Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 24.02.2025
DOI : 10.15672/hujms.1574594

Research Article

M-fuzzifying convexity-preserving mappings and M-fuzzifying closure-preserving mappings

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Abstract

In this paper, M-fuzzifying convexity-preserving mappings between M-fuzzifying convergence spaces, and M-fuzzifying closure-preserving mappings between M-fuzzifying preconvex closure spaces are proposed. The relationships of M-fuzzifying convexity-preserving mappings with M-CP mappings, M-fuzzifying preconvex closure operators, and separation properties in M-fuzzifying convergence spaces are discussed. Moreover, it is proved that S_0, S_1 and S_2 separation properties are preserved by homeomorphisms in M-fuzzifying convergence spaces.

Mathematics Subject Classification (2020). 54A40, 03E72

Keywords. M-fuzzifying convergence space, M-fuzzifying convexity-preserving mapping, M-fuzzifying preconvex closure space, M-fuzzifying closure-preserving mapping, separation axioms

1. Introduction

Axiomatic convexity theory (also called abstract convexity theory in [19]) plays an important role in mathematics. For different mathematical objects, there are so many collections of sets that can form convex structures, such as convexities in lattices [20], convexities in graphs [17], convexities in real vector spaces [18]. Also, convex structures appeared naturally in topology, especially in the theory of supercompact spaces [8].

With the development of fuzzy mathematics, axiomatic convex structures have been endowed with fuzzy set theory. Adopting different fuzzification methods, different types of fuzzy convex structures have been proposed. Rosa [14] first introduced the concept of fuzzy convexity spaces with the real unit interval $I = [0, 1]$ as the lattice background. Maruyama [7] proposed the notion of L-fuzzy convexity spaces by extending the lattice from I to a completely distributive lattice L. Using the current terminology, these two fuzzy convex structures are called L-convex structures. In the sense of L-convex structures, each L-subset can be considered a fuzzy convex set or not. In a different fuzzification method, Shi and Xiu [15] proposed the notion of M-fuzzifying convex structures with a completely distributive De Morgan algebra M being the underlying lattice, where every classical subset is equipped with some degree to be convex. Later, Shi and Xiu [16, 29]

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Received: 27.10.2024; Accepted: 02.12.2024

introduced (L, M)-fuzzy convex structures which can contain L-convex structures and M-fuzzifying convex structures as special cases. Up to now, these three representative types of fuzzy convex structures have been well developed from different aspects, for L-convex structures refer to [3,10,12,13,28], for M-fuzzifying convex structures refer to [21,22] and for (L, M)-fuzzy convex structures refer to [11,26].

In general topology, filter convergence structures can describe spatial properties of topological spaces, such as separation properties, compactness and connectedness, making it an important tool for interpreting topology. Different fuzzy counterparts of filter convergence structures have been extensively investigated in fuzzy topological spaces. Lowen [6] introduced fuzzy convergence structures by means of prefilters. Based on L-filters [4], Jäger [5] introduced stratified L-fuzzy convergence spaces. In a different direction, Yao [24] introduced L-fuzzifying convergence structures by using L-filters of ordinary subsets. Zhang and Pang [25] proposed lattice-valued convergence groups via $\top$-filters. Gao and Pang [2] studied the categorical relationships between various subcategories of $\top$-convergence spaces. Recently, Zhang and Pang [27] introduced stratified (L, M)-semiuniform convergence spaces and stratified (L, M)-semiuniform limit tower spaces.

In order to propose fuzzy convergence theory in M-fuzzifying convex spaces, Pang [9] introduced fuzzy convex convergence structures in fuzzy convex spaces by means of M-fuzzifying convex filters, and established its relationships with fuzzy convex structures. This motivates us to consider the lattice-valued forms of convexity-preserving mappings between M-fuzzifying convergence spaces, and closure-preserving mappings between M-fuzzifying preconvex closure spaces.

This article is arranged as follows. In Section 2, we review some preliminaries that are needed in the subsequent sections. In Section 3, we introduce M-fuzzifying convexity-preserving mappings in M-fuzzifying convergence spaces and establish its relationships with M-CP mappings in M-fuzzifying convex spaces. In Section 4, we introduce M-fuzzifying closure-preserving mappings in M-fuzzifying preconvex closure spaces, and study the relationships between M-fuzzifying preconvex closure operators and M-fuzzifying convexity-preserving mappings, respectively. In Section 5, we consider the relationships between M-fuzzifying convexity-preserving mappings and separation properties in M-fuzzifying convergence spaces.

