BİTLİS EREN ÜNİVERSİTESİ
FEN BİLİMLERİ DERGİSİ

ISSN: 2147-3129 / e-ISSN: 2147-3188

Article Type	: Research Article	**Year**	: 2025
Received	: October 25, 2024	**Volume**	: 14
Revised	: March 11, 2025	**Issue**	: 1
Accepted	: March 16, 2025	**Pages**	: 260-272
DOI	: 10.17798/bitlisfen.1573596		

COMPARISON OF NUMERICAL METHODS FOR THE KUBA OSCILLATOR

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ABSTRACT

In this study, numerical solutions of stochastic differential equation (SDE) systems have been analyzed and three different numerical methods used for solving these systems, the Milstein method, the Simplified Second-Order Taylor Scheme, and the Stochastic Runge-Kutta (SRK) method, have been compared. The Kubo oscillator model has been considered and the stochastic dynamics of this model have been solved using numerical methods. Initially, the general structure of SDEs is introduced, and the theoretical foundations of the methods used for solving these systems are explained.

In the study, the stochastic model of the Kubo oscillator was solved numerically using the Milstein method, the Simplified Second-Order Taylor Scheme, and the SRK method. The results obtained were compared with exact solutions. In the numerical computations, the accuracy of all three methods is analyzed for different discretization counts and the results were supported by graphs and error tables. The comparisons revealed that the Simplified Second-Order Taylor Scheme provided more accurate solutions compared to the Milstein method. It is observed that the Taylor method and the SRK 2-stage method gave close results. Additionally, it was observed that increasing the number of discretizations brought both methods closer to the exact solution.

Keywords: Stochastic differential equation systems, Numerical approximations, Kubo oscillator.

1 INTRODUCTION

Stochastic differential equations (SDEs) are mathematical tools developed and widely used to model the effects of uncertainty in various fields. These equations are employed in disciplines such as physics, biology, engineering, and economics to understand how systems respond to random variables. For instance, SDEs can enhance the realism of modeling changes in population dynamics in biological systems under uncertainty [1]. Similarly, in finance, effective risk management approaches have been developed by modeling market fluctuations using SDEs [2].

The Kubo oscillator is a well-known model used to study uncertainties and random effects when modeling with stochastic differential equations [3]. This model is particularly suitable for evaluating the dynamics of stochastic processes and the performance of various numerical methods applied to these processes.

The Kubo oscillator is a cornerstone model in stochastic dynamics, blending deterministic behavior with random fluctuations to describe systems influenced by noise. Initially introduced by Kubo to model spectral diffusion and line broadening in nuclear magnetic resonance (NMR), it has since evolved into a versatile tool used across various scientific and engineering disciplines [4]. The oscillator's dynamics, governed by the interplay between a damping term and stochastic forcing, enable it to capture the complexity of processes ranging from molecular interactions in fluctuating environments [5].

The Kubo oscillator, a fundamental example of a stochastic Hamiltonian system, is a subject of extensive study for the purpose of comprehending the interplay between deterministic Hamiltonian dynamics and stochastic noise. Its structural characteristics offer insights into the influence of random fluctuations on physical and mathematical systems, thus establishing it as a pivotal model in the field of stochastic dynamics [6].

In addition, the Kubo oscillator effectively models systems affected by high-frequency oscillations and multiplicative noise, making it a cornerstone in the study of stochastic dynamics [7].

The Milstein method is frequently used for solving stochastic differential equations and provides higher accuracy as an extension of the Euler-Maruyama method [8]. This method is especially preferred for reducing the error rate in stochastic models. On the other hand, the Simplified Second-Order Taylor Scheme aims to provide more precise solutions by considering higher-order terms of stochastic equations [9].

The fact that analytical solutions for SDEs are often not feasible increases the importance of numerical solution methods. In fields such as biology and ecology, modeling using these equations provides essential insights into understanding the impact of uncertainties in the real world [10]. However, in such modeling, ensuring accuracy and optimizing computational costs are equally important. Comparing numerical approaches in terms of computational complexity and accuracy is therefore essential [11].

