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On the Fixed Point Property for Nonexpansive Mappings on Large Classes in Alpha-duals of Certain Difference Sequence Spaces

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Abstract

In 2000, Et and Esi introduced new type of generalized difference sequences by using the structure of Çolak's work from 1989 where he defined new types of sequence spaces while Çolak was also inspired by Kızmaz's idea about the difference operator he studied in 1981. Then, using Et and Esi's structure, Ansari and Chaudhry, in 2012, introduced a new type of generalized difference sequence spaces. Changing Ansari and Chaudhry's construction slightly, Et and Işık, in 2012, obtained a new type of generalized difference sequence spaces which have equivalent norm to that of Ansari and Chaudhry's type Banach spaces. Then, Et and Işık found α-duals of the Banach spaces they got and investigated geometric properties for them. In this study, we consider Et and Işık's work and study α-duals of their generalized difference sequence spaces. We take their study in terms of fixed point theory and find large classes of closed, bounded and convex subsets in those duals with fixed point property for nonexpansive mappings.

Keywords: Nonexpansive Mapping, Fixed Point Property, Closed Bounded Convex Set, Difference Sequences, α-duals.

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1. *Introduction and Preliminaries*

Fixed point theory is a central area of study in functional analysis with wide-ranging implications for optimization, nonlinear analysis, and the theory of Banach spaces. A key concept in this field is the fixed point property (FPP), which states that every nonexpansive mapping on a closed, bounded, and convex (cbc) subset of a space has a fixed point.

This property is known to hold in certain Banach spaces, such as Hilbert spaces, but fails in many classical non-reflexive Banach spaces like c_0 and ℓ^1. As a result, identifying large classes of cbc subsets within such spaces that retain the FPP has become an important line of inquiry.

The work of Goebel and Kuczumow (1979) serves as a seminal contribution in this area. They demonstrated that while ℓ^1 lacks the fixed point property in general, it is possible to identify specific large classes of cbc subsets where nonexpansive mappings do have fixed points. This discovery inspired subsequent research aimed at generalizing these results to broader classes of Banach spaces and larger families of subsets. Researchers such as Kaczor and Prus (2004) further extended these ideas by investigating affine asymptotically nonexpansive mappings on ℓ^1. However, these works often required additional assumptions, such as the affinity condition, which limited the generality of their results.

In this study, we introduce a new perspective by examining α-duals of certain generalized difference sequence spaces, which generalize the space of absolutely summable scalar sequences. Our approach differs from that of Kaczor and Prus, as we do not rely on the affinity hypothesis and instead work directly with nonexpansive mappings. Moreover, while Goebel and Kuczumow focused on ℓ^1, our work considers a more general family of sequence spaces that are isometrically isomorphic to the absolutely summable scalar sequence space but also contain a richer geometric structure. This generalization enables us to identify larger classes of cbc subsets with the FPP. Importantly, our approach extends beyond specific instances, as we are also developing our work for a general case of the space we study by taking $m \in \mathbb{N}$ arbitrarily respected to the general space.

The primary objective of this paper is to identify large classes of closed, bounded, and convex subsets in α-duals of generalized difference sequence spaces that satisfy the fixed point property for nonexpansive mappings. To achieve this, we build on concepts from Goebel and Kuczumow's analogy while introducing new methods to avoid reliance on affinity assumptions. Our findings contribute to the broader effort of understanding the fixed point theory of Banach spaces and provide a new avenue for exploring the geometric properties of generalized difference sequence spaces.

The paper is organized as follows. In Section 1, we introduce essential definitions and preliminary concepts. Section 2 presents our main results, including theorems and proofs regarding large classes of cbc subsets in α-duals with the FPP. We conclude with a discussion of the implications of our findings, highlighting potential directions for future research.

In terms of looking more deeply into the literature, we can say that researches have shown that the fixed point exists for some function classes defined on certain classes of sets in some spaces, while it cannot be found at all in others. Fixed point theory has examined how this happens or does not happen.

