Generalized Cubic Aggregation Operators with Application in Decision Making Problem

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Abstract - There are many aggregation operators and their applications have been developed up to date, but in this paper we introduced the idea of generalized aggregation operator. The main idea of this paper is to study the generalized aggregation operators with cubic numbers. In this paper, we introduced three types of cubic aggregation operators called generalized cubic weighted averaging $(GCWA)$ operator, generalized cubic ordered weighted averaging $(GCOWA)$ operator and generalized cubic hybrid averaging $(GCHA)$ operator. We extend the theory of cubic numbers to generalized ordered weighted averaging operators that are characterized by interval membership and exact membership. In last section we provide an application of these aggregation operators to multiple attribute group decision making problem.

Keywords **-** Cubic sets, GCWA Operator, GCOWA Operator, GCHA operator.

1. Introduction

In 1965, Zadeh generalized the classical set theory to fuzzy set theory. Fuzzy set (Fs) has been studied in many fields such that decision making theory, information science, medical diagnosis, pattern recognition, fuzzy algebra and fuzzy topology. Fuzzy set has not explained every concept due to not available of non- membership. In [2] ، Atanassov introduced the concept of intuitionistic fuzzy set (IFs) , intuitionistic fuzzy set is generalized structure of fuzzy set. Intuitionistic fuzzy set characterized by membership and non-membership of an element in a set. The application of intuitionistic fuzzy set has been studied in many fields, logic program, algebra, topology, medical diagnosis and decision making theory. IFs aggregation operator has been studied [3,4,5,6,7] i.e., intuitionistic fuzzy ordered weighted $(IFOW)$ operator, intuitionistic fuzzy ordered weighted

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geometric $(IFOWG)$ operator, intuitionistic fuzzy hybrid averaging $(IFHA)$ operator. The intuitionistic fuzzy set does not explain the problem when arise uncertainty. Therefore Jun et al defined the new concept so called cubic set (CS). In [8] Jun introduced a new theory which is called cubic (CS) set theory. They introduced many concepts of cubic set cubic to deal with uncertainty problem. Cubic set explain all the satisfied, unsatisfied and uncertain information, while fuzzy and intuitionistic fuzzy set fail to explain these terms. In classical fuzzy set, to explain i.e., the experts degree of certainty in various statement, the value of interval $[0,1]$ is used. It is often more difficult for a decision maker's to exactly quantify his certainty. Therefore instead of real number, it is more adequate to explain this degree of certainty by an interval or even by a fuzzy set. In case of cubic set (CS) the membership is represented by interval-valued fuzzy set and non-membership in fuzzy set. Interval - valued fuzzy set has applied to real life application i.e., Sambuc applied it to medical diagnosis in thyroidian pathology. Kohout also applied it to medical, in a system CLINAID [9]. Turlesen [10,11] used interval-valued logic to preference modeling. Cubic set theory applied many areas in BCK/BCI algebra and other structures [12,13,14].

The weighted aggregation (WA) operator and ordered weighted aggregation (OWA) operator are rich area for research and the generalized aggregation operators are new class of aggregation operator. Thus an advantage of the above mentioned aggregation operators. In this paper, we introduced three types of cubic aggregation operators so-called generalized cubic weighted averaging $(GCWA)$ operator, generalized cubic ordered weighted averaging $(GCOWA)$ operator and generalized cubic hybrid averaging $(GCHA)$ operator.

This paper is organized as follows: In section 2, we give some basic definitions and laws of cubic numbers which will be used in our later sections. In section 3, we develop the concept of generalized cubic weighted averaging $(GCWA)$ operator, generalized cubic ordered weighted averaging $(GCOWA)$ operator and generalized cubic hybrid averaging $(GCHA)$ operator. In section 4, we provide an applications of these aggregation operators to multi-criteria decision making. For this purpose we develop a general algorithm for these cubic aggregation operators. In section 5, numerical an application to decision making problems. In section 6, we discuss and compare the proposed operator with $GIFA$ operator. Concluding remarks are made in section 7.

2. Preliminaries

Atanassov generalized the concept of fuzzy set (FS) and defined the concept of IFS as follows [2] .Let X be a fixed set. An IFS A in X is an object having the form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \qquad (1)$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to A, respectively, and for every $x \in X$,

$$0 \le \mu_A(x) + \nu_A(x) \le 1. \qquad (2)$$

For each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \text{ for all } x \in X. \qquad (3)$$

Then, $\pi_A(x)$ is called the degree of indeterminacy of x to A.

Definition 2.1 [8] Let X be a fixed non empty set. A cubic set is an object of the form:

$$\tilde{C} = \{\langle a, A(a), \lambda(a)\rangle : a \in X\},$$

where A is an interval-valued fuzzy set (*IVFS*) and λ is a fuzzy set in X. A cubic set $\tilde{C} = \langle a, A(a), \lambda(a)\rangle$ is simply denoted by $\tilde{C} = \langle \tilde{A}, \lambda\rangle$, the collection of cubic sets is denoted by $\tilde{C}(X)$.

(1) if $\lambda \in \tilde{A}(x)$ for all $x \in X$ so it is called interval cubic set ，

(2) If $\lambda \notin \tilde{A}(x)$ for all $x \in X$ so it is called external cubic set ，

(3) If $\lambda \in \tilde{A}(x)$ or $\lambda \notin \tilde{A}(x)$ its called cubic set for all $x \in X$.

Definition 2.2 [8] Let $\tilde{A} = \langle A, \lambda\rangle$ and $\tilde{B} = \langle B, \mu\rangle$ are two cubic sets in X, such that,

(1) （Equality） $A = B \Leftrightarrow A = B$ and $\lambda = \mu$,

(2) $(P - \text{order})$ $A \subseteq_A B \Leftrightarrow A \subseteq B$ and $\lambda \leq \mu$,

(3) $(R - \text{order})$ $A \subseteq_R B \Leftrightarrow A \subseteq B$ and $\lambda \geq \mu$.

Definition 2.3 [8] The complement of $\tilde{A} = \langle A, \lambda\rangle$ is defined to be the cubic set as follows:

$$A^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}.$$

Cubic Numbers Score and Accuracy Function

In this section, we define some operational laws of cubic numbers. We define score function and accuracy function of a cubic set which will be used in our later sections.

Definition 2.4 Let $\tilde{C} = \langle A_c, \eta_c\rangle$, $\tilde{C}_1 = \langle \bar{A}_{c_1}, \eta_{c_1}\rangle$, and $\tilde{C}_2 = \langle \bar{A}_{c_2}, \eta_{c_2}\rangle$ be any three cubic values (CV). Then, the following operational laws hold:

(1)

$$\tilde{C}_1 \oplus \tilde{C}_2 = \left\langle \begin{bmatrix} \bar{a}_{c_1} + \bar{a}_{c_2} - \bar{a}_{c_1}\bar{a}_{c_2}, \\ a^+_{c_1} + a^+_{c_2} - a^+_{c_1}a^+_{c_2} \end{bmatrix}, \eta_{c_1}\eta_{c_2}\right\rangle,$$

(2)

$$\tilde{C}_1 \otimes \tilde{C}_2 = \left\langle \begin{bmatrix} \bar{a}_{c_1}\bar{a}_{c_2}, & a^+_{c_1}a^+_{c_2} \end{bmatrix}, \eta_{c_1} + \eta_{c_2} - \eta_{c_1}\eta_{c_2}\right\rangle,$$

(3)

$$\delta \tilde{C} = \left\langle \left[1-(1-\bar{a}_c)^\delta \,, (1-(1-a_c^+)^\delta \right], \eta_c^\delta \right\rangle, \quad \delta \geq 0,$$

(4)

$$\tilde{C}^\delta = \left\langle \left[(\bar{a}_c)^\delta \,, (a_c^+)^\delta \right], 1-(1-\eta_c)^\delta \right\rangle, \quad \delta \geq 0.$$

Example 2.5 Let $\tilde{C}_1 = \langle [0.5,0.6], 0.4 \rangle$, $\tilde{C}_2 = \langle [0.4,0.5], 0.7 \rangle$, $\tilde{C}_3 = \langle [0.6,0.8], 0.3 \rangle$, be any three cubic numbers, and let $\delta = 2$. Then, we verify the above results as follows:

(1)$\tilde{C}_1 \oplus \tilde{C}_2$

$$
\begin{aligned}
&= \left\langle \begin{bmatrix} 0.4+0.6-0.4\times0.6, \\ 0.5+0.8-0.5\times0.8 \end{bmatrix}, \right. \\
&\qquad\qquad \left. ,0.3\times0.7 \right\rangle \\
&= \langle [1.0-0.24, 1.3-0.40], 0.21 \rangle, \\
&= \langle [0.76-0.90], 0.21 \rangle.
\end{aligned}
$$

(2)$\tilde{C}_1 \otimes \tilde{C}_2$

$$
\begin{aligned}
&= \langle [0.4\times0.6, 0.5\times0.8], 0.7+0.3-0.7\times0.3 \rangle \\
&= \langle [0.24, 0.40], 1.0-0.21 \rangle \\
&= \langle [0.24-0.40], 0.79 \rangle.
\end{aligned}
$$

(3)$\delta \tilde{C}$

$$
\begin{aligned}
&= \left\langle \left[1-(1-0.5)^2, 1-(1-0.6)^2 \right], (0.4)^2 \right\rangle \\
&= \left\langle \left[1-(0.5)^2, 1-(0.4)^2 \right], 0.16 \right\rangle \\
&= \langle [0.75-0.84], 0.16 \rangle.
\end{aligned}
$$

(4)$\tilde{C}^\delta$

$$
\begin{aligned}
&= \left\langle \left[(0.5)^2, (0.6)^2 \right], 1-(1-0.4)^2 \right\rangle \\
&= \left\langle [0.25, 0.36], 1-(0.6)^2 \right\rangle \\
&= \langle [0.25-0.36], 0.64 \rangle.
\end{aligned}
$$

Theorem 2.6 Let $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$, $\tilde{C}_1 = \left\langle \bar{A}_{c_1}, \eta_{c_1} \right\rangle$, and $\tilde{C}_2 = \left\langle \bar{A}_{c_2}, \eta_{c_2} \right\rangle$, be any three cubic values. Then, the following operational laws hold:

$$\tilde{C}_1^{'} = \tilde{C}_1 \oplus \tilde{C}_2, \tilde{C}_2^{'} = \tilde{C}_1 \otimes \tilde{C}_2, \tilde{C}_3^{'} = \delta\tilde{C}, \tilde{C}_4^{'} = \tilde{C}^{\delta}, \delta > 0$$

then all $\tilde{C}_i^{'}$ $(i = 1,2,3,4)$ are cubic values.