Previous studies on the Kubo oscillator have shown that this model effectively represents the dynamics of stochastic oscillations and provides a suitable foundation for testing numerical methods. Fox, Roy, and Yu (1987) demonstrated that different simulation algorithms, including those for white and colored noise, can be effectively tested using the Kubo oscillator, and that the use of colored noise significantly enhances the accuracy of the numerical solutions compared to white noise simulations, reducing spurious decays in the oscillator amplitude [12].

In this study, the Milstein method, the Simplified Second-Order Taylor Scheme and Stochastic Runge-Kutta method are applied to the Kubo oscillator model, and the obtained numerical results are compared with exact solutions. Additionally, different discretization strategies are used to analyze the accuracy and computational efficiency of both methods.

In conclusion, this study aims to compare the performance of three different numerical methods used for SDEs within the context of the Kubo oscillator model. This comparison highlights the advantages and disadvantages of these methods, providing valuable insights into determining the most suitable method for solving SDEs.

2 MATERIAL AND METHOD

2.1 Systems of Stochastic Differential Equations

Systems of stochastic differential equations are used for vector-valued states or high-order stochastic differential equations. The general form of a d-dimensional stochastic differential equation is described by [9] as following:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t \tag{1}$$

where $W = \{W_t, t \geq 0\}$ Wiener process is m-dimensional and its components $W_t^1, W_t^2, \dots, W_t^m$ are independent Wiener processes with respect to a common family of $\sigma -$ algebras $\{A_t, t \geq 0\}$, d-dimensional vector function is $f: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $d \times m$-matrix function is $g: [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$. Stochastic integral equation form of (1) is

$$X_t = X_{t_0} + \int_{t_0}^{t} f(r, X_r)dr + \int_{t_0}^{t} g(r, X_r)dW_r \tag{2}$$

i.th component of (2) is

$$X_t^i = X_{t_0}^i + \int_{t_0}^{t} f^i(r, X_r)dr + \sum_{k=1}^{m} \int_{t_0}^{t} g^{i,k}(r, X_r)dW_r^k \tag{3}$$

where $i = 1,2, \dots, d$.

The existence and uniqueness for strong solutions of (1) are proven in [9]. There is a relation between the Ito stochastic differential equation and the Stratonovich stochastic differential equation as follows: X_t solution of (1) is also a solution of the Stratonovich differential equation as;

$$dX_t = \bar{f}(t, X_t)dt + g(t, X_t)odW_t \tag{4}$$

where

$$\bar{f}^i(t, X_t) = f^i(t, X_t) - h^i(t, X_t) \tag{5}$$

and

$$h^i(t, X_t) = \frac{1}{2}\sum_{n=1}^{d}\sum_{k=1}^{m} g^{n,k}(t, X_t)\frac{\partial g^{i,k}}{\partial x_n}(t, X_t) \tag{6}$$

for $i = 1,2, \dots, d$.

The general form of d-dimensional linear stochastic differential equations is

$$dX_t = [F(t)X_t + f(t)]dt + \sum_{l=1}^{m} [G^l(t)X_t + g^l(t)]\, dW_t^l \tag{7}$$

where $F(t), G^1(t), G^2(t), \dots, G^m(t)$ are $d \times d$-matrix function and $f(t), g^1(t), g^2(t), \dots, g^m(t)$ are d-dimensional vector functions. Solution of (7) is

$$X_t = \Psi_{t,t_0}\left\{X_{t_0} + \int_{t_0}^{t} \Psi_{r,t_o}^{-1}\left(f(r) - \sum_{l=1}^{m} G^l(r)g^l(r)\right)dr + \sum_{l=1}^{m}\int_{t_0}^{t} \Psi_{r,t_0}^{-1} g^l(r)dW_r^l\right\} \tag{8}$$

where Ψ_{t,t_0} is $d \times d$-fundamental matrix which satisfying $\Psi_{t_0,t_0} = I$ and the homogeneous matrix stochastic differential equation

$$d\Psi_{t,t_0} = \mathrm{F(t)}\Psi_{t,t_0}dt + \sum_{l=1}^{m} G^l(t)\,\Psi_{t,t_0}dW_t^l \tag{9}$$

Because the systems of linear stochastic differential equations cannot be solved analytically. Therefore, we need numerical solutions which are given in [10], [13], [14].

