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Original Article

On Some Maps in Supra Topological Ordered Spaces

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Abstract: In [6] the notion of supra semi open sets was presented and some of its properties were discussed. In this study, we introduce and investigate four main concepts namely supra continuous (supra open, supra closed, supra homeomorphism) maps via supra topological ordered spaces. Our findings in this work generalize some previous results in ([1], [13]). Many examples are considered to show the concepts introduced and main results obtained herein.

Keywords: $I(D,B)$ -supra semi continuous map, $I(D,B)$ -supra semi open map, $I(D,B)$ -supra semi homeomorphism map, Ordered supra semi separation axioms.

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1 Introduction

Nachbin [18] in 1965, initiated the concept of topological ordered spaces and studied its main features. He also investigated the main properties of increasing and decreasing sets. Then McCartan [17], in 1968, carried out a detailed study on ordered separation axioms by utilizing the notions of increasing and decreasing neighborhoods. Mashhour et al. [16] generalized a topology notion to a supra topology and discussed some supra topological notions such as supra continuity and supra separation axioms. In 1991, Arya and Gupta [8] utilized semi open sets [15] to introduce semi separation axioms in topological ordered spaces. In 2002, Kumar [14] introduced and studied the concepts of continuity, openness, closedness and homeomorphism between topological ordered spaces. In 2004, Das [9] introduced and studied ordered separation axioms via some ordered spaces. In 2016, Abo-elhamayel and Al-shami [1] formulated the concepts of x -supra continuous, x -supra open, x -supra closed and x -supra homeomorphism maps in supra topological ordered spaces, for $x = \{I, D, B\}$ and studied their properties. El-Shafei et al. [11] utilized the monotone open sets instead of monotone neighborhoods to present and investigate

strong ordered separation axioms. They also used a notion of supra R -open sets [10] to define several kinds of maps [12] in topological ordered spaces. It is worth noting that the supra R -open sets except for the non-empty set were studied in topological spaces under the name of somewhere dense sets [4]. Recently, some studies on ordered maps via supra topological ordered spaces were done (see for example, [5], [7]).

The aim of the present paper is to establish some types of x -supra semi continuous, x -supra semi open, x -supra semi closed and x -supra semi homeomorphism maps in supra topological spaces, for $x = \{I, D, B\}$. Also, we give necessary and sufficient conditions for these maps and investigate under what conditions these maps preserve some separation axioms. Many of the findings that raised at are generalizations of those findings in supra topological ordered spaces which introduced in [1].

2 Preliminary

Hereinafter, several concepts and results of supra topological ordered spaces are recalled.

Definition 2.1. ([18], [1]) A triple (X, τ, \preceq) is called a topological ordered space, where (X, τ) is a topological space and \preceq is a partial order relation on X . If we replace a topology τ by a supra topology μ , then a triple (X, μ, \preceq) is called a supra topological ordered space.

Remark 2.2. Throughout this paper, (X, τ, \preceq_1) and (Y, τ, \preceq_2) stand for topological ordered spaces and (X, μ, \preceq_1) and (Y, μ, \preceq_2) stand for supra topological ordered spaces. A diagonal relation is denoted by Δ .

Definition 2.3. [18] Let (X, \preceq) be a partially ordered set. Then:

- (i) $i(b) = \{a \in X : b \preceq a\}$ and $d(b) = \{a \in X : a \preceq b\}$.
- (ii) $i(B) = \bigcup \{i(b) : b \in B\}$ and $d(B) = \bigcup \{d(b) : b \in B\}$.
- (iii) A set B is called increasing (resp. decreasing), if $A = i(A)$ (resp. $A = d(A)$).

Definition 2.4. [14] A subset B of a partially ordered set (X, \preceq) is called balancing if $B = i(B) = d(B)$.

Definition 2.5. [16] Let E be a subset of a supra topological space (X, μ) . Then:

- (i) Supra interior of E , denoted by $sint(E)$, is the union of all supra open sets contained in E .
- (ii) Supra closure of E , denoted by $scl(E)$, is the intersection of all supra closed sets containing E .

Definition 2.6. [16]

- (i) A map $g : (X, \tau) \rightarrow (Y, \theta)$ is said to be supra continuous if the inverse image of each open subset of Y is a supra open subset of X .

(ii) Let (X, τ) be a topological space and μ be a supra topology on X . We say that μ is associated supra topology with τ if $\tau \subseteq \mu$.

Definition 2.7. [6] A subset E of a supra topological space (X, μ) is called supra semi open if $E \subseteq scl(sint(E))$ and its complement is called supra semi closed.

Definition 2.8. [6] Let E be a subset of a supra topological space (X, μ) . Then:

(i) Supra semi interior of E , denoted by $ssint(E)$, is the union of all supra semi open sets contained in E .

(ii) Supra semi closure of E , denoted by $sscl(E)$, is the intersection of all supra semi closed sets containing E .

Definition 2.9. [6] A map $g : (X, \tau) \rightarrow (Y, \theta)$ is said to be:

(i) Supra semi continuous if the inverse image of each open subset of Y is a supra semi open subset of X .

(ii) Supra semi open (resp. supra semi closed) if the image of each open (resp. closed) subset of X is a supra semi open (resp. supra semi closed) subset of Y .

(iii) Supra semi homeomorphism if it is bijective, supra semi continuous and supra semi open.

