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A Numerov Type Two Phase Finite Difference Method for the Numerical Solution of the Second Order Boundary Value Problems in Ordinary Differential Equations

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Abstract. In the present work, we propose a two-phase fourth-order method for the approximate numerical solution for second-order non-linear two-point boundary value problems with Dirichlet boundary conditions. Our numerical approach is based on a finite difference and the solution of the problem at discrete points. Our method generates a system of equations, and the solution of the system of equations is considered an approximate solution to the problem. An essential analysis of the method is considered to ensure the performance of the method. A numerical experiment is carried out with model problems to test the performance in terms of efficiency and accuracy of the proposed method.

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1. Introduction

Second order differential equations and corresponding boundary value problems are governing equations for several model problems in science and technology, to describe and verify the observation in physical phenomena. In this article we consider following second order BVPs, which occurs in studies of applied mathematics, physics and engineering,

$$u''(x) = f(x, u(x), u'(x)), \quad a < x < b, \tag{1.1}$$

subject to the Dirichlet boundary conditions

$$u(a) = \beta, \quad u(b) = \beta_1,$$

where β, and β_1 are real constant. Source function $f(x, u, u')$ is real and continuous on $[a, b] \times \mathbb{R} \times \mathbb{R}$.

In literature, there are some problems close to considered problem (1.1) posses exact closed analytical solution [2]. Generally, we face a difficult task in finding the closed analytical solution of the considered problem (1.1) under prescribed condition if it is nonlinear. Usually, we use approximation techniques for an acceptable numerical solution of

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the problem (1.1).

There are many methods reported in the literature for the approximate numerical solution of problems (1.1) using different mathematical approaches. The usually applied mathematical approaches among these methods are finite difference method [5, 6, 8, 17], spline method [4, 16, 18], finite element method [3, 9, 19], Adomian decomposition method [7, 12], and references therein. In the literature, there is evidence of the advantages and disadvantages of all these methods [6]. We prefer to use a finite difference method since it is easy to improve or modify this approach in many cases and problems.

The continuous problem (1.1) posses unique solution if the forcing function $f(x,u,u')$, $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial u'}$ satisfies the following conditions,

(i) Continuous on set $S = \{(x,u,u') : x \in [a,b], u, u' \in (-\infty, \infty)\}$
(ii) There exist $\delta > 0$ such that $\frac{\partial f}{\partial u'} \geq 0$ on S and
(iii) There exist constants K, L such that

$$K = \max_{(x,u,u')\in S} \left|\frac{\partial f}{\partial u}\right| \text{and } L = \max_{(x,u,u')\in S} \left|\frac{\partial f}{\partial u'}\right|.$$

In the present article, for the approximate solution of the problem (1.1), we consider the finite difference method, which involves replacing each of the derivatives in the problem (1.1) with an appropriate difference. The method involving finite differences transfers the continuous problem (1.1) into the discrete problem. In the literature, it is well established by the researchers [1, 10] that discrete problems may either admit spurious solutions or no solutions corresponding to continuous problems (1.1). Hence, we assume that the property of existence and uniqueness of the solution transfer from the continuous problem (1.1) to the discrete problem. So, we will not consider under which condition or any specific assumption on forcing function $f(x,u,u')$ to ensure the existence and uniqueness of the solution to the discrete problem and may refer literature [6, 11] issues related to existence and uniqueness.

In this article, we will consider the development of an approximation technique for the numerical solution of the (1.1). We will define the difference approximation for the derivative term in problem (1.1) and reduce the continuous problem to a discrete problem. We will approximate the solution of the problem using adjacent values of the solution and the forcing function. Finally, we will develop a fourth order method for the approximate numerical solution of the problem (1.1).

We have divided our work and presented it in different sections as follows: in the next section we present difference approximations and technique; the numerical convergence and results in the model test problems in Section 3; and discussion and concluding remarks on the performance and efficiency of the method in Section 4.

2. Two Phase Finite Difference Method

The domain of the problem (1.1) is $[a,b]$, hence the computational domain remains $[a,b]$, with the analytical solution is prescribed at end points of the domain as the boundary conditions. The step length h is described by $h = (b-a)/N$ where N is number of the nodal points in the computation domain. We define the nodal points $x_i = a + ih, \quad i = 0, 1, 2, ..., N$ and we determine the solution $u(x)$ of the (1.1) at the nodal points. The numerical approximation of solution $u(x)$ and forcing function $f(x, u(x), u'(x))$ at node $x = x_i$, we denote as u_i and f_i respectively for $\quad i = 0, 1, 2, ..., N$. We will follow similar notations in the present article. Thus, the continuous problem (1.1) reduced to the following discrete problem at node $x = x_i$,