2. Preliminaries

Throughout this paper, M denotes a frame (or complete Heyting algebra), which means that M is a complete lattice and $a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i)$ holds for all $a,\ b_i \in M$ $(i \in I)$. The bottom and top element of M are denoted by $\perp$ and $\top$, respectively. We equip an order-reversing involution "$'$" on M, and define a residual implication on M by

$$a \to b = \bigvee\{c \in M \mid a \wedge c \leqslant b\}.$$

We list some properties of the residual implication.

Lemma 2.1 ([4]). *Let M be a frame. The following statements hold:*
(H1) $\top \to a = a$;
(H2) $a \leqslant b$ *if and only if* $a \to b = \top$;
(H3) $(a \to b) \wedge (b \to c) \leqslant a \to c$;
(H4) $(a \to b) \to (c \to b) \geqslant c \to a$;
(H5) $a \to (\bigwedge_{j\in J} a_j) = \bigwedge_{j\in J}(a \to a_j)$;
(H6) $(\bigvee_{j\in J} a_j) \to b = \bigwedge_{j\in J}(a_j \to b)$.

For a nonempty set X, let 2^X denote the powerset of X and M^X denote the set of all M-subsets on X. For all $\{A_j\}_{j\in J} \subseteq 2^X$, we say $\{A_j\}_{j\in J}$ is a directed subset of 2^X if for all $B, C \in \{A_j\}_{j\in J}$, there exists $D \in \{A_j\}_{j\in J}$ such that $B \subseteq D$ and $C \subseteq D$, which is denoted by $\{A_j\}_{j\in J} \overset{dir}{\subseteq} 2^X$.

Definition 2.2 ([15]). A mapping $\mathscr{C} : 2^X \longrightarrow M$ is called an M-fuzzifying convex structure on X if it satisfies the following conditions:

(MYC1) $\mathscr{C}(\emptyset) = \mathscr{C}(X) = \top$;

(MYC2) $\mathscr{C}(\bigcap_{k\in K} A_k) \geqslant \bigwedge_{k\in K} \mathscr{C}(A_k)$;

(MYC3) $\mathscr{C}(\bigcup_{j\in J} A_j) \geqslant \bigwedge_{j\in J} \mathscr{C}(A_j), \forall \{A_j\}_{j\in J} \overset{dir}{\subseteq} 2^X$.

The pair $(X, \mathscr{C})$ is called an M-fuzzifying convex space.

Definition 2.3 ([23]). Let $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ be M-fuzzifying convex spaces, and let $f : X \longrightarrow Y$ be a mapping. Then $Cp(f)$ defined by

$$Cp(f) = \bigwedge_{B\in 2^Y} \Big(\mathscr{C}_Y(B) \to \mathscr{C}_X(f^{-1}(B))\Big)$$

is called the M-CP degree of f.

Definition 2.4 ([15]). An M-fuzzifying hull operator on X is a mapping $h : 2^X \longrightarrow M^X$ which satisfies:

(MH1) $h(\emptyset) = \chi_\emptyset$;

(MH2) $\chi_A \leqslant h(A)$;

(MH3) $A \subseteq B$ implies $h(A) \leqslant h(B)$;

(MH4) $h(A)(x) = \bigwedge_{x\notin B\supseteq A} \bigvee_{y\notin B} h(B)(y)$;

(MDF) $h(A)(x) = \bigvee\{h(F)(x) \mid F \in 2^A_{fin}\}$, where 2^A_{fin} denotes the family of all finite subsets of A.

The pair (X, h) is called an M-fuzzifying hull space.

Remark 2.5 ([9]). It is easy to verify that (MDF) in Definition 2.4 is equivalent to

(MDF)* $h(\bigcup_{j\in J} A_j) = \bigvee_{j\in J} h(A_j), \forall \{A_j\}_{j\in J} \overset{dir}{\subseteq} 2^X$.

Definition 2.6 ([9]). An M-fuzzifying preconvex closure operator on X is a mapping $h : 2^X \longrightarrow M^X$ which satisfies (MH1), (MH2) and (MDF) (or (MDF)*). The pair (X, h) is called an M-fuzzifying preconvex closure space.