Definition 2.10. [1] A map $g : (X, \tau) \rightarrow (Y, \theta)$ is said to be supra open (resp. supra closed) if the image of any open (resp. closed) subset of X is a supra open (resp. supra closed) subset of Y .

Definition 2.11. A map $f : (X, \preceq_1) \rightarrow (Y, \preceq_2)$ is called:

(i) Order preserving (or increasing) if $a \preceq_1 b$, then $f(a) \preceq_2 f(b)$.

(ii) Order embedding if $a \preceq_1 b$ if and only if $f(a) \preceq_2 f(b)$.

Definition 2.12. [17] A topological ordered space (X, τ, \preceq) is called:

(i) Lower (Upper) strong T_1 -ordered if for each $a, b \in X$ such that $a \not\preceq b$, there exists an increasing (a decreasing) open set G containing $a(b)$ such that $b(a)$ belongs to G^c .

(ii) Strong T_1 -ordered if it is strong lower T_1 -ordered and strong upper T_1 -ordered.

(iii) Strong T_0 -ordered if it is strong lower T_1 -ordered or strong upper T_1 -ordered.

(iv) Strong T_2 -ordered if for every $a, b \in X$ such that $a \not\preceq b$, there exist disjoint open sets W_1 and W_2 containing a and b , respectively, such that W_1 is increasing and W_2 is decreasing.

Remark 2.13. In definition above, McCartan [17] named the above axioms, T_i -ordered spaces instead of strong T_i -ordered spaces if it is replaced the words open set by neighborhood.

Definition 2.14. [11] A supra topological ordered space (X, μ, \preceq) is called:

- (i) Lower (Upper) SST_1 -ordered if for each $a, b \in X$ such that $a \not\leq b$, there exists an increasing (a decreasing) supra open set G containing $a(b)$ such that $b(a)$ belongs to G^c .
- (ii) SST_1 -ordered space if it is both lower SST_1 -ordered and upper T_1 -ordered space.
- (iii) SST_0 -ordered space if it is lower SST_1 -ordered or upper SST_1 -ordered space.
- (iv) SST_2 -ordered if for every $a, b \in X$ such that $a \not\leq b$, there exist disjoint supra open sets W_1 and W_2 containing a and b , respectively, such that W_1 is increasing and W_2 is decreasing.

3 Supra Semi Continuous Maps in Supra Topological Ordered Spaces

The concepts of I-supra semi continuous, D-supra semi continuous and B-supra semi continuous maps in supra topological ordered spaces are presented and their main properties are investigated. The relationships among them are illustrated with the help of examples. The enough conditions for these three types of supra semi continuous maps to preserve some of ordered supra semi separation axioms are given.

Definition 3.1. A subset E of (X, μ, \preceq_1) is said to be:

- (i) I-supra (resp. D-supra, B-supra) semi open if it is supra semi open and increasing (resp. decreasing, balancing).
- (ii) I-supra (resp. D-supra, B-supra) semi closed if it is supra semi closed and increasing (resp. decreasing, balancing).

Definition 3.2. A map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ is called I-supra (resp. D-supra, B-supra) semi continuous at $p \in X$ if for each open set H containing $f(p)$, there exists an I-supra (resp. a D-supra, a B-supra) semi open set G containing p such that $f(G) \subseteq H$.

Also, the map is called I-supra (resp. D-supra, B-supra) semi continuous if it is I-supra (resp. D-supra, B-supra) semi continuous at each point $p \in X$.

Theorem 3.3. A map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ is I-supra (resp. D-supra, B-supra) semi continuous if and only if the inverse image of each open subset of Y is an I-supra (resp. a D-supra, a B-supra) semi open subset of X .

Proof. We only prove the theorem in case of f is an I-supra semi continuous map and the other follow similar lines.

To prove the necessary part, let G be an open subset of Y , Then we have the following two cases:

- (i) $f^{-1}(G) = \emptyset$ which is an I-supra semi open subset of X .
- (ii) $f^{-1}(G) \neq \emptyset$. By choosing $p \in X$ such that $p \in f^{-1}(G)$, we obtain that $f(p) \in G$. So there exists an I-supra semi open set H_p containing p such that $f(H_p) \subseteq G$. Since p is chosen arbitrary, then $f^{-1}(G) = \bigcup_{p \in f^{-1}(G)} H_p$. Thus $f^{-1}(G)$ is an I-supra semi open subset of X .

To prove the sufficient part, let G be an open subset of Y containing $f(p)$. Then $p \in f^{-1}(G)$. By hypothesis, $f^{-1}(G)$ is an I-supra semi open set. Since $f(f^{-1}(G)) \subseteq G$, then f is an I-supra semi continuous at $p \in X$ and since p is chosen arbitrary, then f is an I-supra semi continuous. \square

Remark 3.4. (i) Every I-supra (D-supra, B-supra) semi continuous map is supra semi continuous.

(ii) Every B-supra semi continuous map is I-supra semi continuous and D-supra semi continuous.

The following two examples illustrate that a supra semi continuous (resp. an I-supra semi continuous) map need not be I-supra semi continuous or D-supra semi continuous or B-supra semi continuous (resp. B-supra semi continuous).

Example 3.5. Let the supra topology $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and the topology $\theta = \{\emptyset, Y, \{x\}\}$ on $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$, respectively. Let the partial order relation $\preceq_1 = \Delta \cup \{(a, b), (b, c), (a, c)\}$ on X and let the map $f : X \rightarrow Y$ be defined as follows $f(a) = f(c) = f(d) = x, f(b) = y$. Obviously, f is supra semi continuous. Now, $\{x\}$ is an open subset of Y , whereas $f^{-1}(\{x\}) = \{a, c, d\}$ is neither a decreasing nor an increasing supra semi open subset of X . Then f is not I-supra (D-supra, B-supra) semi continuous.