2.2 Numerical Methods for Stochastic Differential Equations

We use in our study three numerical methods named with the Milstein method, the Simplified order 2 Taylor scheme, and the SRK method.

2.2.1 Milstein Method

The Milstein method is a numerical method used to approximate solutions to stochastic differential equations (SDEs). It is particularly well-known for achieving strong order 1 convergence under certain conditions, making it more accurate than the other simple methods (which has strong order 0.5). The Milstein scheme has the following recurrence formula for d-dimensional stochastic differential equations when $m = 1$. i.th component of the Milstein scheme for (1) given in [8] as

$$X_{k+1}^i = X_k^i + f^i(t, X_t)\Delta t + g^i(t, X_t)\Delta W_k + \frac{1}{2}\left(\sum_{n=1}^{d} g^n \frac{\partial g^i}{\partial x^n}(t, X_t)\right)[(\Delta W_k)^2 - \Delta t] \tag{10}$$

where

$$\Delta W_k = W_{t_{k+1}} - W_{t_k}, \qquad i = 1,2,\dots,d, \qquad k = 1,2,\dots,N\,.$$

2.2.2 Simplified Order 2 Taylor Scheme

The Simplified Second-Order Taylor Scheme provides significant advantages in solving stochastic differential equations (SDEs) by incorporating higher-order terms that improve accuracy and reduce errors. Unlike first-order methods such as Euler-Maruyama, which have a strong convergence rate of $\left(\mathrm{O}\left(\Delta \mathrm{t}^{1/2}\right)\right)$ or the Milstein Method, which has a strong convergence rate of $\left(\mathrm{O}(\Delta \mathrm{t})\right)$, the Second-order scheme achieves a faster convergence rate of $\left(\mathrm{O}(\Delta \mathrm{t}^2)\right)$. This improvement allows for more precise approximations, especially in systems with nonlinear dynamics and significant stochastic influences. By accounting for higher-order interactions, the method effectively captures the interplay between deterministic and stochastic components,

ensuring a more faithful representation of the system's behavior. Additionally, this scheme mitigates accumulated errors in long-term simulations, enhancing stability and reliability over time [9].

In the d-dimensional case, the i.th component of the Simplified Order 2 Taylor scheme for (1) is given in [15] as follows:

$$X_{k+1}^{i} = X_k^i + \sum_{l=1}^{m} g^{i,l}\Delta W_k^l + f^i\Delta t + \frac{1}{2}\sum_{p,l=1}^{m}\left(\sum_{n=1}^{d} g^{n,p}\frac{\partial g^{i,l}}{\partial x^n}\right)\left(\Delta W_k^p \Delta W_k^l + U_{pl,k}\right)$$

$$+\frac{1}{2}\sum_{l=1}^{m}\left(\sum_{j=1}^{d} g^{j,l}\frac{\partial f^i}{\partial x^j} + \frac{\partial g^{i,l}}{\partial t} + \sum_{j=1}^{d} f^j\frac{\partial g^{i,l}}{\partial x^j} + \frac{1}{2}\sum_{n,j=1}^{d} h^{n,j}\frac{\partial^2 g^{i,l}}{\partial x^n \partial x^j}\right)\Delta t\Delta W_k^l \qquad (11)$$

$$+\frac{1}{2}\left(\frac{\partial f^i}{\partial t} + \sum_{n=1}^{d} f^n\frac{\partial f^i}{\partial x^n} + \frac{1}{2}\sum_{n,j=1}^{d} h^{n,j}\frac{\partial^2 f^i}{\partial x^n \partial x^j}\right)(\Delta t)^2$$

where $k = 1,2,\dots,N$ and $U_{pl,k}$,$p,l = 1,2,\dots,m$ are independent two-point distributed random variables with

$$P(U_{pl,k} = \Delta) = \frac{1}{2} = P(U_{pl,k} = -\Delta) \quad if\ l < p$$

$$U_{pp,k} = -\Delta$$

$$U_{pl,k} = -U_{lp,k} \quad if\ l > p$$

2.2.3 Two Stage Stochastic Runge-Kutta Method

The scheme of the Stochastic Runge-Kutta (SRK) method for a system of stochastic differential equations according to the s −stage ($s \geq 1$) is as follows [15]:

