii})$ If $v = (v_k) \in S_b$, then $(\Delta^m)_u(S_b) \subset (\Delta_v^m)_u(S_b)$
and these inclusions are strict.

Proof: $(\boldsymbol{i})$ Let $u \in S_b$ and $x \in \Delta_v^m(S_b)$. Then there exists numbers $B_u, B_x \geq 0$ such that $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: |u_k| > B_u\}| = 0$ and $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: |\Delta_v^m x_k| > B_x\}| = 0$

Let $B_u \cdot B_x = B$. Thus, we have:

$$\{k \leq n: |u_k \Delta_v^m x_k| > B\} \subseteq \{k \leq n: |\Delta_v^m x_k| > B_x\} \cup \{k \leq n: |u_k| > B_u\}$$

and therefore,

$$\frac{1}{n}|\{k \leq n: |u_k \Delta_v^m x_k| > B\}| \leq \frac{1}{n}|\{k \leq n: |\Delta_v^m x_k| > B_x\}| + \frac{1}{n}|\{k \leq n: |u_k| > B_u\}|$$

is obtained. Taking the limit as $n \to \infty$ on both sides of the above inequality , $x \in (\Delta_v^m)_u(S_b)$ is obtained.

$(\boldsymbol{ii})$ The proof is similar to that of (i).

Corollary 2. If $u, v \in S_b$ then following inclusions are strict.
$(\boldsymbol{i})$ $\Delta^m(S_b) \subset (\Delta_v^m)_u(S_b)$,
$(\boldsymbol{ii})$ $\Delta^m(l_\infty) \subset (\Delta_v^m)_u(S_b)$.

Theorem 3. $(\boldsymbol{i})$ If $|u_k| \leq |u'_k|$ for all $k \in \mathbb{N}$, then $(\Delta_v^m)_{u'}(S_b) \subseteq (\Delta_v^m)_u(S_b)$,
$(\boldsymbol{ii})$ If $|v_k| \leq |v'_k|$ for all $k \in \mathbb{N}$, then $(\Delta_{v'}^m)_u(S_b) \subseteq (\Delta_v^m)_u(S_b)$.

Proof: $(\boldsymbol{i})$ Let $|u_k| \leq |u'_k|$ for all $k \in \mathbb{N}$ and $x \in (\Delta_v^m)_{u'}(S_b)$. So there exists a number $B \geq 0$ such that:

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n: |u'_k \Delta_v^m x_k| > B\}| = 0.$$

Since $|u_k| \leq |u'_k|$ for all $k \in \mathbb{N}$, we have the following inclusion

$$\{k \leq n: |u_k \Delta_v^m x_k| > B\} \subseteq \{k \leq n: |u'_k \Delta_v^m x_k| > B\}$$

and thus

$$|\{k \leq n: |u_k \Delta_v^m x_k| > B\}| \leq |\{k \leq n: |u'_k \Delta_v^m x_k| > B\}|$$

and

$$\frac{1}{n}|\{k \leq n: |u_k \Delta_v^m x_k| > B\}| \leq \frac{1}{n}|\{k \leq n: |u'_k \Delta_v^m x_k| > B\}|.$$

Taking the limit as $n \to \infty$ on both sides of the inequality, we find that $x \in (\Delta_v^m)_u(S_b)$.

$(\boldsymbol{ii})$ The proof is similar to that of (i).

Corollary 3. It is obvious from Theorem 3 that:

(***i***) If $|u_k| \geq 1$ for all $k \in \mathbb{N}$, then $(\Delta_v^m)_u(S_b) \subseteq \Delta_v^m(S_b)$,

(***ii***) If $|v_k| \geq 1$ for all $k \in \mathbb{N}$, then $(\Delta_v^m)_u(S_b) \subseteq (\Delta^m)_u(S_b)$,

(***iii***) If $|v_k| \leq |v'_k|$ and $|u_k| \leq |u'_k|$ for all $k \in \mathbb{N}$, then $\left(\Delta_{v'}^m\right)_{u'}(S_b) \subseteq (\Delta_v^m)_u(S_b)$.

