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On Path Laplacian Eigenvalues and Path Laplacian Energy of Graphs

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Abstaract − We introduce the concept of Path Laplacian Matrix for a graph and explore the eigenvalues of this matrix. The eigenvalues of this matrix are called the path Laplacian eigenvalues of the graph. We investigate path Laplacian eigenvalues of some classes of graph. Several results concerning path Laplacian eigenvalues of graphs have been obtained.

Keywords − *Path, Real symmetric matrix, Laplacian matrix.*

1 Introduction

For a graph G the eigenvalues of G are the eigenvalues of its adjacency matrix. The spectrum of of a graph G is the set of its eigenvalues. Several properties and applications of eigenvalues of graph are useful. For undefined terminology and notations we refer to Lowel W. Beineke [1] and West [2]. For an extensive survey on graph spectra we refer to R. B. Bapat [3], Brouwer A. E. [4] and Verga R. S. [5].

We have defined the path matrix [6, 7] of the graph G as follows. Let G be a graph without loops and let $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of G.
Define the matrix $P = (p_{ij})$ of size $n \times n$ such that

$$p_{ij} = \begin{cases} \text{maximum number of vertex disjoint paths from } v_i \text{ to } v_j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

We call P as Path Matrix of G. The matrix P is real symmetric matrix. Therefore, its eigenvalues are real. We call eigenvalues of P as path eigenvalues of G.

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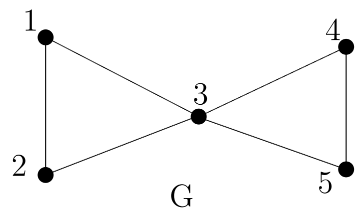
2 Preliminary

We define the path Laplacian matrix of G, $PL(G)$ as follows.

Definition 2.1. The rows and columns of $PL(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i, j)- entry of $PL(G)$ is 0 if there is no path between i and j, and it is $-k$ if the maximum number of vertex disjoint paths between i and j is k. The (i, i) entry of $PL(G)$ is d_i, the degree of the vertex i, $i = 1, 2, 3, ..., n$.

Thus $PL(G)$ is an $n \times n$ matrix. The path Laplacian matrix of G can be defined in an alternative way. Let $D(G)$ be the diagonal matrix of vertex degrees. If $P(G)$ is the path matrix of G, then $PL(G) = D(G) - P(G)$. We call the path eigenvalues of $PL(G)$ as path Laplacian eigenvalues of G.

Example 2.2. Consider the graph G as shown in the following figure.



Then the path Laplacian matrix of G is

$$\mathbf{PL}(G) = \begin{bmatrix} 2 & -2 & -2 & -1 & -1 \\ -2 & 2 & -2 & -1 & -1 \\ -2 & -2 & 4 & -2 & -2 \\ -1 & -1 & -2 & 2 & -2 \\ -1 & -1 & -2 & -2 & 2 \end{bmatrix}.$$

The characteristic polynomial of the matrix $PL(G)$ is
$C_{PL(G)}(x) = |PL - xI| = (x+4)(x-2)(x-4)^2(x-6)$. The path Laplacian eigenvalues of G are $-4, 2, 4, 4$ and 6. The ordinary Laplacian eigenvalues of G are $0, 1, 3, 3$ and 5.

The ordinary Laplacian spectrum of the graph G, consisting of the numbers $\mu_1, \mu_2, ..., \mu_n$ is the spectrum of its Laplacian matrix [8, 9, 10, 11]. In analogy, the path Laplacian spectrum of a graph G is defined as the spectrum of the corresponding path Laplacian matrix.

3 Path Laplacian Eigenvalues of Graphs

In this section, we investigate path Laplacian eigenvalues of some special classes of graphs. In this paper, we define path Laplacian matrix of a graph and investigate the eigenvalues (called path Laplacian eigenvalues) of this matrix. We obtain several properties concerning the path Laplacian eigenvalues. A notion of path Laplacian energy has been introduced and some of its basic properties have been obtained.

Proposition 3.1. Let S_n be a star with n vertices. Then the path Laplacian eigenvalues of S_n are 2 with multiplicity $n-2$, $1+\sqrt{n^2-3n+3}$ with multiplicity 1 and $1-\sqrt{n^2-3n+3}$ with multiplicity 1.