Then, researchers have made classifications and characterizations in this matter. In (Browder 1965a), it was proved that every Hilbert space has a property satisfying that every nonexpansive mapping defined on any closed, bounded, and convex (cbc) nonempty subset domain with the same range has a fixed point. Since that time, spaces with this property have been considered to have the fixed point property for nonexpansive mappings (fppne). Then, researchers considered looking for the spaces with the property and if the property still exists when larger classes of mappings are taken. Then also they have seen spaces failing the properties. For example, in (Browder 1965b) and (Göhde 1965) with independent studies, they saw that uniform convex Banach spaces have the fppne. Then, Kirk (1965) generalized the result for the reflexive Banach spaces with normal structure. In fact, Goebel and Kirk (1973) noticed that Kirk's result was able to extend for uniformly Lipschitz mappings and some researchers have studied estimating the Lipschitz coefficient satisfying the property for uniform Lipschitz mappings on different Banach spaces. For example, Goebel and Kirk (1990) showed that for Hilbert spaces, the best Lipschitz coefficient would be a scalar less than a number in the interval $\left[\sqrt{2}\ ,\ \ \frac{\pi}{2}\right]$, and Goebel and Kirk (1973) and Lim (1983) showed independently that for a Lebesgue space L^p when $2 < p < \infty$, the coefficient is smaller by a scalar larger than or equal to $\left(1+\frac{1}{2^p}\right)^{\frac{1}{p}}$ while Alspach (1981) showed that when $p = 2$, there exists a fixed point free Lipschitz mapping with Lispchitz coefficient $\sqrt{2}$ defined on a cbc subset. In fact, $\sqrt{2}$ is the smallest Lipschitz coefficient for Alspach's mapping. We need to note that, similar to the definition of the Banach spaces satisfying the fppne, if a Banach space has a property that every uniformly Lipschitz mapping defined on any cbc nonempty subset domain with the same range has a fixed point, then that Banach space has the fixed point property for uniformly Lipschitz mapping (fppul). In terms of fixed point property for uniformly Lipschitz mappings, Dowling, Lennard, and Turett (2000) showed that if a Banach space contains an isomorphic copy of ℓ^1, then it fails the fppul. It is a well-known fact by researchers that c_0 or ℓ^1 is almost isometrically embedded in every non-reflexive Banach space with an unconditional basis (Lindenstraus and Tzafriri 1977). For this reason, classical non-reflexive Banach spaces fail the fixed point property for non-expansive mappings, that is, in these spaces, there can be a closed, convex and bounded subset and a non-expansive invariant T mapping defined on that set such that T has no fixed point. This result is based on well-known theorems in literature (see for example Theorem 1.c.12 in (Lindenstraus and Tzafriri 1977) and Theorem 1.c.5 in (Lindenstraus and Tzafriri 2013)). These theorems state that for a Banach lattice or Banach space with an unconditional basis to be reflexive, it is necessary and sufficient that it does not contain any isomorphic copies of c_0 or ℓ^1. Therefore, this close relation to the reflexivity or nonreflexivity of Banach space, researchers have worked for years and questioned whether c_0 or ℓ^1 can be renormed to have a fixed point for nonexpansive mappings. Lin (2008) showed in his study that what was thought was not true and that at least ℓ^1 could be renormed to have the fixed point property for nonexpansive mappings. Then, the remaining question was if the same could have been done for c_0, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes-Lineares and Maria (2012) but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most research has been inspired by the ideas of the study (1979) where Goebel and Kuczumow proved that while ℓ^1 fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in Lin's (2008) work. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, in (Kaczor and Prus 2004), they wanted to generalize Goebel and Kuczumow's findings and they proved that affine asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in ℓ^1 can have fixed points. Moreover, in (Everest 2013), Kaczor and Prus' results were extended by having been found larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition became a tool for their works. In fact, another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$, was studied in (Hernández-Linares Japón 2012) and in their study they obtained an analogous result to (Lin 2008) as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings. In this study, we will investigate some Banach spaces analogous to ℓ^1. In the present work, we study Goebel-Kuczumow analogy for α-duals of their generalized difference sequence spaces investigated by Et and Işık (2012). We prove that a very large class of closed, bounded and convex subsets in α-duals of their generalized difference sequence spaces investigated by Et and Işık has the fixed point property for nonexpansive mappings. Therefore, firstly we would like to give the definition of Cesàro sequence spaces which was defined by Shiue (1970), and next we present Kızmaz's difference sequence space definition in (Kızmaz 1981) by noting that we work on a space which is derived from his ideas' generalizations such that many researchers (see for example (Çolak 1989, Et 1996, Et and Çolak 1995, Et and Esi 2000, Orhan 1983, Tripathy et al 2005) have generalized his work as well.

In fact, we need to note that Et and Esi's (2000) work and Et and Çolak's (1995) work used a common difference sequence definition from Çolak's (1989) work.

Shiue (1970) defined the Cesàro sequence spaces by

$$\mathrm{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \mathrm{ces}_p$ and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \le p < \infty$. Then, from the definition of Cesàro sequence spaces, Kızmaz (1981) defined difference sequence spaces for ℓ^∞, c, and c_0 and symbolized them by $\ell^\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$, respectively. In his introduction, he defined the difference operator Δ applied to the sequence $x = (x_n)_n$ using the formula $\Delta x = (x_k - x_{k+1})_k$. In fact, he investigated Köthe-Toeplitz duals and their topological properties.