Theorem 4. The inclusion $(\Delta_v^m)_u(S_b) \subset (\Delta_v^{m+1})_u(S_b)$ is strict.

$Proof$: For the strictness we can use example which we gave in proof of the Proposition 1. Let $u = v = (1)$ and define the sequence x as follows:

$$x = (x_k) = \begin{cases} k^3, & k \text{ is prime number}, \\ k, & otherwise. \end{cases}$$

For $m = 1$, $(\Delta_v)_u x$ is statistically bounded, but for $m = 0$, is not statistically bounded.

Corollary 4. For $0 < \alpha \leq \beta \leq 1$, the strict inclusion $(\Delta_v^m)_u(S_b^\alpha) \subset (\Delta_v^{m+1})_u\left(S_b^\beta\right)$ follows from Theorem 4 and Theorem 1.

Definition 3. If a sequence $x = (x_k)$ satisfies $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n: |u_k \Delta_v^m x_k - l| \geq \varepsilon\}| = 0$, for all $\varepsilon > 0$ and some l, then x is called $(\Delta_v^m)_u$-statistically convergent of order α. The space of such sequences is denoted by $(\Delta_v^m)_u(Sc^\alpha)$. If in above equation we take $\alpha = 1$, then $x = (x_k)$ is called $(\Delta_v^m)_u$-statistically convergent and we will denote this sequences' space by $(\Delta_v^m)_u(Sc)$. It is obvious that $(\Delta_v^m)_u(Sc^\alpha)$ turns into w for $\alpha > 1$.

Theorem 5. Let $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$. Then $(\Delta_v^m)_u(Sc^\alpha) \subset (\Delta_v^m)_u\left(Sc^\beta\right)$ and this inclusion is strict for certain α and β.

The proof is trivial.

Corollary 5. From Theorem 5, we obtain $(\Delta_v^m)_u(Sc^\alpha) \subset (\Delta_v^m)_u(Sc)$ and this inclusion is strict when $0 < \alpha < 1$.

Proposition 2. $(\Delta_v^m)_u(Sc^\alpha) \subset (\Delta_v^m)_u(S_b^\alpha)$ and this inclusion is strict.

$Proof$: Let $x \in (\Delta_v^m)_u(Sc^\alpha)$. In this case, there exists a number l such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n: |u_k \Delta_v^m x_k - l| \geq \varepsilon\}| = 0 \qquad (1)$$

for a fixed $\varepsilon > 0$. Using the properties of the absolute value,

$$\{k \leq n: |u_k \Delta_v^m x_k| \geq \varepsilon + |l|\} \subseteq \{k \leq n: |u_k \Delta_v^m x_k - l| \geq \varepsilon\}$$

and thus

$$|\{k \leq n: |u_k \Delta_v^m x_k| \geq \varepsilon + |l|\}| \leq |k \leq n: |u_k \Delta_v^m x_k - l| \geq \varepsilon|$$

holds. If we multiply both side of the above inequality by $\frac{1}{n^\alpha}$ and taking the limit for $n \to \infty$, we obtain from (1), $x \in (\Delta_v^m)_u(S_b^\alpha)$. To show that the converse inclusion does not always hold, let $m = 1$, $u = (u_k) = \left(\frac{1}{k+1}\right)$, $v = (v_k) = (k + 1)$ and define the sequence $x = (x_k)$ as:

$$x_k = \begin{cases} k, & k \text{ is odd number}, \\ -k, & k \text{ is even number}. \end{cases}$$

It is easy to see that for $m = 1$;

$$(\Delta_v)_u x = \begin{cases} -2, & k \text{ is odd number}, \\ 2, & k \text{ is even number}. \end{cases}$$

This sequence is statistically bounded but not statistically convergent.

Proposition 3. $(\Delta_v^m)_u(c) \subset (\Delta_v^m)_u(Sc)$ and this inclusion is strict.