Proof. We can write the path Laplacian matrix of S_n as

$$\mathbf{PL(S_n)} = \begin{bmatrix} n-1 & -1 & -1 & ... & -1 & -1 \\ -1 & 1 & -1 & ... & -1 & -1 \\ -1 & -1 & 1 & ... & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & ... & 1 & -1 \\ -1 & -1 & -1 & ... & -1 & 1 \end{bmatrix}$$

The characteristic polynomial of $PL(S_n)$ is

$$C_{PL(S_n)}(x) = (x-2)^{n-2}(x-1-\sqrt{n^2-3n+3})(x-1+\sqrt{n^2-3n+3}).$$

Consequently the path Laplacian eigenvalues of S_n are 2 with multiplicity $n-2$, $1+\sqrt{n^2-3n+3}$ with multiplicity 1 and $1-\sqrt{n^2-3n+3}$ with multiplicity 1. □

Proposition 3.2. Let P_n be a path graph with n vertices. Then the path Laplacian eigenvalues of P_n are 2 with multiplicity 1, 3 with multiplicity $n-3$, $\frac{(-n+5)+\sqrt{n^2-2n+9}}{2}$ with multiplicity 1 and $\frac{(-n+5)-\sqrt{n^2-2n+9}}{2}$ with multiplicity 1.

Proof. The path Laplacian matrix of P_n is

$$\mathbf{PL(P_n)} = \begin{bmatrix} 1 & -1 & -1 & ... & -1 & -1 \\ -1 & 2 & -1 & ... & -1 & -1 \\ -1 & -1 & 2 & ... & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & ... & 2 & -1 \\ -1 & -1 & -1 & ... & -1 & 1 \end{bmatrix}$$

The characteristic polynomial of $PL(P_n)$ is $C_{PL(P_n)}(x) =$

$$(x-2)(x-3)^{n-3}(x-\frac{(-n+5)+\sqrt{n^2-2n+9}}{2})(x-\frac{(-n+5)-\sqrt{n^2-2n+9}}{2}).$$

Consequently the path Laplacian eigenvalues of P_n are 2 with multiplicity 1, 3 with multiplicity $n-3$, $\frac{(-n+5)+\sqrt{n^2-2n+9}}{2}$ with multiplicity 1 and $\frac{(-n+5)-\sqrt{n^2-2n+9}}{2}$ with multiplicity 1. □

Proposition 3.3. Let W_n be a wheel graph with n vertices. Then the path Laplacian eigenvalues of W_n are 6 with multiplicity $n-2$, $-(n-4)+\sqrt{4n^2-11n+16}$ with multiplicity 1 and
$-(n-4)-\sqrt{4n^2-11n+16}$ with multiplicity 1.

Proof. The path Laplacian matrix of W_n is

$$\mathbf{PL(W_n)} = \begin{bmatrix} n-1 & -3 & -3 & \dots & -3 & -3 \\ -3 & 3 & -3 & \dots & -3 & -3 \\ -3 & -3 & 3 & \dots & -3 & -3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -3 & -3 & -3 & \dots & 3 & -3 \\ -3 & -3 & -3 & \dots & -3 & 3 \end{bmatrix}$$

The characteristic polynomial of $PL(W_n)$ is $C_{PL(W_n)}(x) = (x-6)^{n-2}(x+(n-4)-\sqrt{4n^2-11n+16})(x+(n-4)+\sqrt{4n^2-11n+16})$. Consequently the path Laplacian eigenvalues of W_n are 6 with multiplicity $n-2$, $-(n-4)+\sqrt{4n^2-11n+16}$ with multiplicity 1 and
$-(n-4)-\sqrt{4n^2-11n+16}$ with multiplicity 1. □

Proposition 3.4. The path Laplacian eigenvalues of the complete bipartite graph $K_{m,n}$ $(1 < m \leq n)$ are m with multiplicity $n-1$, n with multiplicity $m-1$, $(m+n-mn)+\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1 and $(m+n-mn)-\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1.

Proof. The path Laplacian matrix of $K_{m,n}$ is

$$\mathbf{PL(K_{m,n})} = \begin{bmatrix} n & -n & \dots & -n & -m & -m & \dots & -m \\ -n & n & \dots & -n & -m & -m & \dots & -m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -n & -n & \dots & n & -m & -m & \dots & -m \\ -m & -m & \dots & -m & m & -m & \dots & -m \\ -m & -m & \dots & -m & -m & m & \dots & -m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -m & -m & \dots & -m & -m & -m & \dots & m \end{bmatrix}$$

$$= \begin{bmatrix} 2nI_m - nJ_m & B \\ B' & 2mI_n - mJ_n \end{bmatrix}.$$

where B is $m \times n$ matrix with all entries $-m$ and B' is the transpose of the matrix B. Therefore the path Laplacian eigenvalues of $K_{m,n}$ are $2m$ with multiplicity $n-1$, $2n$ with multiplicity $m-1$, $(m+n-mn)+\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1 and $(m+n-mn)-\sqrt{[m+n-mn]^2+mn[1+3(m-1)]}$ with multiplicity 1. □