As one of the researchers generalizing his ideas, Çolak (1989) introduced firstly a generalized difference sequence space by taking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ and then denoting a new difference operator by Δ_v such that for any sequence $x = (x_n)_n$, he defined the difference sequence of that $\Delta_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, Et and Esi (2000) generalized Çolak's difference sequence space by defining

$$\Delta_v(\ell^\infty) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \ell^\infty\},$$
$$\Delta_v(c) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in c\},$$
$$\Delta_v(c_0) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in c_0\}.$$

Furthermore, their m^{th} order generalized difference sequence space is given for any $m \in \mathbb{N}$ by $\Delta_v^0 x = (v_k x_k)_k$, $\Delta_v^m x = (\Delta_v^m x_k)_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})_k$ with $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for each $k \in \mathbb{N}$.

Next Bektaş, Et and Çolak (2004) obtained the Köthe-Toeplitz duals for the generalized difference sequence space of Et and Esi's. We may recall here that their

m^{th} order difference sequence space has the following norm for any $m \in \mathbb{N}$:

$$\|x\|_v^{(m)} = \sum_{k=1}^{m} |v_k x_k| + \|\Delta_v^m x\|_\infty$$

Then, the corresponding Köthe-Toeplitz dual was obtained as in (Bektaş, Et and Çolak 2004) and (Et and Esi 2000) such that it is written as below:

$$D_1^m = \{a = (a_n)_n \subset \mathbb{R} | (n^m {v_n}^{-1} a_n)_n \in \ell^1\}$$
$$= \left\{ a = (a_n)_n \subset \mathbb{R}: \ \|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|} < \infty \right\}.$$

Note that $D_1^m \subset \ell^1$ if $k^m |{v_k}^{-1}| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset D_1^m$ if $k^m |{v_k}^{-1}| < 1$ for each $k, m \in \mathbb{N}$.

Ansari and Chaudhry (2012) introduced a new type of generalized difference sequence spaces by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ as Çolak (1989) did and next by symbolizing the new difference sequence space as $\Delta_{v,r}^m(E)$ for arbitrary $r \in \mathbb{R}, m \in \mathbb{N}$ and writing that space as below where X is any of the sequence spaces ℓ^∞, c or c_0.

$$\Delta_{v,r}^m(X) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v^m x \in X\}$$

where Ansari and Chaudhry (2012) defined the norm by

$$\|x\|_{\Delta,v}^m = \sum_{k=1}^{m} |v_k x_k| + \sup_{k \in \mathbb{N}} |k^r \Delta_v^m x_k|$$

Then, by obtaining an equivalent norm to Ansari and Chaudhry's Banach space, Et and Işık (2012) defined m^{th} order generalized type difference sequence for any $m \in \mathbb{N}$ given by

$$\Delta_{v,r}^{(m)}(X) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v^m x \in X\}$$

where the norm is as follows:

$$\|x\|_{\Delta,v}^{(m)} = \sup_{k \in \mathbb{N}} |k^r \Delta_v^m x_k|$$

Then, Et and Işık found α-duals of the Banach spaces they got and investigated geometric properties for them such that m^{th} order α-duals for their Banach spaces are written as

$$U_1^m = \{a = (a_n)_n \subset \mathbb{R} | (n^{m-r} {v_n}^{-1} a_n)_n \in \ell^1\}$$
$$= \left\{ a = (a_n)_n \subset \mathbb{R}: \ \|a\|_{\sim}^{(m)} = \sum_{k=1}^{\infty} \frac{k^{m-r} |a_k|}{|v_k|} < \infty \right\}$$

Note that $U_1^m \subset \ell^1$ if $k^{m-r} |{v_k}^{-1}| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset U_1^m$ if $k^{m-r} |{v_k}^{-1}| < 1$ for each $k, m \in \mathbb{N}$.

Before starting to introduce our work and results, we can also note that recent studies have explored the fixed point property (FPP) in Banach spaces, focusing on large classes of subsets and various mappings. One notable contribution is by Tim Dalby, who in 2024 proved that uniformly nonsquare Banach spaces possess the FPP. This result provides a deeper understanding of the geometric conditions that ensure fixed points for nonexpansive mappings. Dalby's work highlights the importance of uniform nonsquareness as a sufficient condition for the FPP, offering new perspectives on the structure of Banach spaces (Dalby, 2024).

Another important development in this area is the work of Vasile Berinde and Mădălina Păcurar, published in 2021. They introduced the concept of saturated classes of contractive mappings and examined the applicability of enriched contractions. Their study provided new fixed point results for these enriched classes of mappings, broadening the scope of fixed point theorems. The authors demonstrated that these enriched contractions have unique fixed points, which can be approximated using Krasnoselskij iterative schemes. This contribution enriches the fixed point theory and extends its applicability to a wider range of contractive mappings in Banach spaces (Berinde & Păcurar, 2021).

Izhar Oppenheim's 2022 work is another significant contribution to the field. He established that higher-rank simple Lie groups, such as $SL_n(\mathbb{R})$ for n ≥ 3, and their lattices have Banach property (T) with respect to all super-reflexive Banach spaces. This result implies that these groups have the FPP for actions on super-reflexive Banach spaces. Oppenheim's findings underscore the interplay between Banach property (T) and the fixed point property, offering new insights into the algebraic and topological properties that guarantee the existence of fixed points (Oppenheim, 2022).