Example 3.6. We replace only the partial order relation in Example 3.5 by $\preceq = \Delta \cup \{(b, c)\}$. Then the map f is I-supra semi continuous, but not B-supra semi continuous.

The relationships among the introduced types of supra continuous maps are illustrated in the following figure.

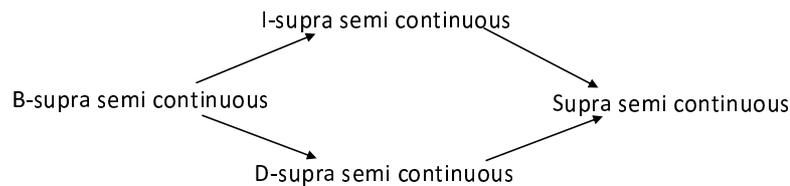


Figure 1: The relationships among types of supra continuous maps

Definition 3.7. Let E be a subset of (X, μ, \preceq) . Then:

- (i) $E^{isso} = \bigcup \{G : G \text{ is an I-supra semi open set included in } E\}$.
- (ii) $E^{dssso} = \bigcup \{G : G \text{ is a D-supra semi open set included in } E\}$.
- (iii) $E^{bssso} = \bigcup \{G : G \text{ is a B-supra semi open set included in } E\}$.
- (iv) $E^{isscl} = \bigcap \{H : H \text{ is an I-supra semi closed set including } E\}$.
- (v) $E^{dsscl} = \bigcap \{H : H \text{ is a D-supra semi closed set including } E\}$.

(vi) $E^{bsscl} = \bigcap \{H : H \text{ is a B-supra semi closed set including } E\}$.

Lemma 3.8. Let E be a subset of (X, μ, \preceq) . Then:

(i) $((E)^{dsscl})^c = ((E)^c)^{isso}$.

(ii) $((E)^{isscl})^c = ((E)^c)^{dsscl}$.

(iii) $((E)^{bsscl})^c = ((E)^c)^{bsscl}$.

Proof. (i) $((E)^{dsscl})^c = \{\bigcup F : F \text{ is a D-supra semi closed set including } E\}^c$
 $= \bigcap \{F^c : F^c \text{ is an I-supra semi open set included in } E^c\}$
 $= ((E)^c)^{isso}$.

The proof of (ii) and (iii) is similar to that of (i). □

Theorem 3.9. Let $g : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:

- (i) g is I-supra semi continuous;
- (ii) The inverse image of each closed subset of Y is a D-supra semi closed subset of X ;
- (iii) $(g^{-1}(H))^{dsscl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;
- (iv) $g(A^{dsscl}) \subseteq cl(g(A))$, for every $A \subseteq X$;
- (v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isso}$, for every $H \subseteq Y$.

Proof. (i) \Rightarrow (ii) Consider H is a closed subset of Y . Then H^c is open. Therefore $g^{-1}(H^c) = (g^{-1}(H))^c$ is an I-supra semi open subset of X . So $g^{-1}(H)$ is D-supra semi closed.

(ii) \Rightarrow (iii) For any subset H of Y , we have that $cl(H)$ is closed. Since $g^{-1}(cl(H))$ is a D-supra semi closed subset of X , then $(g^{-1}(H))^{dsscl} \subseteq (g^{-1}(cl(H)))^{dsscl} = g^{-1}(cl(H))$.

(iii) \Rightarrow (iv): Consider A is a subset of X . Then $A^{dsscl} \subseteq (g^{-1}(g(A)))^{dsscl} \subseteq g^{-1}(cl(g(A)))$. Therefore $g(A^{dsscl}) \subseteq g(g^{-1}(cl(g(A)))) \subseteq cl(g(A))$.

(iv) \Rightarrow (v): Let H be a subset of Y . By Lemma (3.8), we obtain that $g(X - (g^{-1}(H))^{isso}) = g(((g^{-1}(H))^c)^{dsscl})$. By (iv) $g(((g^{-1}(H))^c)^{dsscl}) \subseteq cl(g((g^{-1}(H))^c)) = cl(g(g^{-1}(H^c))) \subseteq cl(Y - H) = Y - int(H)$. Therefore $X - (g^{-1}(H))^{isso} \subseteq g^{-1}(Y - int(H)) = X - g^{-1}(int(H))$. Thus $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isso}$.

(v) \Rightarrow (i): Consider H is an open subset of Y . Then $g^{-1}(H) = g^{-1}(int(H)) \subseteq (g^{-1}(H))^{isso}$. Since $g^{-1}(H)$ is I-supra semi open, then $(g^{-1}(H))^{isso} \subseteq g^{-1}(H)$. Therefore $g^{-1}(H)$ is an I-supra semi open subset of X . Thus g is I-supra semi continuous. □

Theorem 3.10. Let $g : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:

- (i) g is D-supra semi continuous;
- (ii) The inverse image of each closed subset of Y is an I-supra semi closed subset of X ;

(iii) $(g^{-1}(H))^{isscl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;

(iv) $g(A^{isscl}) \subseteq cl(g(A))$, for every $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{dssso}$, for every $H \subseteq Y$.