Theorem 6. If $u = (u_k) \in Sc$, then $\Delta_v^m(Sc) \subset (\Delta_v^m)_u(Sc)$ and this inclusion is strict.

$Proof$: Since (Sc) is a sequence algebra, the proof is obvious. To show the strictness of the inclusion, let $m = 2$, and define $x = (x_k) = (k^2)$, $v = (v_k) = (k)$, and the statistically convergent sequence $u = (u_k)$ as

$$u_k = \begin{cases} (-1)^k, & k \text{ is square}, \\ \dfrac{1}{k}, & otherwise. \end{cases}$$

Then, we have $\Delta_v^2 x = (6k + 6)$, and thus

$$(\Delta_v^2)_u x_k = \begin{cases} (-1)^k(6k + 6), & k \text{ is sequare}, \\ 6 + \dfrac{6}{k}, & otherwise. \end{cases}$$

This sequence is statistically convergent, but $\Delta_v^2 x$ is not statistically convergent.

As a common consequence of Proposition 3 and Theorem 6, we can obtain the following result:

Corollary 6. If $u \in Sc$, then $\Delta_v^m(c) \subseteq (\Delta_v^m)_u(Sc)$.

Definition 4. Let $\alpha \in (0,1]$ and $\lambda \in \Gamma$. We say that a sequence $x = (x_k)$ is $\left(\Delta_{\lambda,v}^m\right)_u$-statistically bounded of order α if there exists a number $B \geq 0$ such that:

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n: |u_k \Delta_v^m x_k| > B\}| = 0.$$

The set of all such sequence is denoted by $\left(\Delta_{\lambda,v}^m\right)_u(S_b^\alpha)$.

Note that this concept reduces to $(\Delta_v^m)_u$-statistical boundedness of order α for $\lambda_n = n$ for all $n \in \mathbb{N}$.

Definition 5. Let $\alpha \in (0,1]$ and $\lambda \in \Gamma$. A sequence $x = (x_k)$ is $\left(\Delta_{\lambda,v}^m\right)_u$-statistically convergent of order α if there exists a number l such that:

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\ |u_k\Delta_v^m x_k-l|>\varepsilon\}\right|=0.$$

By $(\Delta_{\lambda,v}^m)_u(Sc^\alpha)$, we will denote the space of $(\Delta_{\lambda,v}^m)_u$-statistically convergent sequences.

If we get $\lambda_n=n$ for all $n\in\mathbb{N}$, this concept turns into $(\Delta_v^m)_u$-statistical convergence of order α.

Proposition 4. The inclusion $(\Delta_{\lambda,v}^m)_u(Sc^\alpha)\subset(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$ is strict.

Proof: Let $x\in(\Delta_{\lambda,v}^m)_u(Sc^\alpha)$. In this case, there exists a number l such that

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\ |u_k\Delta_v^m x_k-l|>\varepsilon\}\right|=0$$

for fixed $\varepsilon>0$.

Using the properties of the absolute value, we can write:

$$\{k\in I_n:\ |u_k\Delta_v^m x_k|>|l|+\varepsilon\}\subseteq\{k\in I_n:\ |u_k\Delta_v^m x_k-l|>\varepsilon\}$$

and thus

$$\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\ |u_k\Delta_v^m x_k|>|l|+\varepsilon\}\right|\le\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\ |u_k\Delta_v^m x_k-l|>\varepsilon\}\right|.$$

Taking the limit as $n\to\infty$ and setting $|l|+\varepsilon=B$, we obtain $\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|\{k\in I_n:\ |u_k\Delta_v^m x_k|>B\}|=0$. Thus, $x\in(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$. To prove the strictness, by taking $(\lambda_n)=(n)$, we can reconsider the example in Proposition 2.

Theorem 7. Let $\lambda,\mu\in\Gamma$, $\lambda_n\le\mu_n$ for all $n\in\mathbb{N}$ and $0<\alpha\le\beta\le1$.