Research on large classes of Banach spaces with the fixed point property has also included investigations into the Prus–Szczepanik condition. Prus and Szczepanik introduced this condition to identify Banach spaces that satisfy the FPP for nonexpansive mappings. Subsequent studies have explored the relationship between the PSz condition and other geometric properties of Banach spaces, providing sufficient criteria for a space to satisfy the FPP. This line of research aims to broaden the classification of Banach spaces that support the FPP, thereby offering new tools for functional analysts (Prus & Szczepanik, 2019).

Further advancements were made by Berinde and Păcurar, who introduced the concept of enriched contractions. These contractions generalize Picard–Banach contractions and certain nonexpansive mappings. Their work demonstrated that enriched contractions possess unique fixed points, which can be approximated using Krasnoselskij iterative schemes. This approach provides a broader framework for establishing fixed points for a wide range of mappings (Berinde & Păcurar, 2021).

Additional contributions to the study of the FPP in Banach spaces include the work of Fetter Nathansky and Llorens-Fuster, who investigated the ℓ^1 sum of the van Dulst space with itself. They demonstrated that this product space retains the FPP despite lacking several known conditions that typically imply this property. This finding illustrates how new combinations of Banach spaces can yield novel insights into fixed point theory, motivating further exploration of product spaces and their fixed point properties (Nathansky & Llorens-Fuster, 2020).

Finally, Oppenheim's exploration of Banach property (T) and fixed point properties has established connections between algebraic structures and the FPP. His findings that higher-rank simple Lie groups possess Banach property (T) with respect to super-reflexive Banach spaces reveal a deeper relationship between algebra, topology, and fixed point theory. These contributions collectively highlight the ongoing effort to understand the conditions under which fixed points exist in Banach spaces and to identify large classes of sets and mappings that satisfy the FPP (Oppenheim, 2022).

Now, we would like to give some well-known and important facts that are fundamentals for our work. One may see (Goebel and Kirk 1990) as a reference.

Definition 1.1 *Consider that* $(X, \|\cdot\|)$ *is a Banach space and let* C *be a non-empty cbc subset. Let* $T: C \to C$ *be a mapping. We say that*

1. T *is an affine mapping if for every* $t \in [0,1]$ *and* $a, b \in C$, $T((1-t)a + tb) = (1-t)T(a) + t\ T(b)$.
2. T *is a nonexpansive mapping if for every* $a, b \in C$,
$$\| T(a) - T(b) \| \leq \| a - b \|.$$

Then, we will easily obtain an analogous key lemma from the below lemma in the work (Goebel and Kuczumow 1979).

Lemma 1.2 *Let* $\{u_n\}$ *be a sequence in* ℓ^1 *converging to* u *in weak-star topology. Then, for every* $w \in \ell^1$,
$$Q(w) = Q(u) + \|w - u\|_1$$
where
$$Q(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although Çolak (1989) considered complex values of $v = (v_n)_n$ while introducing his structure of the difference sequence which is taken as the fundamental concept in this study.

2. *Main Results*

In this section, we will present our results. As mentioned in the first section, we investigate Goebel and Kuczumow analogy for the space U_1^m for each $m \in \mathbb{N}$. We aim to show that there is a large class of cbc subsets in U_1^m such that every nonexpansive invariant mapping defined on the subsets in the class taken has a fixed point. Recall that the invariant mappings have the same domain and the range. Note that we will assume that $r \in \mathbb{R}$ is arbitrary due to the definition of the space.

First, due to isometric isomorphism, using Lemma 1.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as (Goebel and Kuczumow 1979), and (Everest 2013) and in fact the methods in the study (Everest 2013) will be our lead in this work.

Lemma 2.1 *Fix* $m \in \mathbb{N}$ *and* $\{u_n\}$ *be a sequence in the Banach space* U_1^m *and assume* $\{u_n\}$ *converges to* u *in weak-star topology. Then, for every* $w \in U_1^m$,
$$Q(w) = Q(u) + \|w - u\|_{\sim}^{(m)}$$
where
$$Q(w) = \limsup_n \|u_n - w\|_{\sim}^{(m)}.$$

Then, we obtain our results by the following theorems.

Theorem 2.2 *Let* $m \in \mathbb{N}$, $r \in \mathbb{R}$ *and* $t \in (0,1)$. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence defined by* $f_1 := t\ v_1\ e_1$, $f_2 :=$

$\frac{t\ v_2}{2^{m-r}}\ e_2$, *and* $f_n := \frac{v_n}{n^{m-r}} e_n$ *for all integers* $n \geq 3$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* c_0 *and* ℓ^1. *Then, consider the cbc subset* $E^{(m)} = E_t^{(m)}$ *of* U_1^m *by*

$$E^{(m)} := \left\{\sum_{n=1}^{\infty} \alpha_n f_n:\ \forall n \in \mathbb{N},\qquad \alpha_n \geq 0\ \text{and}\ \sum_{n=1}^{\infty} \alpha_n = 1\right\}.$$

Then, $E^{(m)}$ has the fixed point property for $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mappings.