Proof. The proof is similar to that of Theorem (3.9). \square

Theorem 3.11. Let $g : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ be a map. Then the following five statements are equivalent:

(i) g is B-supra semi continuous;

(ii) The inverse image of each closed subset of Y is a B-supra semi closed subset of X ;

(iii) $(g^{-1}(H))^{bsscl} \subseteq g^{-1}(cl(H))$, for every $H \subseteq Y$;

(iv) $g(A^{bsscl}) \subseteq cl(g(A))$, for every $A \subseteq X$;

(v) $g^{-1}(int(H)) \subseteq (g^{-1}(H))^{bssso}$, for every $H \subseteq Y$.

Proof. The proof is similar to that of Theorem (3.9). \square

Definition 3.12. A supra topological ordered space (X, μ, \preceq) is called:

(i) Lower (Upper) strong supra semi T_1 -ordered (briefly, Lower (Upper) $SSST_1$ -ordered) if for each $a, b \in X$ such that $a \not\preceq b$, there exists an increasing (a decreasing) supra semi open set G containing $a(b)$ such that $b(a)$ belongs to G^c .

(ii) $SSST_0$ -ordered space if it is lower $SSST_1$ -ordered or upper $SSST_1$ -ordered.

(iii) $SSST_1$ -ordered space if it is both lower $SSST_1$ -ordered and upper $SSST_1$ -ordered.

(iv) $SSST_2$ -ordered if for every $a, b \in X$ such that $a \not\preceq b$, there exist disjoint supra semi open sets W_1 and W_2 containing a and b , respectively, such that W_1 is increasing and W_2 is decreasing.

Theorem 3.13. Let a bijective map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ be I-supra semi continuous and f^{-1} be an order preserving map. If (Y, τ, \preceq_2) is a lower T_1 -ordered space, then (X, μ, \preceq_1) is a lower $SSST_1$ -ordered space.

Proof. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a), y = f(b)$. Since f^{-1} is an order preserving map, then $x \not\preceq_2 y$. Since (Y, τ, \preceq_2) is a lower T_1 -ordered space, then there exists an increasing neighborhood W of x in Y such that $x \in W$ and $y \notin W$. Therefore there exists an open set G such that $x \in G \subseteq W$. Since f is bijective I-supra semi continuous, then $a \in f^{-1}(G)$ which is I-supra semi open and $b \notin f^{-1}(G)$. Thus (X, μ, \preceq_1) is a lower $SSST_1$ -ordered space. \square

Theorem 3.14. Let a bijective map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ be D-supra semi continuous and f^{-1} be an order preserving map. If (Y, τ, \preceq_2) is an upper T_1 -ordered space, then (X, μ, \preceq_1) is an upper $SSST_1$ -ordered space.

Proof. The proof is similar to that of Theorem (3.16). □

Theorem 3.15. Let a bijective map $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ be B-supra semi continuous and f^{-1} be an order preserving map. If (Y, τ, \preceq_2) is a T_i -ordered space, then (X, μ, \preceq_1) is an $SSST_1$ -ordered space for $i = 0, 1, 2$.

Proof. We prove the theorem in case of $i = 2$. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f^{-1} is an order preserving map, then $x \not\preceq_2 y$. Since (Y, τ, \preceq_2) is a T_2 -ordered space, then there exist disjoint balancing neighborhoods W_1 and W_2 of x and y , respectively. Therefore there are disjoint open sets G and H containing x and y , respectively. Since f is bijective B-supra semi continuous, then $a \in f^{-1}(G)$ which is an I-supra semi open subset of X , $b \in f^{-1}(H)$ which is a D-supra semi open subset of X and $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. Thus (X, μ, \preceq_1) is a $SSST_2$ -ordered space.

In a similar way, we can prove the theorem in case of $i = 0, 1$. □

Theorem 3.16. Consider $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ is a bijective supra semi continuous map such that f is ordered embedding. If (Y, τ, \preceq_2) is strong T_i -ordered, then (X, μ, \preceq_1) is $SSST_i$ -ordered, for $i = 0, 1, 2$.

Proof. We prove the theorem in case of $i = 2$. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $x = f(a)$ and $y = f(b)$. Since f is ordered embedding, then $x \not\preceq_2 y$. Since (Y, τ, \preceq_2) is strong T_2 -ordered, then there exist disjoint open sets W_1 and W_2 containing x and y , respectively, such that W_1 is increasing and W_2 is decreasing. Since f is bijective supra semi continuous and order preserving, then $f^{-1}(W_1)$ is an I-supra semi open set containing a , $f^{-1}(W_2)$ is a D-supra semi open set containing b and $f^{-1}(W_1) \cap f^{-1}(W_2) = \emptyset$. Thus (X, μ, \preceq_1) is $SSST_2$ -ordered.

Similarly, one can prove theorem in case of $i = 0, 1$. □

Theorem 3.17. Consider $f : (X, \mu, \preceq_1) \rightarrow (Y, \tau, \preceq_2)$ is an injective B-supra semi continuous map. If (Y, τ, \preceq_2) is a T_i -space, then (X, μ, \preceq_1) is an $SSST_i$ -ordered space, for $i = 1, 2$.