Remark: Let G be a graph on n vertices with m edges. Then the sum of the path Laplacian eigenvalues of G is $2m$. For instance, let G be a graph with vertex degrees $d_1, d_2, ..., d_n$ and with path Laplacian eigenvalues $\mu_1, \mu_2, ..., \mu_n$. Then $tracePL(G) = \sum_{i=1}^{n} d_i = 2m$, also $tracePL(G) = \sum_{i=1}^{n} \mu_i$. Thus $\sum_{i=1}^{n} \mu_i = 2m$.

The following theorem gives path Laplacian eigenvalues of r-regular, r-connected graph.

Theorem 3.5. Let G be a r- regular, r-connected graph with n vertices. Then the path Laplacian matrix $PL(G)$ of G is of the form $2rI_n - rJ_n$ and the path Laplacian eigenvalues of G are of the form $2r - nr$ with multiplicity 1 and $2r$ with multiplicity $n-1$.

Proof. We can write $PL(G)$ as

$$\mathbf{PL(G)} = \begin{bmatrix} r & -r & ... & -r \\ -r & r & ... & -r \\ \vdots & \vdots & \ddots & \vdots \\ -r & -r & ... & r \end{bmatrix}$$
$$= 2rI_n - rJ_n.$$

Consequently the path Laplacian eigenvalues of a graph G are $r(2-n)$ with multiplicity 1 and $2r$ with multiplicity $n-1$.

□

Corollary 3.6. Let G_1 be a r_1-regular, r_1-connected graph with n_1 vertices and G_2 be a r_2-regular, r_2-connected graph with n_2 vertices. Then the path Laplacian eigenvalues of their cartesian product are $(r_1+r_2)(2-n)$ with multiplicity 1 and $2(r_1+r_2)$ with multiplicity $n-1$, where $n = n_1.n_2$.

Proof. Let G denote the cartesian product of G_1 and G_2. Then G is r_1+r_2-regular, r_1+r_2-connected with n vertices. By Theorem 3.5, the path Laplacian eigenvalues of G are $(r_1+r_2)(2-n)$ with multiplicity 1 and $2(r_1+r_2)$ with multiplicity $n-1$. □

Remark: Let G be an r-regular, r-connected graph with n vertices. Then $PL(G) + P(G) = rI_n$.

Proposition 3.7. Let G be a r-regular, r-connected graph with n vertices and m edges. Let $\mu_1, ..., \mu_n$ and $d_1, ..., d_n$ be the path Laplacian eigenvalues and degrees of vertices of G, respectively. Then

$$\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + n(n-1)r^2 = \sum_{i=1}^{n} d_i^2 + \frac{4m^2(n-1)}{n}.$$

Proof. Let $PL(G)$ be the path Laplacian matrix of G. Then

$$PL(G)^2 = \begin{bmatrix} nr^2 & (n-4)r^2 & ... & (n-4)r^2 \\ (n-4)r^2 & nr^2 & ... & (n-4)r^2 \\ \vdots & \vdots & \ddots & \vdots \\ (n-4)r^2 & (n-4)r^2 & ... & nr^2 \end{bmatrix}$$

Since G is r-regular, $d_i = r = \frac{2m}{n}$, $\quad i = 1, 2, ..., n$ and $\sum_{i=1}^{n} d_i^2 = nr^2$.

$\sum_{i=1}^{n} \mu_i^2 = trPL(G)^2 = n^2r^2 = nr^2 + n^2r^2 - nr^2 = \sum_{i=1}^{n} d_i^2 + n(n-1)r^2 = \sum_{i=1}^{n} d_i^2 + \frac{4m^2(n-1)}{n}$. □

In the following Proposition, we give the relation between path Laplacian eigenvalues and maximum vertex degree Δ.

Proposition 3.8. Let G be a graph on n vertices with degrees d_i and $PL(G)$ be its path Laplacian matrix. Let $\Delta = \max_i d_i$ and $\mu_1, \mu_2, ..., \mu_n$ be the path Laplacian eigenvalues of PL(G). Then $\sum_i \mu_i \leq n\Delta$.

Proof. We know that $\sum_i \mu_i = \sum_i d_i$ and $\sum_i d_i \leq n\Delta$. Therefore we conclude that $\sum_i \mu_i \leq n\Delta$. □

Proposition 3.9. (Bounds for μ_1 and μ_n:) Let G be a graph on n vertices, m edges with degrees of vertices d_i and $PL(G)$ be its path Laplacian matrix. Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ be the path Laplacian eigenvalues of $PL(G)$. Then $\mu_n \leq \frac{2m}{n} \leq \mu_1$.