$$X_{k+1} = X_k + \sum_{j=1}^{s}\alpha_j f(t_k + \mu_j\Delta t, \eta_j)\Delta t + \sum_{j=1}^{s}\beta_j g(t_k + \mu_j\Delta t, \eta_j)\Delta W_k \qquad (12)$$

where $k = 1,2,\dots,N$ and $\mu_1 = 0, \eta_1 = X_k$ for $j = 1,2,\dots s$

$$\eta_j = X_k + \sum_{l=1}^{j-1}\lambda_{jl} f(t_k + \mu_l\Delta t, \eta_l)\Delta t + \sum_{l=1}^{j-1}\gamma_{jl} g(t_k + \mu_l\Delta t, \eta_l)\Delta W_k$$

and α_j, β_j are to provide $\sum_{j=1}^{s}\alpha_j = \sum_{j=1}^{s}\beta_j = 1$.

The coefficients occurring in (12) can be displayed by generalising Butcher arrays;

$$\begin{array}{c|cccc|cccc} \mu_2 & \lambda_{21} & & & & \gamma_{21} & & & \\ \mu_3 & \lambda_{31} & \lambda_{32} & & & \gamma_{31} & \gamma_{32} & & \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\ \mu_s & \lambda_{s1} & \lambda_{s2} & \cdots & \lambda_{s,s-1} & \gamma_{s1} & \gamma_{s2} & \cdots & \gamma_{s,s-1} \\ \hline & \alpha_1 & \alpha_2 & \cdots & \alpha_{s-1} \quad \alpha_s & \beta_1 & \beta_2 & \cdots & \beta_{s-1} \quad \beta_s \end{array}$$

In our study, the SRK method for 2-stage is used and coefficients are taken from [16]

$$\begin{array}{c|cc|cc} 1 & 1 & & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

The algorithms of the Milstein, the Simplified Order 2 Taylor, and the SRK methods are given in Figure 1.

Our aim in this paper is to compare these numerical methods. We compare the exact solution with numerical solutions that we obtain using the MATLAB program. We aim to demonstrate the errors between the exact solution and each numerical solution. We support our work with graph and error tables.

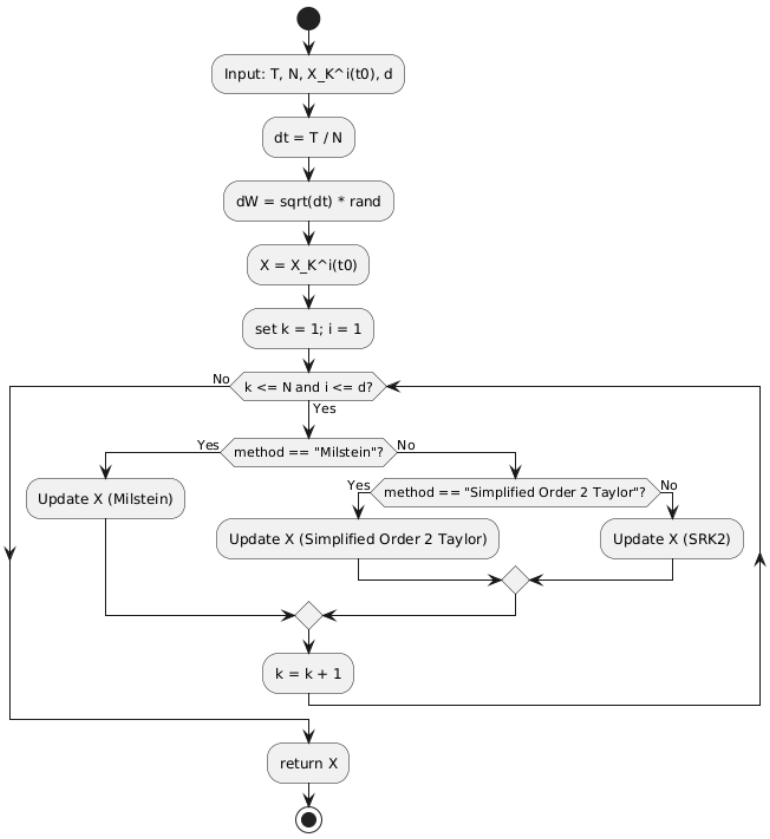

Figure 1. Flowchart of the Milstein, the Simplified Order 2 Taylor, and the SRK methods.