Proof. We prove the theorem in case of $i = 2$ and the other case is similar. Let $a, b \in X$ such that $a \not\preceq_1 b$. Then there exist $x, y \in Y$ such that $f(a) = x, f(b) = y$ and $x \neq y$. Since (Y, τ, \preceq_2) is a T_2 -space, then there exist disjoint open sets G and H such that $x \in G$ and $y \in H$. Therefore $a \in f^{-1}(G)$ which is an I-supra semi open subset of X , $b \in f^{-1}(H)$ which is a D-supra semi open subset of X and $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. Thus (X, μ, \preceq_1) is an $SSST_2$ -ordered space.

Similarly, one can prove the theorem in case of $i = 1$. □

4 Supra Semi Open (Supra Semi Closed) Maps in Supra Topological Ordered Spaces

In this section, we introduce the concepts of I-supra semi open (I-supra semi closed), D-supra semi open (D-supra semi closed) and B-supra semi open (B-supra

semi closed) maps in supra topological ordered spaces. We demonstrate their main properties and illustrate the relationships among them with the help of examples. Finally, some results concerning the image and per image of some separation axioms under these maps are presented.

Definition 4.1. A map $g : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ is said to be:

- (i) I-supra (resp. D-supra, B-supra) semi open if the image of any open subset of X is an I-supra (resp. a D-supra, a B-supra) semi open subset of Y .
- (ii) I-supra (resp. D-supra, B-supra) semi closed if the image of any closed subset of X is an I-supra (resp. a D-supra, a B-supra) semi closed subset of Y .

Remark 4.2. (i) Every I-supra (D-supra, B-supra) semi open map is supra semi open.

(ii) Every I-supra (D-supra, B-supra) semi closed map is supra semi closed.

(iii) Every B-supra semi open (resp. B-supra semi closed) map is I-supra semi open and D-supra semi open (resp. I-supra semi closed and D-supra semi closed).

The following two examples illustrate that a supra semi open (resp. D-supra semi open) map need not be I-supra semi open or D-supra semi open or B-supra semi open (resp. B-supra semi open).

Example 4.3. Let the topology $\tau = \{\emptyset, X, \{1, 2\}\}$ and the partial order relation $\preceq_2 = \Delta \cup \{(1, 3), (3, 2), (1, 2)\}$ on $X = \{1, 2, 3\}$. Let the supra topology associated with τ be $\{\emptyset, X, \{1, 2\}, \{1, 3\}\}$ on X . The identity map $f : (X, \tau) \rightarrow (X, \mu, \preceq_2)$ is a supra semi open map. Now, $\{1, 2\}$ is an open subset of X . Since $f(\{1, 2\}) = \{1, 2\}$ is neither an increasing nor a decreasing supra semi open subset of Y , then f is not x-supra semi open map, for $x = \{I, D, B\}$.

Example 4.4. We replace only the partial order relation in Example (4.3) by $\preceq = \Delta \cup \{(1, 3), (2, 3)\}$. Then the map f is D-supra semi open, but is not B-supra semi open.

The following two examples illustrate that a supra semi closed (resp. an I-supra semi closed) map need not be I-supra semi closed or D-supra semi closed or B-supra semi closed (resp. B-supra semi closed).

Example 4.5. Let the topology $\tau = \{\emptyset, X, \{a, b\}\}$ on $X = \{a, b, c\}$, the supra topology associated with τ be $\{\emptyset, X, \{c\}, \{a, b\}\}$ and the partial order relation $\preceq_2 = \Delta \cup \{(a, c), (c, b), (a, b)\}$ on X . The map $f : (X, \tau) \rightarrow (X, \mu, \preceq_2)$ is defined as follows $f(a) = f(c) = c$ and $f(b) = b$. Obviously, f is supra semi closed. Now, $\{c\}$ is a closed subset of X , but $f(\{c\}) = \{c\}$ is neither a decreasing nor an increasing supra semi closed subset of Y . Then f is not x-supra semi closed map, for $x = \{I, D, B\}$.

Example 4.6. We replace only the partial order relation in Example (4.5) by $\preceq = \Delta \cup \{(b, c)\}$. Then the map f is I-supra semi closed, but is not B-supra semi closed.

The relationships among the introduced types of supra semi open (supra semi closed) maps are illustrated in the following figure.

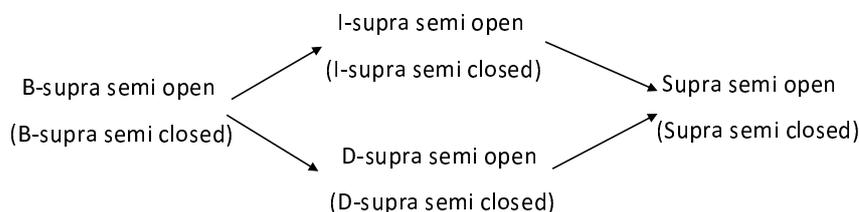


Figure 2: The relationships among types of supra open (supra closed) maps

Theorem 4.7. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$:

- (i) f is I-supra semi open;
- (ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{isso})$, for every $H \subseteq Y$;
- (iii) $f(int(G)) \subseteq (f(G))^{isso}$, for every $G \subseteq X$.

Proof. (i) \Rightarrow (ii): Since $int(f^{-1}(H))$ is an open subset of X , then $f(int(f^{-1}(H)))$ is an I-supra semi open subset of Y . Since $f(int(f^{-1}(H))) \subseteq f(f^{-1}(H)) \subseteq H$, then $int(f^{-1}(H)) \subseteq f^{-1}(H^{isso})$.