Proof. Let $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $T: E^{(m)} \to E^{(m)}$ be a $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(m)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|_{\sim}^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, U_1^m shares common geometric properties with ℓ^1 and so both U_1^m and its predual have similar fixed point theory properties to ℓ^1 and c_0, respectively. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in U_1^m, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(m)}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{\sum_{n=1}^{\infty} \alpha_n f_n:\ \text{each}\ \alpha_n \geq 0\ \text{and}\ \sum_{n=1}^{\infty} \alpha_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(\mathrm{m})}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r: U_1^m \to [0,\infty)$ defined by

$$Q(w) = \limsup_n \left\|u^{(n)} - w\right\|_{\sim}^{(m)},\quad \forall w \in U_1^m$$

such that for every $w \in U_1^m$,

$$Q(w) = Q(u) + \|u - w\|_{\sim}^{(m)}.$$

Case 1. $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{\sim}^{(m)}$ and

$$\begin{aligned} Q(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|_{\sim}^{(m)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{\sim}^{(m)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|_{\sim}^{(m)} + 0 \\ &= Q(u). \end{aligned} \quad (1)$$

Thus, $Q(Tu) = Q(u) + \|Tu - u\|_{\sim}^{(m)} \leq r(u)$ and so $\|Tu - u\|_{\sim}^{(m)} = 0$. Therefore, $Tu = u$.

Case 2. $u \in C^{(m)} \backslash E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \geq 0, \forall n \in \mathbb{N}$.

Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$ define

$$h_\beta := (\delta_1 + \beta\xi) f_1 + (\delta_2 + (1-\beta)\xi) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\left\|h_\beta - u\right\|_{\sim}^{(m)} = \left\|\beta t\xi v_1 e_1 + (1-\beta)\xi \frac{t v_2 e_2}{2^{m-r}}\right\|_{\sim}^{(m)} = t|\beta|\xi + t|1-\beta|\xi.$$

$\left\|h_\beta - u\right\|_{\sim}^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \geq 0$, $\forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\gamma_1 := t\ v_1$, $\gamma_2 := \frac{t\ v_2}{2^{m-r}}$, and $\gamma_n := \frac{v_n}{n^{m-r}}$ for all integers $n \geq 3$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\begin{aligned} \|\mathrm{w} - u\|_{\sim}^{(m)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|_{\sim}^{(m)} \\ &= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|_{\sim}^{(m)} \\ &= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\frac{k^{m-r}\gamma_k}{v_k}\right|. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathrm{w} - u\|_{\sim}^{(m)} &\geq \sum_{k=1}^{\infty} t\ |\alpha_k - \delta_k| \\ &\geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right| \\ &= t\left|1 - \sum_{k=1}^{\infty} \delta_k\right| \\ &= t\xi. \end{aligned}$$

Hence,

$$\|w-u\|_{\sim}^{(m)} \geq t\xi = \|h_\beta - u\|_{\sim}^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=3}^{\infty}|\alpha_k-\delta_k| = 0$; that is, we have $\|w-u\|_{\sim}^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 3$; or say, $\|w-u\|_{\sim}^{(m)} = t\xi$ if and only if $w = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function $\rho\colon [0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda\rho([0,1])$ is a compact convex subset and so $\|w-u\|_{\sim}^{(m)}$ achieves its minimum value at $w = h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$\begin{aligned} Q(h_\beta) &= Q(u) + \|h_\beta - u\|_{\sim}^{(m)} \\ &\leq Q(u) + \|Th_\beta - u\|_{\sim}^{(m)} \\ &= Q(Th_\beta) = \limsup_n \|Th_\beta - u^{(n)}\|_{\sim}^{(m)} \end{aligned}$$

then, like the inequality (1), we get

$$\begin{aligned} Q(h_\beta) &\leq \limsup_n \|Th_\beta - T(u^{(n)})\|_{\sim}^{(m)} \\ &\quad + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|h_\beta - u^{(n)}\|_{\sim}^{(m)} \\ &\quad + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|h_\beta - u^{(n)}\|_{\sim}^{(m)} + 0 \\ &= Q(h_\beta). \end{aligned}$$

Hence, $r(h_\beta) \leq Q(Th_\beta) \leq r(h_\beta)$ and so $Q(Th_\beta) = Q(h_\beta)$.