Theorem 2.7 Let $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$, $\tilde{C}_1 = \left\langle \bar{A}_{c_1}, \eta_{c_1} \right\rangle$, $\tilde{C}_2 = \left\langle \bar{A}_{c_2}, \eta_{c_2} \right\rangle$ and $\tilde{C}_3 = \left\langle \bar{A}_{c_3}, \eta_{c_3} \right\rangle$ be any four (*CVs*), and δ, δ_1, δ_2 are any sclar numbers grater then zero such that,

(1)

$$\delta_1\tilde{C} \oplus \delta_2\tilde{C} = (\delta_1 + \delta_2)\tilde{C},$$

(2)

$$(\tilde{C}_1 \oplus \tilde{C}_2) \oplus \tilde{C}_3 = \tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3),$$

(3)

$$((\tilde{C})^{\delta_1})^{\delta_2} = (\tilde{C})^{\delta_1\,\delta_2}.$$

Example 2.8 Let

$$\tilde{C} = \left\langle [0.3,0.4],0.5 \right\rangle, \tilde{C}_1 = \left\langle [0.4,0.6],0.3 \right\rangle,$$
$$\tilde{C}_2 = \left\langle [0.5,0.7],0.8 \right\rangle, \tilde{C}_3 = \left\langle [0.6,0.3],0.4 \right\rangle$$

be any four cubic numbers, and let $\delta_1 = 2$ and $\delta_2 = 3$. Then, we verify the above results as follows;

(1) $\delta_1\tilde{C} \oplus \delta_2\tilde{C} = (\delta_1 + \delta_2)\tilde{C}$. In this case first we take $\delta_1\tilde{C} \oplus \delta_2\tilde{C}$ and then we take $(\delta_1 + \delta_2)\tilde{C}$. We apply cubic laws to verify the result such that,

$$\delta_1\tilde{C} = \left\langle [1 - (1-0.3)^2, 1 - (1-0.4)^2], (0.5)^2 \right\rangle$$
$$= \left\langle [1 - 0.49, 1 - 0.84], 0.25 \right\rangle$$
$$= \left\langle [0.51, 0.64], 0.25 \right\rangle, \text{ and}$$
$$\delta_2\tilde{C} = \left\langle [1 - (1-0.3)^3, 1 - (1-0.4)^3], (0.5)^3 \right\rangle$$
$$= \left\langle [1 - 0.343, 1 - 0.216], 0.125 \right\rangle$$
$$= \left\langle [0.675, 0.784], 0.125 \right\rangle.$$

By using $\delta_1\tilde{C}$ and $\delta_2\tilde{C}$ such that,

$$(\delta_1 + \delta_2)\tilde{C} = \left\langle \begin{bmatrix} 0.51 + 0.657 - 0.51 \times 0.657, \\ 0.64 + 0.784 - 0.64 \times 0.784 \\ 0.25 \times 0.125 \end{bmatrix}, \right\rangle$$

$$= \left\langle [0.8319 - 0.9224], 0.0312 \right\rangle.$$

Similarly we can find $(\delta_1 + \delta_2)\tilde{C}$ if we use $\delta = 5$ such that,

$$\delta\tilde{C} = \left\langle \left[1 - (1 - 0.3)^5, 1 - (1 - 0.4)^5 \right] (0.5)^5 \right\rangle$$

$$= \left\langle [1 - 0.1680, 1 - 0.0776], 0.0312 \right\rangle$$

$$= \left\langle [0.8319, 0.9224], 0.0312 \right\rangle.$$

(2) $(\tilde{C}_1 \oplus \tilde{C}_2) \oplus \tilde{C}_3 = \tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3)$. In this case first we take $(\tilde{C}_1 \oplus \tilde{C}_2) \oplus \tilde{C}_3$ and then we take $\tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3)$, we apply cubic laws to verify the result such that,

Let

$$\tilde{C} = \left\langle [0.4, 0.6], 0.5 \right\rangle, \tilde{C}_1 = \left\langle [0.4, 0.6], 0.3 \right\rangle,$$

$$\tilde{C}_2 = \left\langle [0.6, 0.7], 0.4 \right\rangle, \tilde{C}_3 = \left\langle [0.5, 0.7], 0.9 \right\rangle.$$

Then,

$$(\tilde{C}_1 \oplus \tilde{C}_2) = \left\langle \begin{bmatrix} 0.4 + 0.6 - 0.4 \times 0.6, \\ 0.5 + 0.8 - 0.5 \times 0.8 \end{bmatrix}, 0.3 \times 0.7 \right\rangle,$$

$$= \left\langle [1.0 - 0.24, 1.3 - 0.40], 0.21 \right\rangle,$$

$$= \left\langle [0.76, 0.90], 0.21 \right\rangle$$

$$(\tilde{C}_1 \oplus \tilde{C}_2) \oplus \tilde{C}_3 = \left\langle [0.76, 0.90], 0.21 \right\rangle \oplus \left\langle [0.5, 0.7], 0.9 \right\rangle$$

$$= \left\langle \begin{bmatrix} 0.76 + 0.5 - 0.76 \times 0.5, \\ 0.90 + 0.7 - 0.90 \times 0.7 \end{bmatrix}, 0.21 \times 0.9 \right\rangle$$

$$= \left\langle [1.26 - 0.38, 1.6 - 0.63], 0.18 \right\rangle$$

$$= \left\langle [0.88, 0.97], 0.18 \right\rangle.$$

Similarly we find $\tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3)$

$$(\tilde{C}_2 \oplus \tilde{C}_3) = \left\langle \begin{bmatrix} 0.6 + 0.5 - 0.6 \times 0.5, \\ 0.7 + 0.7 - 0.7 \times 0.7 \end{bmatrix}, 0.4 \times 0.9 \right\rangle,$$

$$= \langle [1.1 - 0.3, 1.4 - 0.49], 0.36 \rangle,$$

$$= \langle [0.80, 0.91], 0.36 \rangle$$

$$\tilde{C}_1 \oplus (\tilde{C}_2 \oplus \tilde{C}_3) = \langle [0.4, 0.6], 0.3 \rangle \oplus \langle [0.80, 0.91], 0.36 \rangle$$

$$= \left\langle \begin{bmatrix} 0.4 + 0.80 - 0.4 \times 0.80, \\ 0.6 + 0.91 - 0.6 \times 0.91 \end{bmatrix}, 0.3 \times 0.36 \right\rangle$$

$$= \langle [1.2 - 0.32, 1.51 - 0.54], 0.18 \rangle$$

$$= \langle [0.88, 0.97], 0.18 \rangle.$$

(3)　$((\tilde{C})^{\delta_1})^{\delta_2} = (\tilde{C})^{\delta_1 \, \delta_2}$. Let $\tilde{C} = \langle [0.3, 0.4], 0.6 \rangle$ be any cubic number and let $\delta_1 = 0.3$ and $\delta_2 = 0.2$ in this case first we find $((\tilde{C})^{\delta_1})^{\delta_2}$ then we find $\tilde{C}^{\delta_1 \, \delta_2}$ such that,

$$C^{\delta_1} = \left\langle \left[(0.3)^{0.3}, (0.4)^{0.3} \right], 1 - (1 - 0.6)^{0.3} \right\rangle$$

$$= \langle [0.69, 0.83], 0.24 \rangle$$

$$\left(C^{\delta_1} \right)^{\delta_2} = \left\langle \left[(0.69)^{0.2}, (0.83)^{0.2} \right], 1 - (1 - 0.24)^{0.2} \right\rangle$$

$$= \langle [0.93, 0.94], 0.0.05 \rangle \text{ and}$$

$$C^{\delta_1 \delta_2} = \left\langle \left[(0.3)^{0.06}, (0.4)^{0.06} \right], 1 - (1 - 0.6)^{0.06} \right\rangle$$

$$= \langle [0.93, 0.94], 0.0.05 \rangle.$$

Based on the cubic value (CVs) sets · We introduced a score function $s(\tilde{C})$ such that,

Let $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$ be an cubic value, where

$$\bar{A}_c \in [0,1], \; \eta_c \in [0,1]. \tag{4}$$

The score of $\tilde{C}$ can be evaluated by the score function s shown as follows:

$$s(\tilde{C}) = \frac{\bar{A}_c - \eta_c}{3} = \frac{\bar{a} + a^+ - \eta}{3}, \tag{5}$$

where $s(\tilde{C}) \in [-1, 1]$. The function s is used to measure the score of a (CV). Now an accuracy function to evaluate the degree of accuracy of the cubic value $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$ as follows;

$$h(\tilde{C}) = \frac{1 + \bar{A}_c - \eta_c}{3} = \frac{1 + \bar{a} + a^+ - \eta}{3}, \tag{6}$$

where $h(\tilde{C}) \in [0, 1]$.