3 NUMERICAL EXPERIMENTS AND RESULTS

Consider the Kubo oscillator in our study. This oscillator is used in many works to demonstrate the efficiency of numerical methods. The general form of the Kubo oscillator, which is given in [1], is

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} dt + \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} o dW_t \tag{13}$$

with $X_0^1 = x^1, X_0^2 = x^2$ where α , γ are real and $t \in [0, T]$. For initial values $X_0^1 = 1$, $X_0^2 = 0$, the exact solution of (13) is

$$X_t^1 = \cos(\alpha t + \gamma W_t) \tag{14}$$

$$X_t^2 = \sin(\alpha t + \gamma W_t)$$

Coefficients $\alpha = 2, \gamma = 0.3$ and exact solutions are taken from [3] (David Cohen who used these coefficients for testing his method) and initial values are $X_0^1 = 1, X_0^2 = 0$, $T = 10$. Using (5) and (6) formulas we obtain Ito stochastic differential equation system for (13) as follow;

$$dX_t^1 = \left(-2X_t^2 - \frac{(0.3)^2}{2} X_t^1 \right) dt - 0.3X_t^2 dW_t \tag{15}$$

$$dX_t^2 = \left(2X_t^1 - \frac{(0.3)^2}{2} X_t^2 \right) dt + 0.3X_t^1 dW_t$$

where $d = 2, m = 1$.

We solve numerically (15), using the Milstein scheme (10) for $X_0^1 = 1, X_0^2 = 0, T = 10$ and $N = 10000$ discretization by the MATLAB program. There is numerical approximation and an exact solution in the same graph in Figure 2. In this figure, the exact solution is plotted with a red line, and the Milstein approximation is plotted with a blue line. As seen in Figure 2, the approximate solution and the exact solution are very close to each other.

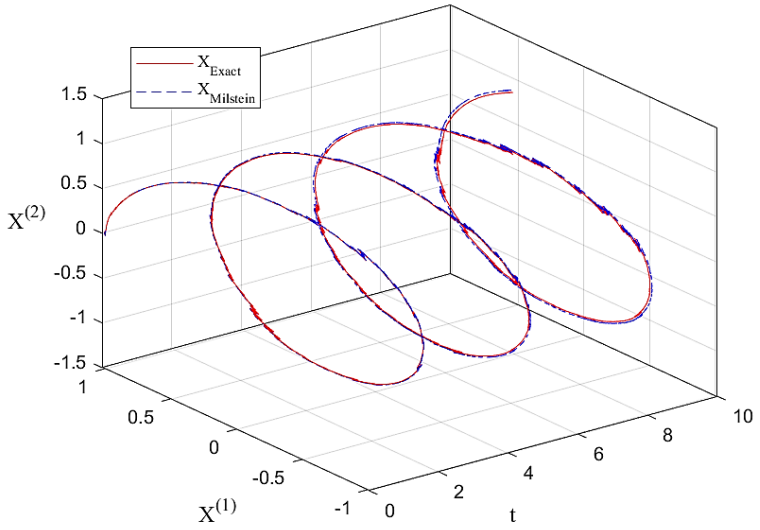

Figure 2. The exact solution with the Milstein Approximation of the Kubo Oscillator for $N = 10000$, T=10, $\alpha = 2, \gamma = 0.3$.

Analogously, using the simplified order 2 Taylor scheme (11) for solving (15), we obtain Figure 3, where the exact solution is plotted with a red line and the Simplified Order 2 Taylor Scheme approximation is plotted with a blue line. It is seen that the solutions are very close.