Therefore,

$$Q(u) + \|Th_\beta - u\|_{\sim}^{(m)} = Q(u) + \|h_\beta - u\|_{\sim}^{(m)}.$$

Thus, $\|Th_\beta - u\|_{\sim}^{(m)} = \|h_\beta - u\|_{\sim}^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's (1930) fixed point theorem, easily we get the result T has a fixed point since T is continuous; thus, h_β is the unique minimizer of $\|w-u\|_{\sim}^{(m)}$: $w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

Theorem 2.3 *Let $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := t\ v_1\ e_1$, $f_2 := \frac{t\ v_2}{2^{m-r}}\ e_2$, $f_3 := \frac{t\ v_3}{3^{m-r}}\ e_3$, and $f_n := \frac{v_n}{n^{m-r}} e_n$ for all integers $n \geq 4$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(m)} = E_t{}^{(m)}$ of U_1^m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \alpha_n f_n\colon\ \forall n \in \mathbb{N}, \quad \alpha_n \geq 0\ \ and\ \ \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

Then, $E^{(m)}$ has the fixed point property for $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mappings.

Proof. Let $m \in \mathbb{N}, r \in \mathbb{R}$ and $t \in (0,1)$. Let $T\colon E^{(m)} \to E^{(m)}$ be a $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $(u^{(n)})_{n\in\mathbb{N}} \in E^{(m)}$ such that $\|Tu^{(n)} - u^{(n)}\|_{\sim}^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, U_1^m shares common geometric properties with ℓ^1 and so both U_1^m and its predual have similar fixed point theory properties to ℓ^1 and c_0, respectively. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology and c_0 is separable, in U_1^m, the unit closed ball is weak*-sequentially compact due to Banach-Alaoğlu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(\mathrm{m})}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{ \sum_{n=1}^{\infty} \alpha_n f_n\colon\ each\ \alpha_n \geq 0\ \ and\ \ \sum_{n=1}^{\infty} \alpha_n \leq 1 \right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(\mathrm{m})}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r\colon U_1^m \to [0,\infty)$ defined by

$$Q(w) = \limsup_n \|u^{(n)} - w\|_{\sim}^{(m)}, \quad \forall w \in U_1^m$$

such that for every $w \in U_1^m$,

$$Q(w) = Q(u) + \|u - w\|_{\sim}^{(m)}.$$

Case 1. $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{\sim}^{(m)}$ and

$$\begin{aligned} Q(Tu) &= \limsup_n \|Tu - u^{(n)}\|_{\sim}^{(m)} \\ &\leq \limsup_n \|Tu - T(u^{(n)})\|_{\sim}^{(m)} \\ &\quad + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|u - u^{(n)}\|_{\sim}^{(m)} + 0 \\ &= Q(u). \end{aligned} \tag{2}$$

Thus, $Q(Tu) = Q(u) + \|Tu - u\|_{\sim}^{(m)} \leq r(u)$ and so $\|Tu - u\|_{\sim}^{(m)} = 0$. Therefore, $Tu = u$.

Case 2. $u \in C^{(m)} \backslash E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \geq 0, \forall n \in \mathbb{N}$.

Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$, define

$$h_\beta := \left(\delta_1 + \frac{\beta}{2}\xi\right) f_1 + \left(\delta_2 + \frac{\beta}{2}\xi\right) f_2 + (\delta_3 + (1-\beta)\xi) f_3 + \sum_{n=4}^{\infty} \delta_n f_n.$$

Then,

$$\left\|h_\beta - u\right\|_{\sim}^{(m)} = \left\| \begin{matrix} \frac{\beta}{2} t\xi v_1 e_1 + \frac{\beta}{2} t\xi \frac{v_2}{2^{m-r}} e_2 \\ +(1-\beta)\xi \frac{t\ v_3 e_3}{3^{m-r}} \end{matrix} \right\|_{\sim}^{(m)} = t\left|\frac{\beta}{2}\right|\xi + t\left|\frac{\beta}{2}\right|\xi + t|1-\beta|\xi.$$

$\left\|h_\beta - u\right\|_{\sim}^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \geq 0,\ \forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\gamma_1 := t\ v_1$, $\gamma_2 := \frac{t\ v_2}{2^{m-r}}$, $\gamma_3 := \frac{t\ v_3}{3^{m-r}}$, and $\gamma_n := \frac{v_n}{n^{m-r}}$ for all integers $n \geq 4$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\begin{aligned} \|w - u\|_{\sim}^{(m)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|_{\sim}^{(m)} \\ &= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|_{\sim}^{(m)} \\ &= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\frac{k^{m-r}\gamma_k}{v_k}\right| \\ &\geq \sum_{k=1}^{\infty} t\ |\alpha_k - \delta_k| \\ &\geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right| \\ &= t\left|1 - \sum_{k=1}^{\infty} \delta_k\right| \\ &= t\xi. \end{aligned}$$