Definition 2.9 Let $\tilde{C} = \left\langle \bar{A}_c, \eta_c \right\rangle$ and $\tilde{D} = \left\langle \bar{A}_D, \eta_D \right\rangle$ be any two cubic set such that,

$$s(\tilde{C}) = \frac{\bar{A}_c - \eta_c}{3} = \frac{\bar{a} + 2a^+ - \eta}{3}$$

$$s(\tilde{D}) = \frac{\bar{A}_D - \eta_D}{3} = \frac{\bar{a} + 2a^+ - \eta}{3}$$

be the scores function of $\tilde{C}$ and $\tilde{D}$, respectively, and be the accuracy degrees of $\tilde{C}$ and $\tilde{D}$, respectively, then

Remarks:

1. If $s(\tilde{C}) < s(\tilde{D})$, then $\tilde{C} < \tilde{D}$,

2. If $s(\tilde{C}) = s(\tilde{D})$, then,

i. If $h(\tilde{C}) = h(\tilde{D})$, then $\tilde{C} = \tilde{D}$,

ii. If $h(\tilde{C}) < h(\tilde{D})$, then $\tilde{C}$ is smaller than $\tilde{D}$, denoted by $\tilde{C} < \tilde{D}$.

3. The GCWA, GCOWA, And GCHA Operators

Definition 3.1 [15] A generalized weighted averaging (GWA) operator of dimension n is a mapping $GWA : (R^+)^n \rightarrow R^+$ $(R$ denotes the set of real number$)$ which has the following form:

$$GWA(a_1, a_2, ..., a_n) = \left(\sum_{j=1}^{n} w_j a_j^{\delta} \right)^{\frac{1}{\delta}} \qquad (7)$$

where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector of the arguments $a_j (j = 1, 2, ..., n)$ with $j \geq 0$, $j = 1, 2, ..., n$ and $\sum_{j=1}^{n} w_j = 1$, R^+ is the set of all non-negative real numbers. Another aggregation operator called the $GOWA$ operators is the generalization of the OWA operator.

Definition 3.2 [15] A generalized ordered weighted averaging $(GOWA)$ operator of dimension n is a mapping $GOWA : R^n \rightarrow R$ which has the following form:

$$GOWA(a_1, a_2, ..., a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\delta} \right)^{\frac{1}{\delta}} \qquad (8)$$

where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector of $(a_1, a_2, ..., a_n)$, $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$, b_j is jth largest of a_i, $I = [0,1]$.

The GCWA Operator

In this section, we define *GCWA* operator and study different results relevant to *GCWA* operator. For our convenience, let $\widetilde{C}$ denotes all of cubic set.

Definition 3.3 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of cubic value set and $GCWA : \widetilde{C}^n \to \widetilde{C}$, if

$$GCWA_w(\widetilde{c}_1, \widetilde{c}_2, ..., \widetilde{c}_n) = (w_1 c_1^{\delta} \oplus w_2 c_2^{\delta} \oplus ... \oplus w_n c_n^{\delta})^{\frac{1}{\delta}}, \qquad (9)$$

then the function *GCWA* is called a *GCWA* operator, where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector associated with the *GCWA* operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$. By using the operation laws of cubic numbers we will prove the following theorems.

Theorem 3.4 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of cubic value set. Then, their aggregated value by using the *GCWA* operator is also cubic value such that,

$$GCWA_w(c_1, c_2, ..., c_n)$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j}^{\delta})^{w_j} \right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^{n}(1 - a_{c_j}^{+\delta})^{w_j} \right)^{\frac{1}{\delta}} \right], \\ 1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j})^{\delta})^{w_j} \right)^{\frac{1}{\delta}} \right\rangle \qquad (10)$$

where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector associated with the *GCWA* operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$.

Proof The first result follows quickly from Definition 6 and Theorem 1. In the following, we first prove

$$w_1 c_1^{\delta} \oplus w_2 c_2^{\delta} \oplus ... \oplus w_n c_n^{\delta}$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j}^{\delta})^{w_j} \right), \left(1 - \prod_{j=1}^{n}(1 - a_{c_j}^{+\delta})^{w_j} \right) \right], \\ \left(\prod_{j=1}^{k}(1 - (1 - \eta_{c_j})^{\delta})^{w_j} \right) \right\rangle \qquad (11)$$

By using mathematically induction on n,

1. For $n = 2$. As we know that

$$\widetilde{C}^{\delta} = \left\langle \left[(\bar{a}_{c_j})^{\delta}, (a_{c_j}^{+})^{\delta} \right], 1 - (1 - \eta_{c_j})^{\delta} \right\rangle.$$

Then

$$\tilde{C}_1^\delta = \left\langle \left[(\bar{a}_{c_1})^\delta, (a_{c_1}^+)^\delta \right], 1 - (1 - \eta_{c_1})^\delta \right\rangle,$$

$$\tilde{C}_2^\delta = \left\langle \left[(\bar{a}_{c_2})^\delta, (a_{c_2}^+)^\delta \right], 1 - (1 - \eta_{c_2})^\delta \right\rangle.$$

Therefore

$$w_1 c_1^\delta \oplus w_2 c_2^\delta$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{2} (1 - \bar{a}_{c_j}^\delta)^{w_j} \right), \left(1 - \prod_{j=1}^{2} (1 - a_{c_j}^{+\delta})^{w_j} \right) \right], \left(\prod_{j=1}^{2} (1 - (1 - \eta_{c_j})^\delta)^{w_j} \right) \right\rangle$$

2. If Eq. 11 holds for $n = k$, then

$$w_1 c_1^\delta \oplus w_2 c_2^\delta \oplus \ldots \oplus w_k c_k^\delta$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{k} (1 - \bar{a}_{c_j}^\delta)^{w_j} \right), \left(1 - \prod_{j=1}^{k} (1 - a_{c_j}^{+\delta})^{w_j} \right) \right], \left(\prod_{j=1}^{k} (1 - (1 - \eta_{c_j})^\delta)^{w_j} \right) \right\rangle.$$

when $n = k + 1$, by the operational laws 1, 2 and 4 such that,

$$w_1 c_1^\delta \oplus w_2 c_2^\delta \oplus \ldots \oplus w_{k+1} c_{k+1}^\delta$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{k} (1 - \bar{a}_{c_j}^\delta)^{w_j} \right), \left(1 - \prod_{j=1}^{k} (1 - a_{c_j}^{+\delta})^{w_j} \right) \right], \left(\prod_{j=1}^{k} (1 - (1 - \eta_{c_j})^\delta)^{w_j} \right) \right\rangle$$

$$\oplus \left\langle \left[\left(1 - (1 - \bar{a}_{c_{k+1}}^\delta)^{w_{k+1}} \right), \left(1 - (1 - a_{c_{k+1}}^{+\delta})^{w_{k+1}} \right) \right], (1 - (1 - \eta_{c_{k+1}})^\delta)^{w_{k+1}} \right\rangle$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{k+1} (1 - \bar{a}_{c_j}^\delta)^{w_j} \right), \left(1 - \prod_{j=1}^{k+1} (1 - a_{c_j}^{+\delta})^{w_j} \right) \right], \left(\prod_{j=1}^{k+1} (1 - (1 - \eta_{c_j})^\delta)^{w_j} \right) \right\rangle$$

i.e. Eq. 11 holds for $n = k + 1$. Thus, Eq. 11 holds for all n such that,

Example 3.5 Let

$$\tilde{C}_1 = \langle [0.3,0.4], 0.5 \rangle, \tilde{C}_2 = \langle [0.2,0.5], 0.3 \rangle, \tilde{C}_3 = \langle [0.4,0.6], 0.3 \rangle$$

be any three cubic numbers, and $w = (0.2, 0.3, 0.5)^T$ be weighting vector of $\tilde{C}_j (j = 1,2,3)$, and $\delta = 2$. Then we have calculated the *GCWA* by applying E.q 10 such that,

$$GCWA_w(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3) = \langle [0.3342, 0.5260], 0.3297 \rangle.$$

On the basis of Theorem 2, we have the following properties of the *GCWA* operators.

Theorem 3.6 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1,2,...,n$) be a collection of cubic value set where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector associated with the *GCWA* operator, with $w_j \geq 0$, $j = 1,2,...,n$, and $\sum_{j=1}^{n} w_j = 1$. If all $C_j (j = 1,2,...,n)$ are equal such that $\tilde{C}_j = \tilde{C}$, for all j, then $GCWA_w(\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n) = \tilde{C}$.

Proof By Theorem 2, we have

$$GCWA_w(\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n) = (w_1 C_1^\delta \oplus w_2 C_2^\delta \oplus ... \oplus w_n C_n^\delta)^{\frac{1}{\delta}}$$

$$= (w_1 C^\delta \oplus w_2 C^\delta \oplus ... \oplus w_n C^\delta)^{\frac{1}{\delta}}$$

$$= ((w_1 + w_2 + ... + w_n) C^\delta)^{\frac{1}{\delta}}$$

$$= (C^\delta)^{\frac{1}{\delta}} = \tilde{C}.$$

Theorem 3.7 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1,2,...,n$) be a collection of cubic value set where $\delta > 0$, and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector associated with the *GCWA* operator with $w_j \geq 0$, $j = 1,2,...,n$, and $\sum_{j=1}^{n} w_j = 1$,

Let

$$\bar{C} = \left(\min_j (\bar{A}_{c_j}), \max_j (\eta_{c_j}) \right), C^+ = \left(\max_j (\bar{A}_{c_j}), \min_j (\eta_{c_j}) \right).$$