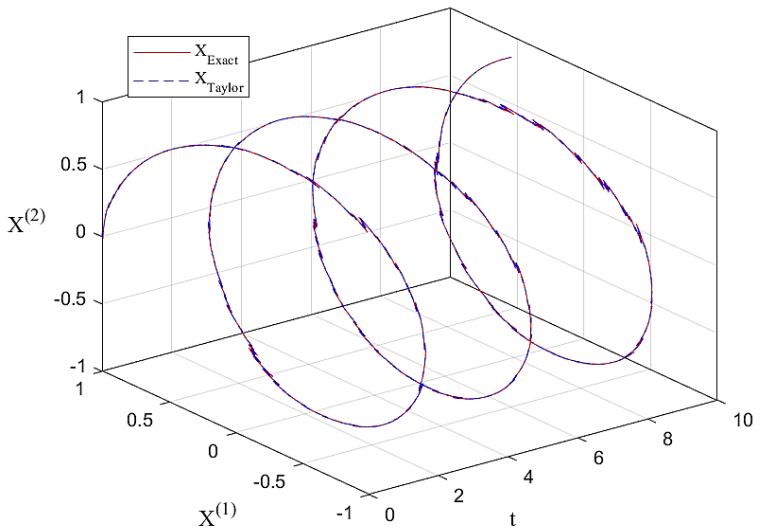

Figure 3. The exact solution and the Simplified 2 order Taylor Approximation of the Kubo Oscillator for $N = 10000$, T=10, $\alpha = 2, \gamma = 0.3$.

In addition, using the SRK method for 2-stage solving (15), we obtain Figure 4, where the exact solution is plotted with a red line and the SRK approximation is plotted with a blue line.

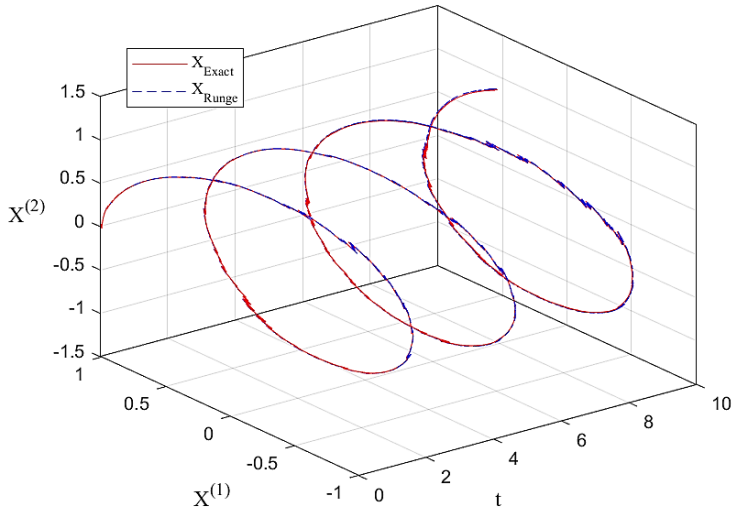

Figure 4. The exact solution and the SRK for 2-Stage Approximation of the Kubo Oscillator for $N = 10000$, T=10, $\alpha = 2, \gamma = 0.3$.

For efficiency both approximations are given in the same graph in Figure 5. There the exact solution is plotted with a red line, the Milstein approximation is plotted with a green line, the simplified order 2 Taylor scheme approximation is plotted with a blue line, and the SRK approximation is plotted with a magenta line.

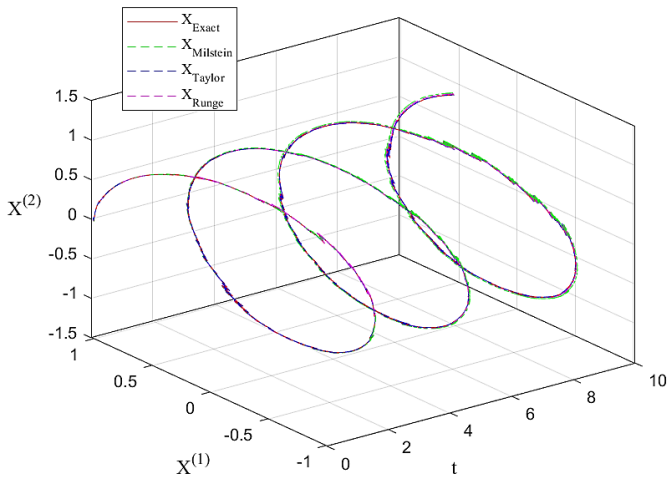

Figure 5. The exact solution with the Milstein Approximation Simplified 2 Order Taylor Approximation and the SRK Approximation of the Kubo Oscillator for $N = 10000$, T=10, $\alpha = 2, \gamma = 0.3$.