$$u_i'' = f_i, \tag{2.1}$$

subject to the boundary conditions

$$u_0 = \beta, \quad u_{N+1} = \beta_1.$$

Let us define following approximations:

$$\overline{u}_i' = \frac{u_{i+1} - u_{i-1}}{2h}, \tag{2.2}$$

$$\overline{u}'_{i+1} = \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}, \tag{2.3}$$

$$\overline{u}'_{i-1} = \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}, \tag{2.4}$$

$$\overline{u}_i = \frac{1}{2}(u_{i+1} + u_{i-1}) - \frac{h^2}{4}(\overline{f}_{i+1} + \overline{f}_{i-1}), \tag{2.5}$$

where

$$\overline{f}_{i\pm1} = f(x_{i\pm1}, u_{i\pm1}, \overline{u}'_{i\pm1}),$$

$$\overline{\overline{u}}'_{i+1} = \frac{1}{2h}(u_{i+1} - u_{i-1}) + h\overline{f}_i, \tag{2.6}$$

$$\overline{\overline{u}}'_{i-1} = \frac{1}{2h}(u_{i+1} - u_{i-1}) - h\overline{f}_i, \tag{2.7}$$

where

$$\overline{f}_i = f(x_i, \overline{u}_i, \overline{u}'_i),$$

$$\overline{\overline{u}}'_i = \overline{u}'_i - \frac{h}{20}(\overline{\overline{f}}_{i+1} - \overline{\overline{f}}_{i-1}), \tag{2.8}$$

where

$$\overline{\overline{f}}_{i\pm1} = f(x_{i\pm1}, u_{i\pm1}, \overline{\overline{u}}'_{i\pm1})$$

Following the ideas in [5, 13–15] and using above approximations (2.2)-(2.8) we discretize the problem (2.1)

$$u_{i+1} - 2u_i + u_{i-1} = \frac{h^2}{12}(\overline{\overline{f}}_{i+1} + 10\overline{\overline{f}}_i + \overline{\overline{f}}_{i-1}), \tag{2.9}$$

where

$$\overline{\overline{f}}_i = f(x_i, \overline{u}_i, \overline{\overline{u}}'_i).$$

Thus, we discretized (2.1) at each nodal point $x_i, i = 1, 2, ..., N$ in the domain of the problem. The solution of the system of equations (2.9) is the approximate numerical solution of the problem (1.1).

3. Derivation of the Method and Truncation Error

In this section, we shall outline the procedure for the derivation and development of the method. Also, we shall estimate local truncation errors in our proposed method. The approximations (2.3) and (2.4) enable us to conclude the following:

$$\overline{u}'_{i\pm1} = u'_{i\pm1} - \frac{2}{3}h^2 u^{(3)}_{i\pm1} + O(h^3), \tag{3.1}$$

i.e., $\overline{u}'_{i\pm1}$ provide $O(h^2)$ approximations for $u'_{i\pm1}$. Hence, using (3.1) and Taylor's series expansion method, we have found the following approximations:

$$\overline{f}_{i\pm1} = f_{i\pm1} + O(h^2) \tag{3.2}$$

i.e., $\overline{f}_{i\pm1}$ provide $O(h^2)$ approximations for $f_{i\pm1}$. Let us expand each term in (2.6) in Taylor's series about node x_i. Using the approximations (3.2) in expansion and simplify, we have obtained the following approximation:

$$\overline{u}_i = u_i + O(h^4). \tag{3.3}$$

Using approximation (3.3) in (2.2), we have obtained following approximation for the source function $f(x, u, u')$ at the node x_i,

$$\overline{f}_i = f_i + O(h^2). \tag{3.4}$$

Using approximations (2.1),(3.4) and in (2.6),(2.7). Hence, we obtained the following approximations:

$$\overline{\overline{u}}'_{i\pm1} = u'_{i\pm1} + O(h^2). \tag{3.5}$$

Using approximations (3.5), we have found following approximations for the source function:

$$\overline{\overline{f}}_{i\pm1} = f_{i\pm1} + O(h^2), \tag{3.6}$$

i.e.,

$$\overline{\overline{f}}_{i\pm1} = f_{i\pm1} + \frac{h^2}{6}(-2u^{(3)}_{i\pm1} \pm hu^{(4)}_{i\pm1} \pm h(u^{(3)}\frac{\partial f}{\partial u'})_i)(\frac{\partial f}{\partial u'})_{i\pm1} + O(h^4).$$