Then,

$$\bar{C} \leq GCWA_w(C_1, C_2, ..., C_n) \leq C^+.$$

Proof Since

$$\min_{j}(\bar{A}_{c_j}) \le (\bar{A}_{c_j}) \le \max_{j}(\bar{A}_{c_j}) \text{ and}$$

$$\min_{j}(\eta_{c_j}) \le \eta_{c_j} \le \max_{j}(\eta_{c_j}), \forall \ j,. \text{ Then,}$$

$$\Rightarrow \min_{j}(\bar{a}_{c_j}) \le \bar{a}_{c_j} \le \max_{j}(\bar{a}_{c_j}) \text{ and}$$

$$\min_{j}(a_{c_j}^+) \le a_{c_j}^+ \le \max_{j}(a_{c_j}^+)$$

$$\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j} \ge \prod_{j=1}^{n}\left(1-(\max_{j}(\bar{a}_{c_j}^{\delta}))\right)^{w_j} = 1-(\max_{j}(\bar{a}_{c_j})^{\delta}$$

$$\text{and } \prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j} \ge \prod_{j=1}^{n}\left(1-(\max_{j}(a_{c_j}^{+\delta}))\right)^{w_j} = 1-(\max_{j}(a_{c_j}^+)^{\delta}$$

$$\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \le \max_{j}(\bar{a}_{c_j}^{\delta}) \text{ and}$$

$$\left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} \le \max_{j}(a_{c_j}^{+\delta})$$

$$= \left[\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}\right]$$

$$\le \max_{j}(\bar{a}_{c_j}^{\delta}, a_{c_j}^{+\delta}) \qquad (13)$$

Similarly

$$\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \ge \min_{j}(\bar{a}_{c_j}^{\delta})$$

$$\text{and} \left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} \ge \max_{j}(a_{c_j}^{+\delta})$$

$$= \left[\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}\right]$$

$$\ge \max_{j}(\bar{a}_{c_j}^{\delta}, a_{c_j}^{+\delta}) \qquad (14)$$

$$\prod_{j=1}^{n}\left(1-(1-\eta_{c_j})^{\delta}\right)^{w_j} \le \prod_{j=1}^{n}\left(1-(1-\max_{j}(\eta_{c_j}))^{\delta}\right)^{w_j}$$

$$= 1-\left(1-\max_{j}(\eta_{c_j})\right)^{\delta}$$

$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \ge \left(1-\max_{j}(\eta_{c_j})\right)^{\delta}$$

$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \ge 1-\max_{j}(\eta_{c_j})$$

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \le \max_{j}(\eta_{c_j}) \qquad (15)$$

Similarly

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \ge \min_{j}(\eta_{c_j}). \qquad (16)$$

Let $GCWAw \ (c_1,c_2,,...,c_n)=\tilde{C}=\left\langle \bar{A}_{c_j},\eta_{c_j}\right\rangle$, where $\left\langle \bar{A}_c,\eta_c\right\rangle=\left\langle\left[\bar{a_c},a_c^+\right],\eta_c\right\rangle$. Then,

$$S(\tilde{C})=\frac{A_c-\eta_c}{3}=\frac{\bar{a}+2a^+-\eta}{3} \le \max_{j}(A_{c_j})-\min_{j}(\eta_{c_j})=S(C^+)$$

$$S(\tilde{C})=\frac{A_c-\eta_c}{3} \ge \min_{j}(A_{c_j})-\max_{j}(\eta_{c_j})=S(\bar{C}).$$

If $S(\tilde{C}) \le S(\tilde{C}^+)$ and $S(\tilde{C}) \ge S(\bar{C})$, then by Definition 5, we have

$$\bar{C} < GCWA_w(C_1,C_2,...,C_n) < C^+. \qquad (17)$$

If

$$S(\tilde{C})=S(\tilde{C}^+), i.e. A_c-\eta_c = \max_{j}(A_{c_j})-\min_{j}(\eta_{c_j}).$$

Then, by Eq. 13 and Eq. 16 such that,

$$A_c = \max_{j}(A_{c_j}), \ \eta_c = \min_{j}(\eta_{c_j}).$$

Hence,

$$h(\tilde{C})=\frac{1+A_c-\eta_c}{4}=\frac{1+\bar{a}+2a^+-\eta}{4}$$

$$= \max_{j}(A_{c_j})+\min_{j}(\eta_{c_j})=h(\tilde{C}^+).$$

Then, by Definition 5, we have

$$GCWA_w(\tilde{C}_1,\tilde{C}_2,...,\tilde{C}_n)=\tilde{C}^+. \qquad (18)$$

If $S(\tilde{C}) = S(\bar{\tilde{C}})$ such that,

$$A_c - \eta_c = \min_j(A_{c_j}) - \max_j(\eta_{c_j}).$$

Then by Eq. 14 and Eq. 15 we have

$$A_c = \min_j(A_{c_j}), \ \eta_c = \max_j(\eta_{c_j}).$$

Therefor,

$$h(\tilde{C}) = \frac{1 + A_c - \eta_c}{4} = \frac{1 + \bar{a} + 2a^+ - \eta}{4}$$

$$h(\tilde{C}) = \min_j(A_{c_j}) + \max_j(\eta_{c_j}) = h(\bar{C}).$$

Thus, from Definition 5, we have

$$GCWA_w(\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n) = \bar{C} \qquad (19)$$

From Eqs. $17-19$, Eq. 12, always hold.

Theorem 3.8 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) and $\tilde{C}_j^* = \left\langle \bar{A}_{c_j^*}, \eta_{c_j^*} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of any two cubic value set and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector related to the $GCWA$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$, and $\delta > 0$. If $\bar{A}_{c_j} \leq \bar{A}_{c_j^*}$ and $\eta_{c_j} \geq \eta_{c_j^*}$, for all j, such that,

$$GCWA_w(c_1, c_2, ..., c_n) \leq GCWA_w(c_1^*, c_2^*, ..., c_n^*). \qquad (20)$$

Proof As we know that

$$\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle = \left\langle \left[\bar{a}_{c_j}, a^+_{c_j} \right], \eta_{c_j} \right\rangle \text{ and}$$

$$\tilde{C}_j^* = \left\langle \bar{A}_{c_j^*}, \eta_{c_j^*} \right\rangle = \left\langle \left[\bar{a}_{c_j^*}, a^+_{c_j^*} \right], \eta_{c_j^*} \right\rangle$$

Therefor $\bar{A}_{c_j} \leq \bar{A}_{c_j^*}$ and $\eta_{c_j} \geq \eta_{c_j^*}$, for all j, such that,

$$\left[\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}, \prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j} \right]$$

$$\geq \left[\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\delta})^{w_j}, \prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{w_j} \right]$$

$$\left[1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}, 1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j} \right]$$

$$\leq \left[1-\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\delta})^{w_j}, 1-\prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{w_j} \right]$$

$$\left[1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}, 1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j} \right]^{\frac{1}{\delta}}$$

$$\leq \left[1-\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\delta})^{w_j}, 1-\prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{w_j} \right]^{\frac{1}{\delta}}$$

and

$$\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j} \geq \prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}$$

$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right) \leq \left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)$$

$$\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \leq \left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}$$

$$1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}} \geq 1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}$$

Hence

$$\frac{\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1-\prod_{j=1}^{n}(1-a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}$$

$$\leq \frac{\left(1-\prod_{j=1}^{n}(1-\bar{a}_{c_j^*}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1-\prod_{j=1}^{n}(1-a_{c_j^*}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1-\left(1-\prod_{j=1}^{n}(1-(1-\eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3} \qquad (21)$$

Let

$$\tilde{C} = GCWA_w(c_1, c_2, ,..., c_n), \tilde{C}^* = GCWA_w(c_1^*, c_2^*, ,...., c_n^*).$$

Then by Eq. 21, we have $S(\tilde{C}) \le S(\tilde{C}^*)$. If $S(\tilde{C}) < S(\tilde{C}^*)$, then by Definition 5, we have

$$GCWA_w(\tilde{c}_1, \tilde{c}_2, ,..., \tilde{c}_n), < GCWA_w(\tilde{c}_1^*, \tilde{c}_2^*, ,..., \tilde{c}_n^*) \quad (22) \text{ If } S(\tilde{C}) = S(\tilde{C}^*), \text{ such that,}$$

$$= \frac{\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1 - \prod_{j=1}^{n}(1 - a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}$$

$$= \frac{\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j^*}^{\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1 - \prod_{j=1}^{n}(1 - a_{c_j^*}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{3}$$

Since $\bar{A}_{c_j} \le \bar{A}_{c_j^*}$ and $\eta_{c_j} \ge \eta_{c_j^*}$, for all j, such that,

$$\left[\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^{n}(1 - a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}\right]$$

$$= \left[\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j^*}^{\delta})^{w_j}\right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^{n}(1 - a_{c_j^*}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}}\right]$$

and

$$1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}$$

$$= 1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j^*})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}.$$

Hence

$$h(C) = \frac{1 + \left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j}^{-\delta})^{w_j}\right)^{\frac{1}{\delta}} + \left(1 - \prod_{j=1}^{n}(1 - a_{c_j}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} - \left(1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j})^{\delta})^{w_j}\right)^{\frac{1}{\delta}}\right)}{4}$$

$$h(C^*) = \frac{1 + \left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_j^*}^{-\delta})^{wj}\right)^{\frac{1}{\delta}} + \left(1 - \prod_{j=1}^{n}(1 - a_{c_j^*}^{+\delta})^{wj}\right)^{\frac{1}{\delta}} - \left(1 - \left(1 - \prod_{j=1}^{n}(1 - (1 - \eta_{c_j^*})^{\delta})^{wj}\right)^{\frac{1}{\delta}}\right)}{4}$$

By Definition 5, such that,

$$GCWA_w(\tilde{c}_1, \tilde{c}_2,,..., \tilde{c}_n) = GCWA_w(\tilde{c}_1^*, \tilde{c}_2^*,,..., \tilde{c}_n^*). \quad (23)$$

From Eq. 22 and Eq. 23, we know that Eq. 20 always holds.

Now we have some special cases which obtained by using choices of the parameters w and δ.