(***i***) If $\liminf_{n\to\infty}\frac{\lambda_n^\alpha}{\mu_n^\beta}>0$, then $(\Delta_{\mu,v}^m)_u(S_b^\beta)\subset(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$,

(***ii***) If $\lim_{n\to\infty}\frac{\lambda_n^\alpha}{\mu_n^\beta}=1$ and $\lim_{n\to\infty}\frac{\mu_n}{\mu_n^\beta}=1$, then $(\Delta_{\mu,v}^m)_u(S_b^\beta)=(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$.

Proof: The proof can be referred to Temizsu and Et, Theorem 4 [17].

From Theorem 7 we obtain the following results:

Corollary 7. Let $\lambda,\mu\in\Gamma$, $\lambda_n\le\mu_n$ for all $n\in\mathbb{N}$ and $\alpha\in(0,1]$.

(***i***) If $\liminf_{n\to\infty}\frac{\lambda_n}{\mu_n}>0$, then $(\Delta_{\mu,v}^m)_u(S_b)\subset(\Delta_{\lambda,v}^m)_u(S_b)$

(***ii***) If $\lim_{n\to\infty}\frac{\lambda_n}{\mu_n}=1$, then $(\Delta_{\mu,v}^m)_u(S_b)=(\Delta_{\lambda,v}^m)_u(S_b)$.

Corollary 8. Let $\lambda\in\Gamma$, $\lambda_n\le n$ for all $n\in\mathbb{N}$ and $\alpha\in(0,1]$.

(***i***) If $\liminf_{n\to\infty}\frac{\lambda_n^\alpha}{n}>0$, then $(\Delta_v^m)_u(S_b)\subset(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$,

(***ii***) If $\lim_{n\to\infty}\frac{\lambda_n^\alpha}{n}=1$, then $(\Delta_v^m)_u(S_b)=(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$.

Theorem 8. Let $\lambda,\mu\in\Gamma$, $\lambda_n\le\mu_n$ for all $n\in\mathbb{N}$ and $0<\alpha\le\beta\le1$.

(***i***) If $\liminf_{n\to\infty}\frac{\lambda_n^\alpha}{\mu_n^\beta}>0$, then $(\Delta_{\mu,v}^m)_u(Sc^\beta)\subseteq(\Delta_{\lambda,v}^m)_u(Sc^\alpha)$,

(***ii***) If $\lim_{n\to\infty}\frac{\lambda_n^\alpha}{\mu_n^\beta}=1$ and $\lim_{n\to\infty}\frac{\mu_n}{\mu_n^\beta}=1$, then $(\Delta_{\mu,v}^m)_u(Sc^\beta)=(\Delta_{\lambda,v}^m)_u(Sc^\alpha)$.

The proof is trivial.

From Theorem 8 we can get the following result:

Corollary 9. Let $\lambda\in\Gamma$, $\lambda_n\le n$ for all $n\in\mathbb{N}$ and $\alpha\in(0,1]$.

(***i***) If $\liminf_{n\to\infty}\frac{\lambda_n^\alpha}{n}>0$, then $(\Delta_v^m)_u(Sc)\subseteq(\Delta_{\lambda,v}^m)_u(Sc^\alpha)$,

(***ii***) If $\lim_{n\to\infty}\frac{\lambda_n^\alpha}{n}=1$, then $(\Delta_v^m)_u(Sc)=(\Delta_{\lambda,v}^m)_u(Sc^\alpha)$.

Proposition 5. $(\Delta_v^m)_u(l_\infty)\subset(\Delta_{\lambda,v}^m)_u(S_b^\alpha)$.

Proof: Let $x\in(\Delta_v^m)_u(l_\infty)$. Thus there exists $B\ge0$ such that $\{k:|u_k\Delta_v^m x_k|>B\}=\emptyset$. Thus, we have

$$\{k:|u_k\Delta_v^m x_k|>B\}\supseteq\{k\in I_n:|u_k\Delta_v^m x_k|>B\}.$$

Therefore, we obtain

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\ |u_k\Delta_v^m x_k|>B\}\right|=0.$$

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