Definition 2.7 ([9]). A mapping $\mathcal{F} : 2^X \longrightarrow M$ is called an M-fuzzifying convex filter on X if it satisfies:

(MF1) $\mathcal{F}(\emptyset) = \bot$, $\mathcal{F}(X) = \top$;

(MF2) $\mathcal{F}(\bigcap_{j\in J} A_j) = \bigwedge_{j\in J} \mathcal{F}(A_j)$ for each $\{A_j\}_{j\in J} \overset{cdir}{\subseteq} 2^X$.

The family of all M-fuzzifying convex filters on X is denoted by $\mathcal{F}_M(X)$.

Proposition 2.8 ([9]). *Suppose that $f : X \longrightarrow Y$ is a mapping and $\mathcal{F} \in \mathcal{F}_M(X)$. Define $f^{\Rightarrow}(\mathcal{F}) : 2^Y \longrightarrow M$ by*

$$f^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(f^{-1}(B)), \forall B \in 2^Y.$$

Then $f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}_M(Y)$, which is called the image of $\mathcal{F}$ under f.

Definition 2.9 ([9]). For all $x \in X$, define $[x] : 2^X \longrightarrow M$ by

$$[x](A) = \chi_A(x), \forall A \in 2^X.$$

Then $[x] \in \mathcal{F}_M(X)$, which is called the point M-fuzzifying convex filter of x.

Definition 2.10 ([1]). A fuzzy inclusion order on M^X is a mapping $\mathcal{S} : M^X \times M^X \to M$ which is defined by

$$\mathcal{S}(U, V) = \bigwedge_{x\in X} \Big(U(x) \to V(x)\Big), \forall U,\ V \in M^X.$$

Definition 2.11 ([9])**.** An M-fuzzifying convergence structure on X is a mapping $\lim : \mathcal{F}_M(X) \longrightarrow M^X$ which satisfies:

(MC1) $\lim([x])(x) = \top$;

(MC2) $\mathcal{S}_{\mathcal{F}}(\mathcal{F}, \mathcal{G}) \leqslant \mathcal{S}(\lim(\mathcal{F}), \lim(\mathcal{G}))$, where $\mathcal{S}_{\mathcal{F}}(\mathcal{F}, \mathcal{G})$ denotes the fuzzy inclusion order on $\mathcal{F}_M(X)$.

The pair $(X, \lim)$ is called an M-fuzzifying convergence space.

3. M-fuzzifying convexity-preserving mappings

In this section, we introduce M-fuzzifying convexity-preserving mappings between M-fuzzifying convergence spaces, and establish its relationships with M-CP mappings in M-fuzzifying convex spaces.

Definition 3.1. Let $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ be a mapping between two M-fuzzifying convergence spaces. Then the degree $Con(f)$ to which f is M-fuzzifying convexity-preserving is defined by

$$Con(f) = \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \bigwedge_{x\in X} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))).$$

Remark 3.2. If $Con(f) = \top$, by Lemma 2.1, we know

$$lim_X(\mathcal{F})(x) \leqslant lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)), \forall \mathcal{F} \in \mathcal{F}_M(X), x \in X,$$

which is exactly the definition of M-fuzzifying convexity-preserving mapping in M-fuzzifying convergence spaces [9].

Proposition 3.3. *(1) If $id : (X, lim_X) \longrightarrow (X, lim_X)$ is the identify mapping, then $Con(id) = \top$.*

(2) For all $\alpha \in Y$, let $\overline{\alpha} : (X, lim_X) \longrightarrow (Y, lim_Y)$ be the constant mapping, i.e., $\forall x \in X, \overline{\alpha}(x) = \alpha$. Then $Con(\overline{\alpha}) = \top$.

Proof. (1) The proof is straightforward and omitted.