Theorem 3.9 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2,..., n$) be a collection of cubic value set $\delta > 0$, and $w = (w_1, w_2,..., w_n)^T$ be the weighting vector related to the $GCWA$ operator, with $w_j \geq 0$ ($j = 1, 2,..., n$), and $\sum_{j=1}^{n} w_j = 1$.

1. If $\delta = 1$, then the $GCWA$ operator (9) is reduced to the following:

$$CWA_w(\tilde{c}_1, \tilde{c}_2,,..., \tilde{c}_n) = w_1 c_1 \oplus w_2 c_2 \oplus ... \oplus w_n c_n,$$

which is called cubic weighted average operator.

2. $\delta \to 0$, then the $GCWA$ operator (9) is reduced to the following:

$$CWG_w(\tilde{c}_1, \tilde{c}_2,,..., \tilde{c}_n) = c_1^{w_1} \otimes c_2^{w_2} \otimes ... \otimes c_n^{w_n},$$

which is called cubic weighted geometric operator.

3. $\delta \to +\infty$, then the $GCWA$ operator (9) is reduced to the following:

$$CMAX_w(\tilde{c}_1, \tilde{c}_2,,..., \tilde{c}_n) = \max_j(\tilde{C}_j),$$

which is called cubic maximum operator.

4. If $w = (\frac{1}{n}, \frac{1}{n},..., \frac{1}{n})^T$ and $\delta = 1$, then the $GCWA$ operator (9) is reduced to the following:

$$CA_w(\tilde{c}_1, \tilde{c}_2,,..., \tilde{c}_n) = \frac{1}{n}(c_1 \oplus c_2 \oplus ... \oplus c_n),$$

which is called cubic average operator.

5. If $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ and $\delta \to 0,$ then the $GCWA$ operator (9) is reduced to the following:

$$CG_w(\tilde{c}_1, \tilde{c}_2, ,...., \tilde{c}_n) = (c_1 \otimes c_2 \otimes ... \otimes c_n),$$

which is called cubic geometric operator.

The GCOWA Operator

In this section we shall define $GCOWA$ operator and study different results relevant to $GCOWA$ operator.

Definition 3.10 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of cubic value set and $GCOWA : C^n \to C,$ if

$$GCOWA_w(\tilde{c}_1, \tilde{c}_2, ,...., \tilde{c}_n) = (w_1 c_{\sigma_{(1)}}^\delta \oplus w_2 c_{\sigma_{(2)}}^\delta \oplus ... \oplus w_n c_{\sigma_{(n)}}^\delta)^{\frac{1}{\delta}}, \qquad (24)$$

where $\delta > 0,$ and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector such that $w_j \geq 0,$ $j = 1, 2, ..., n,$ and $\sum_{j=1}^n w_j = 1,$ $\tilde{C}$ is the jth largest of $\tilde{C}_j,$ then the function $GCOWA$ is called a $GCOWA$ operator.

The $GCOWA$ operator has some properties similar to those of the $GCWA$ operator.

Theorem 3.11 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of cubic value set then their aggregated value by using the $GCOWA$ operator is also a cubic value such that,

$$GCOWA_w (\tilde{c}_1, \tilde{c}_2, ,...., \tilde{c}_n)$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^n (1 - \bar{a}_{c_{\sigma(j)}}^\delta)^{w_j} \right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^n (1 - \overset{+}{a}_{c_{\sigma(j)}}^\delta)^{w_j} \right)^{\frac{1}{\delta}} \right], \\ 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \eta_{c_{\sigma(j)}})^\delta)^{w_j} \right)^{\frac{1}{\delta}} \right\rangle. \quad (25)$$

where $\delta > 0,$ and $w = (w_1, w_2, ..., w_n)^T$ be weighting vector associated with the $GCOWA$ operator, with $w_j \geq 0,$ $j = 1, 2, ..., n$ and $\sum_{j=1}^n w_j = 1,$ $\tilde{c}_{\sigma(j)}$ is the jth largest of $C_j.$

Example 3.12 Let

$$\tilde{C}_1 = \left\langle [0.2, 0.4], 0.3 \right\rangle, \tilde{C}_2 = \left\langle [0.3, 0.4], 0.5 \right\rangle, \tilde{C}_3 = \left\langle [0.3, 0.6], 0.2 \right\rangle,$$
$$\tilde{C}_4 = \left\langle [0.4, 0.6], 0.6 \right\rangle, \text{ and } \tilde{C}_5 = \left\langle [0.6, 0.8], 0.4 \right\rangle,$$

be any five cubic numbers and $w = (0.2, 0.3, 0.12, 0.16, 0.22)^T$ be the weighting vector of $\tilde{C}j$ ($j = 1, 2, 3, 4, 5$). Let $\delta = 2.$ We calculate the scores of $\tilde{C}_j$ ($j = 1, 2, 3, 4, 5$).

$$S(\widetilde{C}_1) = 0.2333, \ S(\widetilde{C}_2) = 0.20, \ S(\widetilde{C}_3) = 0.4333,$$
$$S(\widetilde{C}_4) = 0.3333, \ \text{and} \ S(\widetilde{C}_5) = 0.60$$

Since

$$S(\widetilde{C}_5) > S(\widetilde{C}_3) > S(\widetilde{C}_4) > S(\widetilde{C}_1) > S(\widetilde{C}_2).$$

then

$$\widetilde{C}_{\sigma_{(1)}} = \langle [0.6, 0.8], 0.4 \rangle, \ \widetilde{C}_{\sigma_{(2)}} = \langle [0.3, 0.6], 0.2 \rangle,$$
$$\widetilde{C}_{\sigma_{(3)}} = \langle [0.4, 0.6], 0.6 \rangle, \ \widetilde{C}_{\sigma_{(4)}} = \langle [0.2, 0.4], 0.3 \rangle,$$
$$\widetilde{C}_{\sigma_{(5)}} = \langle [0.3, 0.4], 0.5 \rangle.$$

and thus, by Eq. 25, we have

$$GCOWA_w(\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4, \widetilde{C}_5) = \langle [0.3910, 0.6063], 0.3332 \rangle.$$

Theorem 3.13 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of cubic value set and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector related to the $GCOW$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$. If all $\widetilde{C}_j (j = 1, 2, ..., n)$ are equal, i.e. $\widetilde{C}_j = \widetilde{C}$, for all j, then $GCOWA_w(\widetilde{C}_1, \widetilde{C}_2, ,..., \widetilde{C}_n) = \widetilde{C}$.

Theorem 3.14 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of cubic value set and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector related to the $GCOWA$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$.

Let

$$\bar{C} = (\min_j(\bar{A}_{c_j}), \ \max_j(\eta_{c_j})), \ C^+ = (\max_j(\bar{A}_{c_j}), \ \min_j(\eta_{c_j})).$$

Then,

$$\bar{C} \leq GCOWA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) \leq C^+.$$

Theorem 3.15 Let $\widetilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j = 1, 2, ..., n$) and $\widetilde{C}_j^* = \left\langle \bar{A}_{c_j^*}, \eta_{c_j^*} \right\rangle$ ($j = 1, 2, ..., n$) be a collection of two cubic value set and $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector related to the $GCWA$ operator, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^{n} w_j = 1$, if $\bar{A}_{c_j} \leq \bar{A}_{c_j^*}$ and $\eta_{c_j} \geq \eta_{c_j^*}$, for all j, such that,

$$GCOWA_w(\widetilde{c}_1, \widetilde{c}_2, ,..., \widetilde{c}_n) \leq GCOWA_w(\widetilde{c}_1^*, \widetilde{c}_2^*, ,..., \widetilde{c}_n^*).$$

Theorem 3.16 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j=1,2,...,n$) and $\tilde{C}'_j = \left\langle \bar{A}_{c'_j}, \eta_{c'_j} \right\rangle$ ($j=1,2,...,n$)

be a collection of two cubic value set $,$ $\delta > 0$ and $w = (w_1, w_2,..., w_n)^T$ be the weighting vector related to the *GCOWA* operator, with $w_j \geq 0,$ $j=1,2,...,n,$ and $\sum_{j=1}^{n} w_j = 1$ such that,

$$GCOWA_w(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n) = GCOWA_w(\tilde{c}'_1, \tilde{c}'_2,,...,\tilde{c}'_n). \qquad (26)$$

where $(c'_1, c'_2,,...,c'_n)^T$ is any permutation of $(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n)^T$.

$$GCOWA_w(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n) = \left(w_1 c_{\alpha(1)}^{\delta} \oplus w_2 c_{\alpha(2)}^{\delta} \oplus ... \oplus w_n c_{\alpha(n)}^{\delta} \right)^{\frac{1}{\delta}}$$

$$GCOWA_w(c'_1, c'_2,,...,c'_n) = \left(w_1 (c'_{\alpha(1)})^{\delta} \oplus w_2 (c'_{\alpha(2)})^{\delta} \oplus ... \oplus w_n (c'_{\alpha(n)})^{\delta} \right)^{\frac{1}{\delta}}.$$

Since $(c'_1, c'_2,,...,c'_n)^T$ is any permutation of $(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n)^T$. Then, $c_{\alpha(j)} = c'_{\alpha(j)}, j=1,2,...,n.$ From E.q 26, we now take a look at some the *GCOWA* operator has commutativity property that we desire. It is worth noting that the *GCWA* operator does not have this property. We now take a look at some special cases obtained by using different choices of the parameter w and δ.

Theorem 3.17 Let $\tilde{C}_j = \left\langle \bar{A}_{c_j}, \eta_{c_j} \right\rangle$ ($j=1,2,...,n$) be a collection of cubic value set $,$

$\delta > 0$ and $w = (w_1, w_2,..., w_n)^T$ be the weighted vector related to the *GCOWA* operator, with $w_j \geq 0$ ($j=1,2,...,n$), $\sum_{j=1}^{n} w_j = 1,$ then

1. If $\delta = 1,$ then the *GCOWA* operator (24) is reduced to the following:

$$COWA_w(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n) = w_1 c_{\sigma(1)} \oplus w_2 c_{\sigma(2))} \oplus ... \oplus w_n c_{\sigma(n)},$$

which is called cubic ordered weighted average operator.