Proof. We know, $\sum_i \mu_i = 2m$ and $n\mu_n \leq \sum_i \mu_i \leq n\mu_1$. This implies that $\mu_n \leq \frac{2m}{n}$ and $\mu_1 \geq \frac{2m}{n}$. Thus $\mu_n \leq \frac{2m}{n} \leq \mu_1$. □

4 Path Laplacian Energy of Graphs

In this section, we find path Laplacian energy of some graphs.
Definition: Let G be a graph with n vertices and m edges. Let $\mu_1,\ \mu_2, ...,\ \mu_n$ be the path Laplacian eigenvalues of G. We define the path Laplacian energy as

$$PLE(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|.$$

In the following table, we explore the path Laplacian energy of some classes of graphs which have just two distinct path Laplacian eigenvalues denoted by μ_1 and μ_2.

Graphs	μ_1	μ_2	Path Laplacian Energy
K_n	$(n-1)(2-n)$	$2(n-1)$	$3(n-1)^2$
C_n	$2(2-n)$	4	$3(n-1)$
Q_n	$n(2-2^n)$	$2n$	$2n(2^n-1)$
Petersen Graph	6	-24	54

From Propositions 3.1-3.4, we get the path Laplacian energies of S_n, P_n, W_n and $K_{m,n}$ as follows.
The path Laplacian energy of the star graph S_n is $\frac{2(n-2)}{n} + 2\sqrt{n^2-3n+3}$.
The path Laplacian energy of the path graph P_n is $\frac{n^2-n-4}{n} + \sqrt{n^2-2n+9}$.
The path Laplacian energy of the wheel graph W_n is $\frac{2(n^2-4)}{n} + 2\sqrt{4n^2-11n+16}$.
The path Laplacian energy of the complete bipartite graph $K_{m,n}$ $(1 < m \leq n)$ is $\frac{2mn(n-m)}{m+n} + (m-n) + \sqrt{[m+n-mn]^2 + mn[1+3(m-1)]}$.

The following result follows from the definitions of the path energy and path Laplacian energy.

Proposition 4.1. Let G be a r-regular, r-connected graph on n vertices $(1 \leq r \leq n-1)$ and m edges. Then $PE(G) = PLE(G) = \frac{4(n-1)}{n}m$.

Proof. By [6], the path eigenvalues of G are $r(n-1)$ with multiplicity 1 and $-r$ with multiplicity $n-1$. Since G is r-regular, $r = \frac{2m}{n}$, this implies that

$$PE(G) = |r(n-1)| + (n-1)|-r| = 2r(n-1) = \frac{4(n-1)}{n}m.$$

By Theorem 3.5, the path Laplacian eigenvalues of G are $2r - nr$ with multiplicity 1 and $2r$ with multiplicity $n-1$. Thus

$$PLE(G) = |r(2-n)-r|+(n-1)|2r-r| = |r-nr|+(n-1)|r| = 2r(n-1) = \frac{4(n-1)}{n}m.$$

□

Let G be a disconnected graph with two components G_1 and G_2, then $PLE(G)$ need not be equal to $PLE(G_1) + PLE(G_2)$. Consider the following example.

Example 4.2. Consider the graph G with two connected components P_4 and C_3, then $PLE(G) \neq PLE(P_4) + PLE(C_3)$ as the value of LHS is 13.982 and the value of RHS is 12.123. We observe that average vertex degree of $P_4 = 1.5 \neq 2 =$ average vertex degree of C_3.

In the following Proposition, we give a sufficient condition so that $PLE(G) = PLE(G_1) + PLE(G_2)$.

Proposition 4.3. If the graph G consists of disconnected components G_1 and G_2, and if G_1 and G_2 have equal average vertex degrees, then $PLE(G) = PLE(G_1) + PLE(G_2)$.

Proof. Let G, G_1, and G_2 be $(n,\ m)$, $(n_1,\ m_1)$, and $(n_2,\ m_2)$-graphs, respectively. Then from $2m_1/n_1 = 2m_2/n_2$ it follows $2m/n = 2m_i/n_i,\ i = 1, 2$. Therefore

$$PLE(G) = \sum_{i=1}^{n_1+n_2} |\mu_i - \frac{2m}{n}| = \sum_{i=1}^{n_1} |\mu_i - \frac{2m_1}{n_1}| + \sum_{i=n_1+1}^{n_1+n_2} |\mu_i - \frac{2m_2}{n_2}|$$

$= PLE(G_1) + PLE(G_2)$. □

Let G_1 and G_2 be two graphs with disjoint vertex sets. Let V_i and E_i be the vertex and edge sets of G_i $(i = 1, 2)$, respectively. The union of G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. If G_1 is an $(n_1,\ m_1)$-graph and G_2 is an $(n_2,\ m_2)$-graph then $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges.