Using approximations (3.6) and (2.1), we have obtained following approximations for solution function and source function at node x_i;

$$\overline{\overline{u}}'_i = u'_i + O(h^2) \quad \text{and} \quad \overline{\overline{f}}_i = f_i + O(h^2), \tag{3.7}$$

i.e.,

$$\overline{\overline{f}}_i = f_i + \frac{h^2}{60}(4u^{(3)}_i - h(u^{(3)}(\frac{\partial f}{\partial u'})^2)_i)(\frac{\partial f}{\partial u'})_i + O(h^4).$$

Using approximations (3.6) and (3.7), we have obtained following approximation;

$$\overline{\overline{f}}_{i+1} + 10\overline{\overline{f}}_i + \overline{\overline{f}}_{i-1} = f_{i+1} + 10f_i + f_{i-1} + O(h^4), \tag{3.8}$$

i.e., the expression $\overline{\overline{f}}_{i+1} + 10\overline{\overline{f}}_i + \overline{\overline{f}}_{i-1}$ provides $O(h^4)$ for the expression $f_{i+1} + 10f_i + f_{i-1}$. Using approximation (3.8), we have established that our proposed method (2.9),

$$u_{i+1} - 2u_i + u_{i-1} = \frac{h^2}{12}(\overline{\overline{f}}_{i+1} + 10\overline{\overline{f}}_i + \overline{\overline{f}}_{i-1})$$

has truncated remainder term of at least fourth order. Following similar algorithms in [5], it is easy to establish the property of convergency of the finite difference method (2.9) under appropriate conditions and simple manipulations. The approximations used in the development and discussion of the algorithm suggest that the order of accuracy of the proposed method is at least $O(h^4)$.

4. Numerical Results

To test the computational efficiency of our proposed method (2.9), we have considered linear and nonlinear model problems. In each model problem, we took a uniform step size h. In Table 1 - Table 6, we have shown MAE the maximum absolute error in the solution $u(x)$ of the problem (1.1) for different values of N. In computation following formulas were used,

$$MAE = \max_{1\le i\le N} |U(x_i) - u(x_i)|,$$

where $U(x_i)$ and $u(x_i)$ are respectively exact and computed solution of the problem. Since the solution $u(x_i)$ of the problem (2.9) approximates the solution u of the problem (1.1) up to order of the accuracy of the discretization which is four in present case. So, in many cases it does not make much sense to solve the problem (2.9) exactly. Hence, we have used Gausss-Seidel and Newton-Raphson iteration method to solve respectively the system of linear and non-linear equations arise from equation (2.9). All computations were performed on a Windows 7 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iteration reached 10^3.

Problem 1. The model non-linear problem in [5] given by

$$u''(x) = \frac{1}{2}(\exp(2u(x)) + (u'(x))^2), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u(1) = -\ln(2).$$

The analytical solution of the problem is $u(x) = \ln(\frac{1}{1+x})$. The MAE computed by method (2.9) for different values of N. In Table 1, there are numerically computed order of convergence and elapsed time needed to converge presented. For the sake of comparison, numerical results using other methods are found in the literature presented by us.

Problem 2. The model non-linear problem in [6] given by

$$u''(x) = -(1.0 + a^2(u'(x))^2), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u(1) = 1.0.$$

The analytical solution of the problem is $u(x) = \frac{\ln(\frac{\cos(a(x-\frac{1}{2}))}{\cos(\frac{a}{2})})}{a^2}$ and a is parameter. The *MAE* computed by method (2.9) for different values of N, and a are presented in Tables 2-4. In Tables 2-4, we have presented numerical results *MAE* obtained using finite difference method [6] in our numerical experiment.

Problem 3. The model non-linear problem in [5] given by

$$u''(x) = \frac{(u(x) + x(u'(x))^2)}{1+x}, \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 1 \quad , \quad u(1) = \exp(2).$$

The analytical solution of the problem is $u(x) = \exp(x)$. The *MAE* computed by method (2.9) for different values of N are presented in Table 5. In Table 5, there are numerically computed order of convergence and elapsed time needed to converge presented. For the sake of comparison, numerical results using other methods are found in the literature presented by us.

Problem 4. The model non-linear problem in [6] given by

$$u''(x) = -\sin(u'(x)) - \cos(u(x)) + \cos(x(4x^2 - 1)) - \sin(1 - 12x^2) + 24x,$$
$$\frac{-1}{2} < x < \frac{1}{2}$$

subject to boundary conditions

$$u(-\frac{1}{2}) = 0 \quad , \quad u(\frac{1}{2}) = 0.$$

The constructed analytical solution of the problem is $u(x) = x(4x^2 - 1)$. The *MAE* computed by method (2.9) for different values of N are presented in Table 6. In Table 6, we have presented numerical results *MAE* obtained using finite difference method [6] in our numerical experiment.

Table 1. Maximum absolute error (Problem 1).