2. $\delta \to 0,$ then the *GCOWA* operator (24) is reduced to the following:

$$COWG_w(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n) = c_{\sigma(1)}^{w_1} \otimes c_{\sigma(2)}^{w_2} \otimes ... \otimes c_{\sigma(n)}^{w_n},$$

which is called cubic ordered weighted geometric operator.

3. $\delta \to +\infty,$ then the *GCOWA* operator (24) is reduced to the following:

$$CMAX_w(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n) = \max_j(C_j),$$

which is called cubic maximum operator.

4. If $w = (\frac{1}{n}, \frac{1}{n},..., \frac{1}{n})^T$ and $\delta = 1,$ then the *GCOWA* operator (24) is reduced to the following:

$$CA_w(\tilde{c}_1, \tilde{c}_2,,...,\tilde{c}_n) = \frac{1}{n}(c_1 \oplus c_2 \oplus ... \oplus c_n),$$

which is calld cubic averaging operator.

5. If $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ and $\delta \to 0$, then the *GCOWA* operator (24) is reduced to the following:

$$CG_w(\tilde{c}_1, \tilde{c}_2,, ..., \tilde{c}_n) = (c_1 \otimes c_2 \otimes ... \otimes c_n)^{\frac{1}{n}},$$

which is called cubic geometric operator.

The GCHA Operator

Consider that the *GCWA* operator weights only the cubic value set whereas the *GCOWA* operator weights only the ordered positions of the *CVs* instead of the weighting the cubic value set themselves. To overcome this limitation, motivated by the idea of combining the *WA* and *OWA* operators, in what follows, we developed a generalized cubic hybrid aggregation (*GCHA*) operator, which weights both the given cubic value and its ordered position.

Definition 3.18 *GCHA* operator of dimension n is a mapping $GCHA : C^n \to C$, which has an associated vector $w = (w_1, w_2, ..., w_n)^T$, with $w_j \geq 0$, $j = 1, 2, ..., n$, and $\sum_{j=1}^n w_j = 1$, such that,

$$GCHA_{w,w}(\tilde{c}_1, \tilde{c}_2,, ..., \tilde{c}_n) = \left(w_1 (c_{\sigma(1)}^{\cdot})^\delta \oplus w_2 (c_{\sigma(2)}^{\cdot})^\delta \oplus ... \oplus w_n (c_{\sigma(n)}^{\cdot})^\delta \right)^{\frac{1}{\delta}}. \quad (27)$$

where $\delta > 0$, $c_{\sigma(j)}^{\cdot}$ is the *jth* largest of the weighted *CVs* $c_j^{\cdot} (c_j^{\cdot} = nw_j c_j$, $j = (1, 2, ..., n)$, and $w = (w_1, w_2, ..., w_n)^T$ is the weighting vector of $c_{\cdot j}$ ($j = 1, 2, ..., n$) with $w_j \geq 0$, and $\sum_{j=1}^n w_j = 1$, and n is balancing coefficient, which plays a role of balance if the vector $w = (w_1, w_2, ..., w_n)^T$ approaches $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$. Then, the vector $(nw_1 c_1, nw_2 c_2, ..., nw_n c_n)^T$ approaches $(c_1, c_2, ..., c_n)^T$. Let $C_{\alpha(j)} = \left\langle \bar{A}_{c_{\sigma(j)}^{\cdot}}, \eta_{c_{\sigma(j)}^{\cdot}} \right\rangle$, then, similar to Theorem 3, such that,

$$GCHA_{w,w}(c_1, c_2, ..., c_n)$$

$$= \left\langle \begin{array}{c} \left[\left(1 - \prod_{j=1}^n (1 - \bar{a}_{c_{\sigma(j)}^{\cdot}}^\delta)^{w_j}\right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^n (1 - a_{c_{\sigma(j)}^{\cdot}}^{+\delta})^{w_j}\right)^{\frac{1}{\delta}} \right], \\ 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \eta_{c_{\sigma(j)}^{\cdot}})^\delta)^{w_j}\right)^{\frac{1}{\delta}} \end{array} \right\rangle \quad (28)$$

and the aggregated value derived by using the *GCHA* operator is also *CVs* . Especially, if $\delta = 1$, then (28) is reduced to the following form:

$$CHA_{w,w}(c_1, c_2, ..., c_n)$$

$$= \left\langle \left[\left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_{\alpha(j)}}^{-\delta})^{w_j}\right), \left(1 - \prod_{j=1}^{n}(1 - \bar{a}_{c_{\alpha(j)}}^{+\delta})^{w_j}\right) \right], \left(\prod_{j=1}^{n}(\eta_{c_{\alpha(j)}})^{w_j}\right) \right\rangle,$$

which is called cubic hybrid averaging (*CHA*) operator.

Theorem 3.19 The *GCOWA* operator is a special case of the *GCHA* operator.

Proof Let $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then $\dot{C}_j = C_j (j = 1, 2, ..., n)$, so we have

$$GCHA_{w,w}(c_1, c_2, ..., c_n) = \left(w_1(\dot{c}_{\sigma(1)})^{\delta} \oplus w_2(\dot{c}_{\sigma(2)})^{\delta} \oplus ... \oplus w_n(\dot{c}_{\sigma(n)})^{\delta}\right)^{\frac{1}{\delta}}$$

$$= (w_1(c_{\sigma(1)}^{\delta}) \oplus w_2(c_{\sigma(2)}^{\delta}) \oplus ... \oplus w_n(c_{\sigma(n)}^{\delta}))^{\frac{1}{\delta}}$$

$$= GCOWA_w(c_1, c_2, ..., c_n).$$

which completes the proof of Theorem.

Example 3.20 Let $\tilde{C}_1 = \langle [0.2, 0.3], 0.5 \rangle$, $\tilde{C}_2 = \langle [0.4, 0.6], 0.2 \rangle$, $\tilde{C}_3 = \langle [0.5, 0.7], 0.3 \rangle$, and $\tilde{C}_4 = \langle [0.6, 0.7], 0.1 \rangle$, be any four cubic numbers and let $w = (0.1, 0.3, 0.2, 0.4)^T$ be the weighting vector of $\tilde{C}_j$ $(j = 1, 2, 3, 4)$, and $\delta = 2$, then by applying operational law 3, and Definition 4 we get

$$\tilde{C}_1^{\cdot} = \langle [0.0853, 0.1329], 0.7578 \rangle, \quad \tilde{C}_2^{\cdot} = \langle [0.4582, 0.6669], 0.1449 \rangle,$$
$$\tilde{C}_3^{\cdot} = \langle [0.4256, 0.6183], 0.3816 \rangle, \quad \tilde{C}_4^{\cdot} = \langle [0.7691, 0.8543], 0.0251 \rangle.$$

By using Eq. 5, we calculate the scores of $\tilde{C}_j (j = 1, 2, 3, 4)$

$$S(\tilde{C}_1^{\cdot}) = -0.1355, S(\tilde{C}_2^{\cdot}) = 0.5490, S(\tilde{C}_3^{\cdot}) = 0.4268, S(\tilde{C}_4^{\cdot}) = 0.8275,$$

$$S(\tilde{C}_4^{\cdot}) > S(\tilde{C}_2^{\cdot}) > S(\tilde{C}_3^{\cdot}) > S(\tilde{C}_1^{\cdot}).$$

Then,

$$\tilde{C}_{\sigma(1)}^{\cdot} = \langle [0.7691, 0.8543], 0.0251 \rangle, \quad \tilde{C}_{\sigma(2)}^{\cdot} = \langle [0.4582, 0.6669], 0.1449 \rangle,$$
$$\tilde{C}_{\sigma(3)}^{\cdot} = \langle [0.4256, 0.6183], 0.3816 \rangle, \quad \tilde{C}_{\sigma(4)}^{\cdot} = \langle [0.0853, 0.1329], 0.7578 \rangle.$$

Now we find the weighting vector of *GCHA* operator by means of the normal distribution based method such that, $w = (0.1550, 0.3450, 0.3450, 0.1550)^T$. Then, by Eq. 28 it follows that,

$$GCHA_{w,w}(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4) = \langle [0.5020, 0.6612], 0.1842 \rangle.$$

4. Applications in Decision Making Problem

In this section, we provide an application of proposed score function, accuracy function and aggregation operators. We develop a general algorithm frame work of proposed aggregation operators and their application. In decision support system (DSS) the group decision making problem under consideration is explained as follows;

Algorithm 1. Let $X = \{A_1, A_2, ..., A_n\}$ be the set of n alternatives, and $C = \{C_1, C_2, ..., C_m\}$ be set of criteria of the each alternative with weighting vector of m criteria $w = (w, w, ...w_m)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^m w_j = 1$. Let $D_{ij} = \left\langle \bar{A}_{ij}, \eta_{ij} \right\rangle$ be cubic matrices, where $\left\langle A_{ij}, \eta_{ij} \right\rangle$ is an evaluation in term of cubic sets provided by decision maker related to the alternative $A_i \in A$ based on the criterion $C_j \in C$. The main goal of decision maker is finding the best alternative or ranking the alternative given information. In decision making process it depends on the weights of criteria of the alternatives. In this method we proposed an algorithm to rank the alternative or find out the best one of alternatives. our method is based on more knowledge about the criteria of each alternative. Then decision making method consists of the following steps.

Step 1. The decision makers give their opinions related to each alternative with respect to each criterion. The evaluation of each alternative with respect to each given criterion is listed in decision matrices .