(ii) \Rightarrow (iii): By replacing H by $f(G)$ in (ii), we obtain that $int(f^{-1}(f(G))) \subseteq f^{-1}((f(G))^{isso})$. Since $int(G) \subseteq f^{-1}(f(int(f^{-1}(f(G)))) \subseteq f^{-1}((f(G))^{isso})$, then $f(int(G)) \subseteq (f(G))^{isso}$.

(iii) \Rightarrow (i): Let G be an open subset of X . Then $f(int(G)) = f(G) \subseteq (f(G))^{isso}$. So f is an I-supra semi open map. □

In a similar way, one can prove the following two theorems.

Theorem 4.8. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$:

- (i) f is D-supra semi open;
- (ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{dss})$, for every $H \subseteq Y$;
- (iii) $f(int(G)) \subseteq (f(G))^{dss}$, for every $G \subseteq X$.

Theorem 4.9. The following statements are equivalent, for a map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$:

- (i) f is B-supra semi open;
- (ii) $int(f^{-1}(H)) \subseteq f^{-1}(H^{bss})$, for every $H \subseteq Y$;
- (iii) $f(int(G)) \subseteq (f(G))^{bss}$, for every $G \subseteq X$.

Theorem 4.10. Let $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be a map. Then we have the following results.

- (i) f is I-supra semi closed if and only if $(f(G))^{isscl} \subseteq f(cl(G))$, for any $G \subseteq X$.
- (ii) f is D-supra semi closed if and only if $(f(G))^{dsscl} \subseteq f(cl(G))$, for any $G \subseteq X$.
- (iii) f is B-supra semi closed if and only if $(f(G))^{bsscl} \subseteq f(cl(G))$, for any $G \subseteq X$.

Proof. (i) Necessity: Consider f is an I-supra semi closed map. Then $f(cl(G))$ is an I-supra semi closed subset of Y . Since $f(G) \subseteq f(cl(G))$, then $(f(G))^{isscl} \subseteq f(cl(G))$. Sufficiency: Consider B is a closed subset of X . Then $f(B) \subseteq (f(B))^{isscl} \subseteq f(cl(B)) = f(B)$. Therefore $f(B) = (f(B))^{isscl}$ is an I-supra semi closed set. Thus f is an I-supra semi closed map.

The proof of (ii) and (iii) is similar to that of (i). \square

Theorem 4.11. Let $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be a bijective map. Then we have the following results.

- (i) f is I-supra semi open if and only if f is D-supra semi closed.
- (ii) f is D-supra semi open if and only if f is I-supra semi closed.
- (iii) f is B-supra semi open if and only if f is B-supra semi closed.

Proof. (i) Necessity: Let f be an I-supra semi open map and let G be a closed subset of X . Then G^c is open. Since f is bijective, then $f(G^c) = (f(G))^c$ is I-supra semi open. Therefore $f(G)$ is a D-supra semi closed subset of Y . Thus f is D-supra semi closed.

Sufficiency: Let f be a D-supra semi closed map and let B be an open subset of X . Then B^c is closed. Since f is bijective, then $f(B^c) = (f(B))^c$ is D-supra semi closed. Therefore $f(B)$ is I-supra semi open. Thus f is I-supra semi open.

The proof of (ii) and (iii) is similar to that of (i). \square

Theorem 4.12. The following two statements hold.

- (i) If the maps $f : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$ is open and $g : (Y, \theta, \preceq_2) \rightarrow (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi open, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) semi open.
- (ii) If the maps $f : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$ is closed and $g : (Y, \theta, \preceq_2) \rightarrow (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi closed, then a map $g \circ f$ is I-supra (resp. D-supra, B-supra) semi closed.

Proof. It is clear. \square

Theorem 4.13. If the maps $g \circ f$ is I-supra (resp. D-supra, B-supra) semi open and $f : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$ is surjective continuous, then a map $g : (Y, \theta, \preceq_2) \rightarrow (Z, \nu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi open.

Proof. Consider $g \circ f$ is I-supra semi open and let G be an open subset of Y . Then $f^{-1}(G)$ is an open subset of X . Since $g \circ f$ is I-supra semi open and f is surjective, then $(g \circ f)(f^{-1}(G)) = g(G)$ is an I-supra semi open subset of Z . Therefore g is I-supra semi open.

A similar proof can be given for the cases between parentheses. □

Theorem 4.14. If the maps $g \circ f : (X, \tau, \preceq_1) \rightarrow (Z, \mu, \preceq_3)$ is closed and $g : (Y, \theta, \preceq_2) \rightarrow (Z, \mu, \preceq_3)$ is I-supra (resp. D-supra, B-supra) semi continuous injective, then a map $f : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$ is D-supra (resp. I-supra, B-supra) semi closed.

Proof. Consider g is I-supra semi continuous. Let G be a closed subset of X . Then $(g \circ f)(G)$ is a closed subset of Z . Since g is injective and I-supra semi continuous, then $g^{-1}(g \circ f)(G) = f(G)$ is a D-supra semi closed subset of Y . Therefore f is D-supra semi closed.

A similar proof can be given for the cases between parentheses. □

Theorem 4.15. We have the following results for a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$.

- (i) f is I-supra (resp. D-supra, B-supra) semi open if and only if f^{-1} is I-supra (resp. D-supra, B-supra) semi continuous.
- (ii) f is D-supra (resp. I-supra, B-supra) semi closed if and only if f^{-1} is I-supra (resp. D-supra, B-supra) semi continuous.

Proof. (i) We prove (i) when f is B-supra semi open, and the other cases follow similar lines.