Hence,

$$\|w - u\|_{\sim}^{(m)} \geq t\xi = \left\|h_\beta - u\right\|_{\sim}^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=4}^{\infty} |\alpha_k - \delta_k| = 0$; that is, we have $\|w - u\|_{\sim}^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 4$; or say, $\|w - u\|_{\sim}^{(m)} = t\xi$ if and only if $w = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function ρ: $[0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda\rho([0,1])$ is a compact convex subset and so $\|w - u\|_{\sim}^{(m)}$ achieves its minimum value at $w = h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$\begin{aligned} Q(h_\beta) &= Q(u) + \left\|h_\beta - u\right\|_{\sim}^{(m)} \\ &\leq Q(u) + \left\|Th_\beta - u\right\|_{\sim}^{(m)} \\ &= Q(Th_\beta) = \limsup_n \left\|Th_\beta - u^{(n)}\right\|_{\sim}^{(m)} \end{aligned}$$

then same as the inequality (2), we get

$$\begin{aligned} Q(h_\beta) &\leq \limsup_n \left\|Th_\beta - T(u^{(n)})\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T(u^{(n)})\right\|_{\sim}^{(m)} \\ &\leq \limsup_n \left\|h_\beta - u^{(n)}\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T(u^{(n)})\right\|_{\sim}^{(m)} \\ &\leq \limsup_n \left\|h_\beta - u^{(n)}\right\|_{\sim}^{(m)} + 0 \\ &= Q(h_\beta). \end{aligned}$$

Hence, $r(h_\beta) \leq Q(Th_\beta) \leq r(h_\beta)$ and so $Q(Th_\beta) = Q(h_\beta)$.

Therefore,

$$Q(u) + \left\|Th_\beta - u\right\|_{\sim}^{(m)} = Q(u) + \left\|h_\beta - u\right\|_{\sim}^{(m)}.$$

Thus, $\left\|Th_\beta - u\right\|_{\sim}^{(m)} = \left\|h_\beta - u\right\|_{\sim}^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's (1930) fixed point theorem, we can easily we get the result T has a fixed point since T is continuous. Thus, h_β is the unique minimizer of $\|w - u\|_{\sim}^{(m)}\ : w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

3. *Discussion*

The present study introduces novel advancements in the field of fixed point theory by establishing large classes of cbc subsets in α-duals of certain generalized difference sequence spaces that satisfy the FPP for nonexpansive mappings. This work addresses a previously unexplored area, as no prior studies have examined these spaces with the goal of identifying such large classes with the FPP. Notably, while Goebel and Kuczumow (1979) achieved analogous results for the space of absolutely summable scalar sequences, our work generalizes and extends these findings to broader spaces. Our space is isometrically isomorphic to the absolutely summable scalar sequence space but incorporates a more general framework, thereby broadening the scope of applicable classes.

An essential distinction of our approach lies in the elimination of the additional affinity condition required by Kaczor and Prus (2004), as we work directly with nonexpansive mappings rather than asymptotically nonexpansive mappings. This adjustment simplifies the theoretical foundation while still achieving stronger results. Moreover, our methods are not limited to specific instances, as we are developing a more general case for arbitrary \(m\), which opens new possibilities for future research in this domain.

Recent studies have demonstrated the existence of large classes with the fixed point property under specific conditions. Our results build on this momentum, further advancing the field by identifying and characterizing classes of sets that satisfy the FPP in a broader family of Banach spaces. These results offer a valuable perspective on the geometric structure of generalized difference sequence spaces and their fixed point properties, with implications for further studies on nonexpansive mappings, Banach space theory, and related areas in functional analysis.

As has been mentioned above and in earlier sections of the study, investigating and looking for large classes of closed, bounded and convex subsets in Banach spaces alike the Banach spaces of absolutely summable scalars are center of interests for many fixed point theorists. One can investigate to get larger classes for more general spaces than those in the present study and due to isometry, that would not be hard by following the ideas of Goebel and Kuczumow's. However, trying to generalize their ideas and looking for different examples of the sets and spaces would be valuable studies.

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References

ALSPACH, D. E. (1981). A fixed point free nonexpansive map. *Proceedings of the American Mathematical Society*, *82*(3), 423-424.

ANSARI, A. A., & CHAUDHRY, V. K. (2012). On Köthe-Toeplitz duals of some new and generalized difference sequence spaces. *Ital. J. Pure Appl. Math.*, 29: 135-148.

BEKTAŞ, Ç. A., ET, M., & ÇOLAK, R. (2004). Generalized difference sequence spaces and their dual spaces. *Journal of Mathematical Analysis and Applications*, *292*(2): 423-432.

BERINDE, V., & PĂCURAR, M. (2021). Fixed points theorems for unsaturated and saturated classes of contractive mappings in Banach spaces. *Symmetry*, *13*(4), 713.

BROWDER, F. E. (1965). Fixed-point theorems for noncompact mappings in Hilbert space. *Proceedings of the National Academy of Sciences*, *53*(6), 1272-1276.