<p align="center">Cubic Decision Matrix</p>

$$(D_{ij})_{nm} = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_m \\ \hline A_1 & \left\langle \bar{A}_{11}, \eta_{11} \right\rangle & \left\langle \bar{A}_{12}, \eta_{12} \right\rangle & \cdots & \left\langle \bar{A}_{1m}, \eta_{1m} \right\rangle \\ A_2 & \left\langle \bar{A}_{21}, \eta_{21} \right\rangle & \left\langle \bar{A}_{22}, \eta_{22} \right\rangle & \cdots & \left\langle \bar{A}_{2m}, \eta_{2m} \right\rangle \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_n & \left\langle \bar{A}_{n1}, \eta_{n1} \right\rangle & \left\langle \bar{A}_{n2}, \eta_{n2} \right\rangle & \cdots & \left\langle \bar{A}_{nm}, \eta_{nm} \right\rangle \end{array}$$

Step 2. Applying generalized cubic weighted aggregation $(GCWA)$ operator to cubic decision matrices, the aggregated information of each alternative with respect the criteria.

Step 3. In this step, we calculate the scores to aggregate the value of each alternative. If there is no difference between two or more than two scores then we have must to find out the accuracy degrees of the aggregated values of each alternative.

Step 4. In this step, we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value.

Algorithm 2. Let $X = \{A_1, A_2, ..., A_n\}$ be the set of n alternatives and $C = \{C_1, C_2, ..., C_m\}$ be set of criteria of the each alternative with weighting vector of m criteria

$w = (w, w, \ldots w_m)^T$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$. Let $D_{ij} = \left\langle \bar{A}_{ij}, \eta_{ij} \right\rangle$ be cubic

matrices, where $\left\langle \bar{A}_{ij}, \eta_{ij} \right\rangle$ is an evaluation in term of cubic sets provided by decision

maker related to the alternative $A_i \in A$ based on the criterion $C_j \in C$. The main goal of decision maker is finding the best alternative or ranking the alternative given information. In decision making process it depends on the weights of criteria of the alternative. In this method we proposed an algorithm to rank the alternative or find out the best one of alternatives. Our method is based on more knowledge about the criteria of each alternative. Then decision making method consists of the following steps.

Step 1. The decision makers give their opinions related to each alternative with respect to each criterion. The evaluation of each alternative with respect to each given criterion is listed in decision matrices

<div align="center">Cubic Decision Matrix</div>

$$(D_{ij})_{nm} = \begin{array}{c|cccc} & C_1 & C_2 & \ldots & C_m \\ \hline A_1 & \left\langle \bar{A}_{11}, \eta_{11} \right\rangle & \left\langle \bar{A}_{12}, \eta_{12} \right\rangle & \ldots & \left\langle \bar{A}_{1m}, \eta_{1m} \right\rangle \\ A_2 & \left\langle \bar{A}_{21}, \eta_{21} \right\rangle & \left\langle \bar{A}_{22}, \eta_{22} \right\rangle & \ldots & \left\langle \bar{A}_{2m}, \eta_{2m} \right\rangle \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ A_n & \left\langle \bar{A}_{n1}, \eta_{n1} \right\rangle & \left\langle \bar{A}_{n2}, \eta_{n2} \right\rangle & \ldots & \left\langle \bar{A}_{nm}, \eta_{nm} \right\rangle \end{array}$$

Step 2. In this step, we apply the known weightted vector by using operational law 3 in Definition 4, and score function to order the cubic values in cubic decision matrix.

Step 3. Applying generalized cubic hybrid aggregation (*GCHA*) operator to cubic decision matrix the aggregated information of each alternative with respect the criteria.

Step 4. In this step, we calculate the scores of the aggregated values of each alternative. If there is no difference between two or more than two scores then we have to find out the accuracy degrees of the aggregated values of each alternative.

Step 5. In this step, we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value.

Step 6. End

5. Illustrative Example

In this section, we are going to present an illustrative example of the new approach in a decision-making problem. We analyze a company that operates in Europe and North America that wants to invest some money in a new market. They consider four possible alternatives

- A_1 = Invest in the Asian market.

- A_2 = Invest in the South American market.

- A_3 = Invest in the African market.

- A_4 = Invest in all three markets.

To evaluate these alternatives, the investor has brought together a group of three alternatives. After analyzing the information, this group considers that the key factor is the economic situation of the world economy for the next period. They consider five main possible states of nature that could happen in the future:

Let C_1, C_2, C_3, C_4, C_5 be criteria for these four markets, In the process of choosing one of the market, five factor are considered;

- C_1 = Very bad economic situation.

- C_2 = Bad economic situation.

- C_3 = Regular economic situation.

- C_4 = Good economic situation.

- C_5 = Very good economic situation.

Suppose that the weighting vector of $C_j (j = 1, 2, ..., 5)$ is and $w = (0.2, 0.3, 0.13, 0.17, 0.20,)^T$, $\delta = 2$, the cubic values of the alternatives $A_i (i = 1, 2, 3, 4)$ are represented by the cubic decision matrix $a_{ij} (i = 1, 2, 3, 4,; j = 1, 2, 3, 4, 5)$ listed in Table 1.

Step 1. The decision makers give their opinions in Table 1.

Table 1. Cubic decision matrix

$D_{ij} = $

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.6,0.7],0.9\rangle$	$\langle[0.4,0.5],0.2\rangle$	$\langle[0.4,0.6],0.7\rangle$	$\langle[0.6,0.7],0.3\rangle$	$\langle[0.5,0.6],0.6\rangle$
A_2	$\langle[0.5,0.8],0.7\rangle$	$\langle[0.6,0.7],0.8\rangle$	$\langle[0.4,0.7],0.5\rangle$	$\langle[0.6,0.8],0.2\rangle$	$\langle[0.2,0.4],0.1\rangle$
A_3	$\langle[0.3,0.4],0.5\rangle$	$\langle[0.6,0.8],0.3\rangle$	$\langle[0.7,0.8],0.9\rangle$	$\langle[0.4,0.6],0.5\rangle$	$\langle[0.3,0.7],0.4\rangle$
A_4	$\langle[0.7,0.9],0.6\rangle$	$\langle[0.4,0.6],0.2\rangle$	$\langle[0.5,0.6],0.8\rangle$	$\langle[0.4,0.6],0.3\rangle$	$\langle[0.4,0.8],0.6\rangle$

Step 2. Now we normalized the decision making matrices by using normalized procedure.

Table 2. Normalized cubic decision matrix $R_{ij} = $

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.3,0.4],0.1\rangle$	$\langle[0.5,0.6],0.8\rangle$	$\langle[0.4,0.6],0.3\rangle$	$\langle[0.3,0.4],0.7\rangle$	$\langle[0.4,0.5],0.4\rangle$
A_2	$\langle[0.2,0.5],0.3\rangle$	$\langle[0.3,0.4],0.2\rangle$	$\langle[0.3,0.6],0.5\rangle$	$\langle[0.2,0.4],0.8\rangle$	$\langle[0.6,0.8],0.9\rangle$
A_3	$\langle[0.6,0.7],0.5\rangle$	$\langle[0.2,0.4],0.7\rangle$	$\langle[0.2,0.3],0.1\rangle$	$\langle[0.4,0.6],0.5\rangle$	$\langle[0.3,0.7],0.6\rangle$
A_4	$\langle[0.1,0.3],0.4\rangle$	$\langle[0.4,0.6],0.8\rangle$	$\langle[0.4,0.5],0.2\rangle$	$\langle[0.4,0.6],0.7\rangle$	$\langle[0.2,0.6],0.4\rangle$

Step 2. Now using generalized cubic weighted aggregation operator by using Eq. 10, we have the aggregated values of the normalized cubic decision matrix is given in Table 3.

Table 3. Aggregated values

A_1	$\langle[0.4511, 0.5641], 0.2361\rangle$
A_2	$\langle[0.4608, 0.6398], 0.2938\rangle$
A_3	$\langle[0.4919, 0.6965], 0.3227\rangle$
A_4	$\langle[0.3637, 0.6127], 0.2983\rangle$

Step 3. In this step, we calculate the scores to aggregate the value of each alternative. If there is no difference between two or more than two scores then we have to find out the accuracy degrees of the aggregated values of each alternative.

$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$
0.4477	0.4822	0.5207	0.4302

Step 4. In this step we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value. Here $A_3 > A_2 > A_1 > A_4$. Thus most wanted alternative is A_3.