' \Rightarrow ' Let f be a B-supra semi open map and let G be an open subset of X . Then $(f^{-1})^{-1}(G) = f(G)$ is a B-supra semi open subset of Y . Therefore f^{-1} is a B-supra semi continuous.

' \Leftarrow ' let G be an open subset of X and f^{-1} be a B-supra semi continuous. Then $f(G) = (f^{-1})^{-1}(G)$ is a B-supra semi open subset of Y . Therefore f is B-supra semi open.

(ii) Similarly, one can prove (ii). □

Theorem 4.16. Let a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be I-supra semi open (D-supra semi closed) and order preserving. If (X, τ, \preceq_1) is a lower T_1 -ordered space, then (Y, μ, \preceq_2) is a lower $SSST_1$ -ordered space.

Proof. We prove the theorem when a map f be I-supra semi open.
 Let $x, y \in Y$ such that $x \not\preceq_2 y$. Since f is bijective, then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$ and since f is an order preserving map, then $a \not\preceq_1 b$. By hypotheses (X, τ, \preceq_1) is a lower T_1 -ordered space, then there exists an increasing neighborhood W in X such that $a \in W$ and $b \notin W$. Therefore there exists an open set G such that $a \in G \subseteq W$. Thus $x \in f(G)$ which is an I-supra semi open and $y \notin f(G)$. Hence (Y, μ, \preceq_2) is a lower $SSST_1$ -ordered space.
 The proof for a D-supra semi closed map is achieved similarly. □

Theorem 4.17. Let a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be D-supra semi open (I-supra semi closed) and order preserving. If (X, τ, \preceq_1) is an upper T_1 -ordered space, then (Y, μ, \preceq_2) is an upper $SSST_1$ -ordered space.

Proof. The proof is similar to that of Theorem (4.16). \square

Theorem 4.18. Let a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be B-supra semi open (B-supra semi closed) and order preserving. If (X, τ, \preceq_1) is a T_i -ordered space, then (Y, μ, \preceq_2) is an $SSST_i$ -ordered space for $i = 0, 1, 2$.

Proof. When a map f is B-supra semi open and $i = 2$.

For all $x, y \in Y$ such that $x \not\preceq_2 y$, there are $a, b \in X$ such that $a = f^{-1}(x), b = f^{-1}(y)$. Since f is an order preserving, then $a \not\preceq_1 b$. Since (X, τ, \preceq_1) is a T_2 -ordered space, then there exist disjoint neighborhoods W_1 and W_2 of a and b , respectively, such that W_1 is increasing and W_2 is decreasing. Therefore there are disjoint open sets G and H such that $a \in G \subseteq W_1$ and $b \in H \subseteq W_2$. Thus $x \in f(G)$ which is a balancing supra semi open, $y \in f(H)$ which is a balancing supra semi open and $f(G) \cap f(H) = \emptyset$. Thus (Y, μ, \preceq_2) is an $SSST_2$ -ordered space.

In a similar way, we can prove the theorem in case of $i = 0, 1$.

The proof for a B-supra semi closed map is achieved similarly. \square

Theorem 4.19. Consider a bijective map $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ is supra semi open such that f and f^{-1} are order preserving. If (X, τ, \preceq_1) is strong T_i -ordered, then (Y, μ, \preceq_2) is $SSST_i$ -ordered, for $i = 0, 1, 2$.

Proof. We prove the theorem in case of $i = 2$. Let $x, y \in Y$ such that $x \not\preceq_2 y$. Then there exist $a, b \in X$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since f is an order preserving, then $a \not\preceq_1 b$. Since (X, τ, \preceq_1) is strong T_2 -ordered space, then there exist disjoint an increasing open set W_1 containing a and a decreasing open set W_2 containing b such that $a \in W_1$ and $b \in W_2$. Since f is a bijective supra semi open and f^{-1} is an order preserving, then $f(W_1)$ is an I-supra semi open set containing x , $f(W_2)$ is a D-supra semi open set containing y and $f(W_1) \cap f(W_2) = \emptyset$. Therefore (Y, μ, \preceq_2) is $SSST_2$ -ordered.

Similarly, one can prove theorem in case of $i = 0, 1$. \square

Theorem 4.20. Let $f : (X, \tau, \preceq_1) \rightarrow (Y, \mu, \preceq_2)$ be a bijective supra open map such that f and f^{-1} are order preserving. If (X, τ, \preceq_1) is strong T_i -ordered, then (Y, μ, \preceq_2) is $SSST_i$ -ordered, for $i = 0, 1, 2$.

Proof. The proof is similar to that of Theorem (4.19). \square

5 Supra Semi Homeomorphism Maps in Supra Topological Ordered Spaces

The concepts of I-supra semi homeomorphism, D-supra semi homeomorphism and B-supra semi homeomorphism maps are introduced and many of their properties are established. Some illustrative examples are provided.

Definition 5.1. Let τ^* and θ^* be associated supra topologies with τ and θ , respectively. A bijective map $g : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$ is called I-supra (resp. D-supra, B-supra) semi homeomorphism if it is I-supra semi continuous and I-supra semi open (resp. D-supra semi continuous and D-supra semi open, B-supra semi continuous and B-supra semi open).

Remark 5.2. (i) Every I-supra (D-supra, B-supra) semi homeomorphism map is supra semi homeomorphism.

(ii) Every B-supra semi homeomorphism map is I-supra semi homeomorphism and D-supra semi homeomorphism.