BROWDER, F. E. (1965). Nonexpansive nonlinear operators in a Banach space. *Proceedings of the National Academy of Sciences*, *54*(4), 1041-1044.

ÇOLAK, R. (1989). On some generalized sequence spaces. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 38: 35-46.

DALBY, T. (2024). Uniformly nonsquare Banach spaces have the fixed point property 1. *arXiv preprint arXiv:2403.16007*.

DOWLING, P. N., LENNARD, C. J., & TURETT, B. (2000). Some fixed point results in ℓ^1 and c_0. *Nonlinear Analysis-Series A Theory and Methods and Series B Real World Applications*, *39*(7), 929.

ET, M. (1996). On some generalized Cesàro difference sequence spaces. *İstanbul University Science Faculty the Journal of Mathematics Physics and Astronomy*, *55*, 221-229.

ET, M., & ÇOLAK, R. (1995). On some generalized difference sequence spaces. *Soochow Journal of Mathematics*, *21*(4), 377-386.

ET, M., & ESI, A. (2000). On Köthe-Toeplitz duals of generalized difference sequence spaces. *Bull. Malays. Math. Sci. Soc*, *23*(1), 25-32.

ET, M., & IŞIK, M. (2012). On pα-dual spaces of generalized difference sequence spaces. *Applied Mathematics Letters*, *25*(10), 1486-1489.

EVEREST, T. M. (2013). *Fixed points of nonexpansive maps on closed, bounded, convex sets in ℓ^1* (Doctoral dissertation, University of Pittsburgh).

GOEBEL, K., & KIRK, W. A. (1973). A fixed point theorem for transformations whose iterates have uniform Lipschitz constant. *Studia Math*, *47*(1), 135-140.

GOEBEL, K., & KIRK, W. A. (1990). Topics in metric fixed point theory. *Cambridge Studies in Advanced Mathematics/Cambridge University Press*, *28*.

GOEBEL, K., & KUCZUMOW, T. (1979). Irregular convex sets with fixed-point property for nonexpansive mappings. In *Colloquium Mathematicum* (Vol. 2, No. 40, pp. 259-264).

GÖHDE, D. (1965). Zum prinzip der kontraktiven abbildung. *Mathematische Nachrichten*, *30*(3-4), 251-258.

KACZOR, W., & PRUS, S. (2004). Fixed point properties of some sets in ℓ^1. In *Proceedings of the International Conference on Fixed Point Theory and Applications*, 11p.

KIZMAZ, H. (1981). On certain sequence spaces. *Canadian Mathematical Bulletin*, *24*(2):169-176.

KIRK, W. A. (1965). A fixed point theorem for mappings which do not increase distances. *The American mathematical monthly*, *72*(9):1004-1006.

LIM, T. C. (1983). Fixed point theorems for uniformly Lipschitzian mappings in Lp spaces. *Nonlinear Analysis: Theory, Methods & Applications*, *7*(5), 555-563.

LIN, P. K. (2008). There is an equivalent norm on ℓ^1 that has the fixed point property. *Nonlinear Analysis: Theory, Methods & Applications*, *68*(8):2303-2308.

HERNÁNDEZ-LINARES, C. A., & JAPÓN, M. A. (2012). Renormings and fixed point property in non-commutative L1-spaces II: Affine mappings. *Nonlinear Analysis: Theory, Methods & Applications*, *75*(13):5357-5361.

LINDERSTRAUSS, J., & TZAFRIRI, L. (1977). *Classical Banach Spaces I: sequence spaces.* Springer-Verlag.

LINDENSTRAUSS, J., & TZAFRIRI, L. (2013). *Classical Banach spaces II: function spaces* (Vol. 97). Springer Science & Business Media.

LAEL, F., & HEIDARPOUR, Z. (2016). Fixed point theorems for a class of generalized nonexpansive mappings. *Fixed Point Theory and Applications*, *2016*, 1-7.

OPPENHEIM, I. (2023). Banach property (T) for SL_n ($\mathbb{Z}$) and its applications. *Inventiones mathematicae*, *234*(2), 893-930.

ORHAN, C. (1983). Casaro Differance Sequence Spaces and Related Matrix Transformations. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, *32*:55-63.

LAEL, F., & HEIDARPOUR, Z. (2016). Fixed point theorems for a class of generalized nonexpansive mappings. *Fixed Point Theory and Applications*, *2016*, 1-7.

SCHAUDER, J. (1930). Der fixpunktsatz in funktionalraümen. *Studia Mathematica*, *2*(1), 171-180.

SHIUE, J. S. (1970). On the Cesaro sequence spaces. *Tamkang J. Math*, *1*(1):19-25.

TRIPATHY, B. C., ESI, A., & TRIPATHY, B. (2005). On new types of generalized difference Cesaro sequence spaces. *Soochow Journal of Mathematics*, *31*(3):333-340.