Figure 1
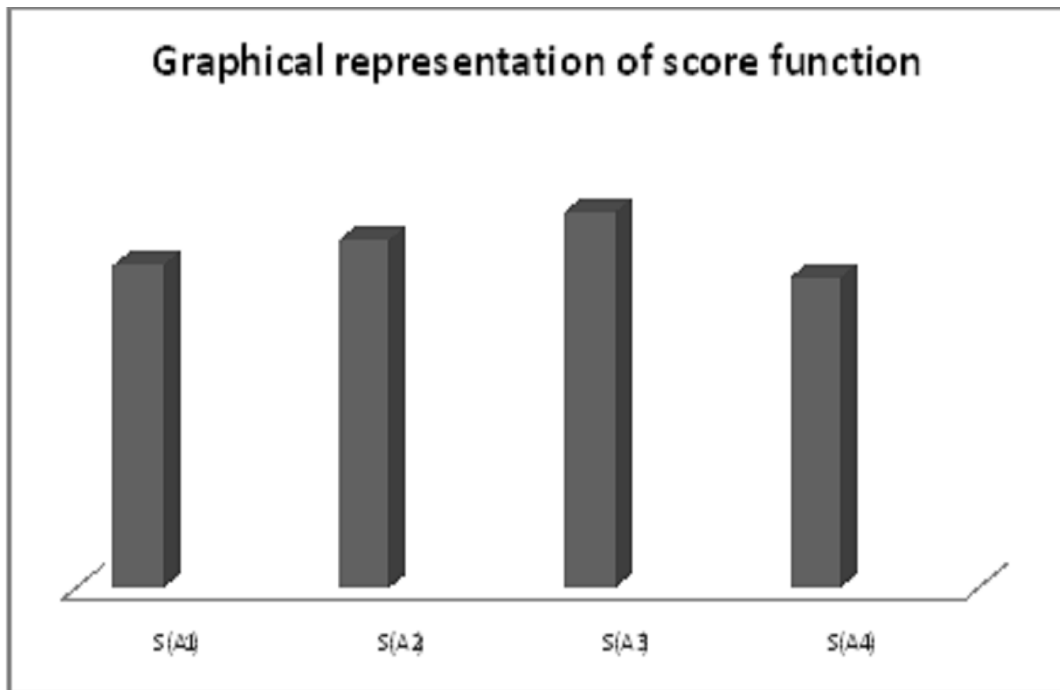
a. **Illustrative Example**

A computer center in a university desires to select an information system to improve the product, for this purpose suppose A_1, A_2, A_3, A_4 are four alternatives $A_i(i = 1, 2, 3, 4)$ have remained the list of candidate. There are four experts from a committee to act the decision makers having weighting vector $\lambda = (0.3, 0.2, 0.4, 0.1)^T$. Consider there are four attributes C_1, C_2, C_3, C_4 such that $C_j(j = 1, 2, 3, 4)$,

(i) C_1 is cost for software investment,

(ii) C_2 is contribution for organization performance,

(iii) C_3 is effort to transformation current system,

(iv) C_4 is for out sourcing software reliability.

Consider that the weighting vector of $C_j(j=1,2,...,4)$ is $w=(0.1,0.3,0.2,0.4,)^T$, $\delta=2$, and the cubic values of the alternatives $A_i(i=1,2,3,4)$ are represented by the cubic decision matrix $a_{ij}(i=1,2,3,4 \quad ; \quad j=1,2,3,4)$ listed in Table 1. (Cubic decision matrix), to rank the given four projects, we first weight all the (CVs) $a_{ij}(i=1,2,3,4 \quad ;$ $j=1,2,3,4)$ by the weighting vector $w=(0.1,0.3,0.2,0.4)^T$ of the attribute $C_j(j=1,2,...,4)$ and multiply these values by the balancing coefficient $n=4$, and we get (CVs) $4w_j a_{ij}$, listed in Table 2. Then, we utilize the $GCHA$ operator $w=(0.1550,0.3450,0.3450,0.1550)^T$ be the weighting vector derived by the normal distribution based method to get the overall values.

Step 1. The decision makers give their opinions in table 1.

Table 1. Cubic Decision Matrix

		C_1	C_2	C_3	C_4
$D_{ij}=$	A_1	$\langle[0.2,0.3],0.5\rangle$	$\langle[0.4,0.6],0.2\rangle$	$\langle[0.5,0.7],0.3\rangle$	$\langle[0.6,0.7],0.1\rangle$
	A_2	$\langle[0.1,0.2],0.3\rangle$	$\langle[0.3,0.5],0.4\rangle$	$\langle[0.6,0.8],0.6\rangle$	$\langle[0.3,0.5],0.3\rangle$
	A_3	$\langle[0.4,0.5],0.9\rangle$	$\langle[0.8,0.9],0.3\rangle$	$\langle[0.5,0.6],0.3\rangle$	$\langle[0.5,0.7],0.2\rangle$
	A_4	$\langle[0.3,0.8],0.2\rangle$	$\langle[0.6,0.7],0.5\rangle$	$\langle[0.6,0.8],0.2\rangle$	$\langle[0.3,0.4],0.3\rangle$

Step 2. Using known weighting vector by applying Definition 4 and operational law 3 in Table 2.

Table 2. Order cubic decision matrix $R_{ij}=$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.76,0.85],0.02\rangle$	$\langle[0.45,0.66],0.14\rangle$	$\langle[0.42,0.61],0.38\rangle$	$\langle[0.08,0.13],0.75\rangle$
A_2	$\langle[0.66,0.85],0.54\rangle$	$\langle[0.34,0.56],0.23\rangle$	$\langle[0.34,0.56],0.33\rangle$	$\langle[0.11,0.23],0.23\rangle$
A_3	$\langle[0.72,0.84],0.38\rangle$	$\langle[0.42,0.61],0.27\rangle$	$\langle[0.42,0.51],0.38\rangle$	$\langle[0.33,0.42],0.94\rangle$
A_4	$\langle[0.76,0.92],0.07\rangle$	$\langle[0.76,0.85],0.32\rangle$	$\langle[0.43,0.92],0.07\rangle$	$\langle[0.43,0.55],0.14\rangle$

Step 3. Now using generalized cubic hybrid aggregation $(GCHA)$ operator by using Eq. 28, we have the aggregated values of the cubic decision matrix is given in Table 3.

Table 3. Aggregated values

| A_1	$\langle [0.5020, 0.6612], 0.1842 \rangle$
A_2	$\langle [0.4084, 0.6161], 0.2982 \rangle$
A_3	$\langle [0.4878, 0.6256], 0.3736 \rangle$
A_4	$\langle [0.6516, 0.8774], 0.2812 \rangle$

Step 4. In this step, we calculate the scores of the aggregated values of each alternative. If there is no difference between two or more than two scores then we have to find out the accuracy degrees of the aggregated values of each alternative.

$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$
0.5467	0.4462	0.4551	0.7084

Step 5. In this step, we arrange all the score values of the alternatives in the form of descending order and select the best alternative which has the highest degree of the score value. Here $S(A_4) > S(A_1) > S(A_3) > S(A_2)$. Thus most wanted alternative is (A_4).

Step 6. End,

Figure 2

Graphical Representation of Score Function
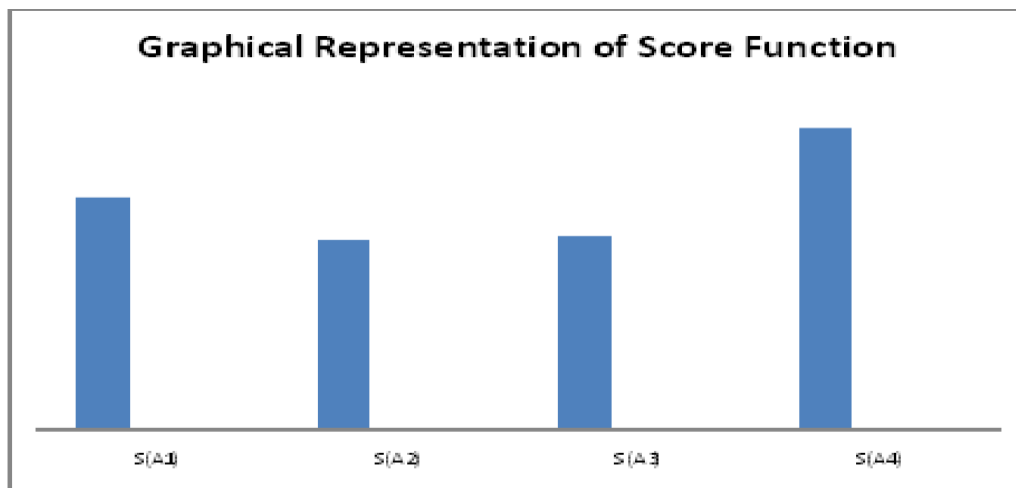
6. Further Discussion

In order to show the validity of the proposed methods, we utilize intuitionistic fuzzy (IFs) sets to solve the same problem described above. We apply the proposed aggregation operators developed in this paper. After simplification we obtained the ranking result as $A_4 > A_1 > A_3 > A_2$, and we find that A_4 is best alternative. In the above example, if we use IFs sets to express the decision maker's evaluations then the decision matrix D_{ij} can be written as decision matrix $D_{ij}^{(1)}$ by applying intuitionistic fuzzy numbers. In [16] the proposed $GIFW$ operators to deal with multiple attribute decision making with intuitionistic fuzzy information respectively;

Table 1. Cubic Decision Matrix

$$D_{ij}^{(1)} =$$

	C_1	C_2	C_3	C_4
A_1	$\langle 0.2, 0.5 \rangle$	$\langle 0.4, 0.2 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.6, 0.1 \rangle$
A_2	$\langle 0.1, 0.3 \rangle$	$\langle 0.3, 0.4 \rangle$	$\langle 0.6, 0.6 \rangle$	$\langle 0.3, 0.3 \rangle$
A_3	$\langle 0.4, 0.9 \rangle$	$\langle 0.8, 0.3 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.5, 0.2 \rangle$
A_4	$\langle 0.3, 0.2 \rangle$	$\langle 0.6, 0.5 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.3, 0.3 \rangle$

We further explain to find the best alternative of IFs, after the computation process of the aggregated values of each alternative $D_{ij}^{(1)}$ as follows. By applying score function of such that,

$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$
0.3178	0.1102	0.1142	0.3704

Now we find the ranking as $A_4 > A_1 > A_3 > A_2$. In this case A_4 is the best alternative.

It is noted that the ranking orders obtained by this paper and by [16] are very different. Therefore, $CFNs$ may better reflect the decision information than $IFNs$, hence our proposed approach is more better than $IFNs$

7. Conclusion

In this paper, we constructed new kinds of aggregation operators, consists of the $GCWA$ operator, the $GCOWA$ operator and the $GCH\!\ell$ operator which extend the $GOWA$ operator. We also discussed some basic properties of these operators, the weighting vector of $GCOWA$ operator and $GCHA$ operator can be determined by the normal distribution based method. At the end of this paper we have developed two numerical example by applying these operators to multiple attribute group decision making $(MAGD)$ problem based on cubic sets. We can extend this to various field.

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