In the following Theorem, we obtain bound for the path Laplacian energy of the union of two graphs.

Theorem 4.4. If G_1 be an $(n_1,\ m_1)$-graph and G_2 be an $(n_2,\ m_2)$-graph, such that $\frac{2m_1}{n_1} > \frac{2m_2}{n_2}$. Then

$$PLE(G_1)+PLE(G_2)-\frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2} \leq PLE(G_1 \cup G_2) \leq PLE(G_1)+PLE(G_2)+ \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2}.$$

Proof. Let $G = G_1 \cup G_2$. Then G is an $(n_1+n_2,\ m_1+m_2)$-graph. By the definition of path Laplacian energy,

$$PLE(G_1 \cup G_2) = \sum_{i=1}^{n_1+n_2} |\mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=n_1+1}^{n_1+n_2} |\mu_i(G) - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$= \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2m_1}{n_1} + \frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2m_2}{n_2} + \frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2}|$$

$$\leq \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2m_1}{n_1}| + n_1|\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2m_2}{n_2}| + n_2|\frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2}|.$$

Since $n_2m_1 > n_1m_2$, above inequality becomes
$PLE(G_1 \cup G_2) \leq PLE(G_1) + n_1(\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}) + PLE(G_2) + n_2(-\frac{2m_2}{n_2} + \frac{2(m_1+m_2)}{n_1+n_2}) = PLE(G_1) + PLE(G_2) + \frac{4(n_2m_1 - n_1m_2)}{n_1+n_2}$ which is an upper bound for path Laplacian energy of $G_1 \cup G_2$.
To get the lower bound, we just have to note that in full analogy to the above arguments,
$PLE(G_1 \cup G_2) \geq \sum_{i=1}^{n_1} |\mu_i(G_1) - \frac{2m_1}{n_1}| - n_1|\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}| + \sum_{i=1}^{n_2} |\mu_i(G_2) - \frac{2m_2}{n_2}| - n_2|\frac{2m_2}{n_2} - \frac{2(m_1+m_2)}{n_1+n_2}|$.
Since $n_2m_1 > n_1m_2$, above inequality becomes
$PLE(G_1 \cup G_2) \geq PLE(G_1) - n_1(\frac{2m_1}{n_1} - \frac{2(m_1+m_2)}{n_1+n_2}) + PLE(G_2) - n_2(-\frac{2m_2}{n_2} + \frac{2(m_1+m_2)}{n_1+n_2}) = PLE(G_1) + PLE(G_2) - \frac{4(n_2m_1 - n_1m_2)}{n_1+n_2}$
which is a lower bound for path Laplacian energy of $G_1 \cup G_2$. □

Corollary 4.5. Let G_1 be an r_1 regular graph on n_1 vertices and G_2 be an r_2 regular graph on n_2 vertices, such that $r_1 > r_2$. Then
$PLE(G_1) + PLE(G_2) - \frac{2n_1n_2(r_1-r_2)}{n_1+n_2} \leq PLE(G_1 \cup G_2) \leq PLE(G_1) + PLE(G_2) + \frac{2n_1n_2(r_1-r_2)}{n_1+n_2}$.

Proof. Since G_1 is r_1 regular, the number of edges in G_1 is $m_1 = \frac{n_1r_1}{2}$ and since G_2 is r_2 regular, the number of edges in G_2 is $m_2 = \frac{n_2r_2}{2}$. Now $\frac{2m_1}{n_1} = r_1 > r_2 = \frac{2m_2}{n_2}$. By Theorem 4.4, we get the required inequality. □

Corollary 4.6. Let G_1 be an $(n,\ m)$-graph and G_2 be the graph obtained from G_1 by removing k edges, $0 \leq k \leq m$. Then
$PLE(G_1) + PLE(G_2) - 2k \leq PLE(G_1 \cup G_2) \leq PLE(G_1) + PLE(G_2) + 2k$.

Proof. The number of vertices of G_2 is n and the number of edges in G_2 is $m - k$. By Theorem 4.4, the result follows. □

5 Conclusion

In the present paper, the concepts of path Laplacian matrix, path Laplacian eigenvalues and path Laplacian energy of a graph are given and studied. Also, some bounds on Path Laplacian Energy of graphs are given and studied.

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