To compare the solutions obtained by the Milstein, the Simplified 2 Order Taylor methods, and the SRK methods, the mean absolute error and mean relative error were calculated. For different N discretizations, there are the mean absolute errors (MAE) obtained with the Milstein Simplified 2 Order Taylor and SRK methods using following scheme;

$$MAE = \frac{1}{N}\sum_{k=1}^{N}\left|X_k - \tilde{X}_k\right| \tag{16}$$

where X_k and $\tilde{X}_k$ represent the exact and approximate solutions, respectively, in iteration k.

Table 1. Mean absolute errors of the Milstein, Simplified 2 Ordet Taylor and SRK methods for different discretizations.

N	**Milstein Method**		**Simplified 2 Order Taylor Method**		**SRK Method for 2-Stage**	
	X_t^1	X_t^2	X_t^1	X_t^2	X_t^1	X_t^2
10^4	0.0059	0.0057	0.0036	0.0035	0.00357	0.00362
10^5	6.8179e-04	7.6157e-04	1.6650e-04	2.0252e-04	1.6650e-04	2.0252e-04
10^6	6.6710e-05	7.5789e-05	6.5196e-05	7.2913e-05	6.5197e-05	7.2911e-05

Table 1 shows the mean absolute errors of the three methods for different N discretizations. It is observed that the solutions obtained by the Simplified 2 Order Taylor method are more effective than the solutions obtained by the Milstein method. Runge-kutta 2-stage and Simplified 2 Order Taylor solutions are close to each other.

In addition, for different N discretizations, there are the mean relative errors (MRE) obtained with the three methods using the following scheme:

$$MRE = \frac{1}{N}\sum_{k=1}^{N}\left|\frac{X_k - \tilde{X}_k}{X_k}\right| \tag{17}$$

where X_k and $\tilde{X}_k$ represent the exact and approximate solutions, respectively, in iteration k.

Table 2. Mean relative errors of the Milstein, Simplified 2 Ordet Taylor and SRK method for different discretizations.

N	**Milstein Method**		**Simplified 2 Order Taylor Method**		**SRK Method for 2-Stage**	
	X_t^1	X_t^2	X_t^1	X_t^2	X_t^1	X_t^2
10^4	2.0440e-05	9.4782e-06	2.1132e-08	7.7765e-09	2.1135e-08	7.7769e-09
10^5	3.4300e-07	1.1335e-06	5.4836e-10	1.6983e-09	5.4839e-10	1.6988e-09
10^6	1.8974e-09	1.4597e-09	2.1355e-12	1.5529e-12	2.1357e-12	1.5531e-12

Table 2 shows the mean relative errors of the three methods for different N discretizations. It can be seen that the solutions obtained with the simplified 2 Ordet Taylor method are better than the solutions obtained with the Milstein method.

4 RESULTS AND DISCUSSION

In this paper we compare the Milstein scheme and the Simplified 2 Order Taylor scheme for the Kubo oscillator. For each scheme we show the approximate solution and the exact solution. We also calculate the mean absolute error and the mean relative error between the numerical solutions and the exact solution. According to our results, we can say that the numerical solution obtained from the Simplified 2 Order Taylor scheme is closer to the exact solution than the numerical solution obtained from the Milstein method. From the error table, we can say that the numerical solutions get closer to the exact solution as the number of discretizations increases for each numerical scheme. It is also observed that the mean relative error is smaller than the mean absolute error for both methods. The reason is that Simplified 2 Order Taylor's strong convergence order is greater than Milstein's strong convergence order. The errors obtained from the SRK 2-stage and Simplified 2 Order Taylor solutions are lower than the Milstein solution. The Taylor and SRK 2-stage solutions are very close to each other.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

Gülşen Orucova Büyüköz: Conceptualization, Methodology, Formal Analysis, Writing – Original Draft, Writing – Review & Editing.

Tuğçem Partal: Methodology, Software, Validation, Data Curation, Writing – Review & Editing.

Mustafa Bayram: Supervision, Project Administration, Resources, Writing – Review & Editing.

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