The following two examples illustrate that a supra semi homeomorphism (resp. D-supra semi homeomorphism) map need not be I-supra semi homeomorphism or D-supra semi homeomorphism or B-supra semi homeomorphism (resp. B-supra semi homeomorphism).

Example 5.3. Let the topology $\tau = \{\emptyset, X, \{a, c\}\}$ on $X = \{a, b, c\}$, the supra topology associated with τ be $\{\emptyset, X, \{a\}, \{a, c\}\}$ and the partial order relation $\preceq_1 = \Delta \cup \{(c, a), (c, b)\}$. Let the topology $\theta = \{\emptyset, Y, \{y, z\}\}$ on $Y = \{x, y, z\}$, the supra topology associated with θ be $\{\emptyset, Y, \{y\}, \{y, z\}\}$ and the partial order relation $\preceq_2 = \Delta \cup \{(y, z)\}$ on Y . The map $f : (X, \tau, \preceq_1) \rightarrow (Y, \theta, \preceq_2)$ is defined as $f(a) = y, f(b) = z$ and $f(c) = x$. Now, f is supra semi homeomorphism, but is not x -supra semi homeomorphism, for $x = \{I, D, B\}$.

Example 5.4. We replace only the partial order relation \preceq_1 in Example (5.3) by $\preceq = \Delta \cup \{(a, c)\}$. Then the map f is D-supra semi homeomorphism, but not B-supra semi homeomorphism.

The relationships among the presented types of supra semi homeomorphism maps are illustrated in the following figure.

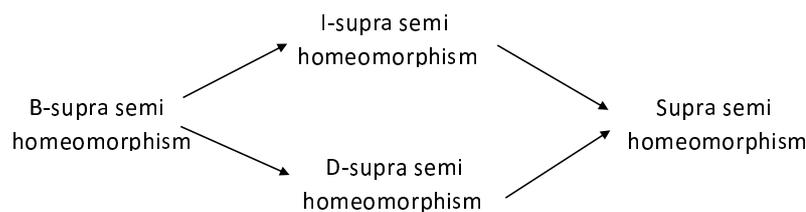


Figure 3: The relationships among types of supra homeomorphism maps

Theorem 5.5. Let a map $f : X \rightarrow Y$ be bijective and I-supra semi continuous. Then the following statements are equivalent:

- (i) f is I-supra semi homeomorphism;
- (ii) f^{-1} is I-supra semi continuous;
- (iii) f is D-supra semi closed.

Proof. (i) \Rightarrow (ii) Let G be an open subset of X . Then $(f^{-1})^{-1}(G) = f(G)$ is an I-supra semi open set in Y . Therefore f^{-1} is I-supra semi continuous.

(ii) \Rightarrow (iii) Let G be a closed subset of X . Then G^c is an open subset of X and $(f^{-1})^{-1}(G^c) = f(G^c) = (f(G))^c$ is an I-supra semi open set in Y . Therefore $f(G)$ is a D-supra semi closed subset of Y . Thus f is D-supra semi closed.

(iii) \Rightarrow (i) Let G be an open subset of X . Then G^c is a closed set and $f(G^c) = (f(G))^c$ is D-supra semi closed. Therefore $f(G)$ is an I-supra semi open subset of Y . Thus f is I-supra semi open. Hence f is an I-supra semi homeomorphism map. \square

In a similar way one can prove the following two theorems.

Theorem 5.6. Let a map $f : X \rightarrow Y$ be bijective and D-supra semi continuous. Then the following statements are equivalent:

(i) f is D-supra semi homeomorphism;

(ii) f^{-1} is D-supra semi continuous;

(iii) f is I-supra semi closed.

Theorem 5.7. Let a map $f : X \rightarrow Y$ be bijective and B-supra semi continuous. Then the following statements are equivalent:

(i) f is B-supra semi homeomorphism;

(ii) f^{-1} is B-supra semi continuous;

(iii) f is B-supra semi closed.

Theorem 5.8. Consider (X, τ, \preceq_1) and (Y, θ, \preceq_2) are two topological ordered spaces, and τ^* and θ^* are associated supra topologies with τ and θ , respectively. Let $f : X \rightarrow Y$ be a supra semi homeomorphism map such that f and f^{-1} are order preserving. If X (resp. Y) is strong T_i -ordered, then Y (resp. X) is $SSST_i$ -ordered, for $i = 0, 1, 2$.

Proof. (i) Let (X, τ, \preceq_1) be a strong T_i -ordered space, then by Theorem (4.19), (Y, θ, \preceq_2) is an $SSST_i$ -ordered space, for $i = 0, 1, 2$.

(ii) Let (Y, θ, \preceq_2) be a strong T_i -ordered space, then by Theorem (3.16), (X, τ, \preceq_1) is an $SSST_i$ -ordered space, for $i = 0, 1, 2$. \square

Conclusion

In the present paper, the concepts of I-supra (D-supra, B-supra) semi continuous, I-supra (D-supra, B-supra) semi open, I-supra (D-supra, B-supra) closed and I-supra (D-supra, B-supra) semi homeomorphism maps are given and studied. The sufficient conditions for maps to preserve some separation axioms (which introduced in [9], [11] and [17]) are determined. In particular, we investigate the equivalent conditions for each concept and present their properties. Apart from that, we point out the relationships among them with the help of illustrative examples. In the end, the presented concepts in this paper are fundamental background for studying several topics in supra topological ordered spaces.

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