	(2.9)			[5]		
h	MAE	Etime	Order	MAE	Etime	Order
2^{-6}	.66232991e -9	.2808	-	.15459572e -9	.2028	-
2^{-7}	.41349257e -10	2.3088	4.0016	.96124773e -11	.7488	4.0076
2^{-8}	.24006424e -11	5.0700	4.1064	.41843677e -12	.1185	4.5218
2^{-9}	.19141874e -13	1.0296	6.9705	.70672363e -15	.1248	9.2096
2^{-10}	.18594067e -15	.0468	6.6857	.18594067e -15	.0312	1.9263

In order to develop the an efficient finite difference method, it is necessary to consider at both, its convergence and its costs. In practice, the most interesting quantity to solve the problem is the time needed. In Table 1, we show the elapsed times for the convergence of the two different finite difference method and the order of convergence. The numerically measured convergence of the finite difference method should be independent of the nodes or step size in the method in absence of discretization error. But, we have the discretization error there always. Hence, the order of convergence varies in some extent and this characteristic property is observed by the results in Table 1. Our method has at least biquadratic order of convergence. Our method has higher elapsed time than other considered method. So, higher cost but better rate of convergent there in proposed method. Our mission in developing algorithms is to provide a good approximation to the solution of the continuous problem (1.1). In numerical result presented in Table 1, proposed method (2.9) confirmed the mission.

Table 2. Maximum absolute error and $a = \frac{0.5}{7}$ (Problem 2).

	h				
	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
(2.9)	.35306137e -10	.21969826e -11	.15350240e -12	.30836770e -13	.27048134e -13
[6]	.16615182e -5	.41538120e -6	.10384539e -6	.25961344e -7	.64903154e -8

Table 3. Maximum absolute error and $a = \frac{1}{7}$ (Problem 2).

	h				
	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
(2.9)	.56552662e -9	.35348278e -10	.22125898e -11	.13844342e -12	.79465988e -14
[6]	.66544525e -5	.16636394e -5	.41591149e -6	.103977974e -6	.25994492e -7

Table 4. Maximum absolute error and $a = \frac{2}{7}$ (Problem 2).

	h				
	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
(2.9)	.90779412e -8	.56731552e -9	.35454074e -10	.22112489e -11	.12095439e -12
[6]	.26752456e -4	.66885232e -5	.16721563e -5	.41804067e -6	.10451025e -6

Table 5. Maximum absolute error (Problem 3).

	(2.9)			[5]		
h	MAE	Etime	Order	MAE	Etime	Order
2^{-4}	.21853300e -6	.0312	-	.47419998e -7	.0156	-
2^{-5}	.13617942e -7	.0468	4.0046	.29678280e -8	.0624	3.9980
2^{-6}	.85060907e -9	.1248	4.0009	.18540482e -9	.1560	4.0007
2^{-7}	.52898551e -10	.6396	4.0072	.11314300e -10	.3900	4.0345
2^{-8}	.24274390e -11	1.0452	4.4457	.94673618e -14	.0156	10.2229

It is observed in the tabulated results for the model linear problem in Table 5 that our method converges, and the order of the convergence is $O(h^4)$. The order of convergence does not vary substantially as the number of nodes increases. The proposed method has a comparable elapsed time to other finite difference methods for the convergence.

Table 6. Maximum absolute error (Problem 4).

	h				
	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
(2.9)	.17349783e -4	.10890199e -5	.68136124e -7	.42611526e -8	.26627359e -9
[6]	.14097179e -2	.35085561e -3	.87579349e -4	.21889093e -4	.54717447e -5

The approximate solution of the system of equations (2.9) is an approximation to the solution of the continuous problem (1.1). The maximum absolute errors in the approximate solution are computed and presented in Tables. The order of accuracy increases with a decrease in step size h, i.e., the number of nodes in the domain of the solution. The tabulated results confirm the order of accuracy is at least biquadratic.

5. Conclusion

We considered a second-order differential equation and the corresponding two-point boundary value problem for the approximate numerical solution. The technique developed for the approximate solution of the problem with Dirichlet's boundary conditions employed the method of finite difference. The finite difference method developed in two stages, and each stage is an algorithm. The continuous derivatives are replaced by the difference approximations. Hence, a continuous problem transformed into a discrete problem, and finally a system of algebraic equations (2.9) was solved for the approximate solution of the considered problem. The numerical results presented in tables approve that the proposed method provides a good approximation to the solution of the continuous problem (1.1). Hence, the proposed method is computationally efficient, and the order of accuracy is at least biquadratic. There are possibilities to apply thoughts in the present article to develop and improve the order of finite difference methods for other boundary value problems. Works in these directions are in progress.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Both authors are equally contributed in conceptualization, presentation and computation.

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