(2) For all $\mathcal{F} \in \mathcal{F}_M(X)$ and $B \in 2^Y$, we have

$$\overline{\alpha}^{-1}(B) = \{x \mid \overline{\alpha}(x) \in B\} = \begin{cases} X, & \alpha \in B; \\ \emptyset, & \alpha \notin B. \end{cases}$$

This implies

$$\begin{aligned} \overline{\alpha}^{\Rightarrow}(\mathcal{F})(B) &= \mathcal{F}(\overline{\alpha}^{-1}(B)) \\ &= \begin{cases} \mathcal{F}(X), & \alpha \in B, \\ \mathcal{F}(\emptyset), & \alpha \notin B, \end{cases} \\ &= \begin{cases} \top, \alpha \in B, \\ \bot, \alpha \notin B. \end{cases} \end{aligned}$$

Then

$$\overline{\alpha}^{\Rightarrow}(\mathcal{F})(B) = \chi_B(\alpha) = [\alpha](B).$$

This shows $\overline{\alpha}^{\Rightarrow}(\mathcal{F}) = [\alpha]$. Therefore, by Definition 3.1, we can obtain

$$\begin{aligned} Con(\overline{\alpha}) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \bigwedge_{x\in X} (lim_X(\mathcal{F})(x) \to lim_Y(\overline{\alpha}^{\Rightarrow}(\mathcal{F}))(\overline{\alpha}(x))) \\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \bigwedge_{x\in X} (lim_X(\mathcal{F})(x) \to lim_Y([\alpha])(\alpha)) \\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)} \bigwedge_{x\in X} (lim_X(\mathcal{F})(x) \to \top) \end{aligned}$$

$$= \top,$$

as desired. $\square$

Proposition 3.4. *Let $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ and $g : (Y, lim_Y) \longrightarrow (Z, lim_Z)$ be mappings between M-fuzzifying convergence spaces. Then $Con(f) \wedge Con(g) \leqslant Con(g \circ f)$.*

Proof. For all $B \in 2^Z$ and $\mathcal{F} \in \mathcal{F}_M(X)$, we have

$$\begin{aligned} g^{\Rightarrow}(f^{\Rightarrow}(\mathcal{F}))(B) &= f^{\Rightarrow}(\mathcal{F})(g^{-1}(B)) \\ &= \mathcal{F}(f^{-1}(g^{-1}(B))) \\ &= \mathcal{F}((g \circ f)^{-1}(B)) \\ &= (g \circ f)^{\Rightarrow}(\mathcal{F})(B). \end{aligned}$$

This implies $g^{\Rightarrow}(f^{\Rightarrow}(\mathcal{F})) = (g \circ f)^{\Rightarrow}(\mathcal{F})$. Therefore

$$\begin{aligned} & Con(f) \wedge Con(g) \\ = & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\ & \wedge \bigwedge_{\mathcal{G} \in \mathcal{F}_M(Y)} \bigwedge_{y \in Y} (lim_Y(\mathcal{G})(y) \to lim_Z(g^{\Rightarrow}(\mathcal{G}))(g(y))) \\ = & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} \bigwedge_{\mathcal{G} \in \mathcal{F}_M(Y)} \bigwedge_{y \in Y} \Big((lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\ & \wedge (lim_Y(\mathcal{G})(y) \to lim_Z(g^{\Rightarrow}(\mathcal{G}))(g(y))) \Big) \\ \leqslant & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} \Big((lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\ & \wedge (lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to lim_Z(g^{\Rightarrow}(f^{\Rightarrow}\mathcal{F}))(g(f(x)))) \Big) \\ = & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} \Big((lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\ & \wedge (lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to lim_Z((g \circ f)^{\Rightarrow}(\mathcal{F}))((g \circ f)(x))) \Big) \\ \leqslant & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} (lim_X(\mathcal{F})(x) \to lim_Z((g \circ f)^{\Rightarrow}(\mathcal{F}))((g \circ f)(x))) \\ = & Con(g \circ f), \end{aligned}$$

as desired. $\square$

Next, we disscuss the relationships between M-fuzzifying convexity-preserving mappings and M-CP mappings.

Proposition 3.5 ([9])**.** *Let $(X, \mathscr{C})$ be an M-fuzzifying convex space and define $\lim^{\mathscr{C}} : \mathcal{F}_M(X) \longrightarrow M^X$ as follows:*

$$\forall \mathcal{F} \in \mathcal{F}_M(X), \forall x \in X,\ \lim^{\mathscr{C}}(\mathcal{F})(x) = \bigwedge_{x \in A} \Big(\mathscr{C}(X - A) \to \mathcal{F}(A) \Big).$$

Then $\lim^{\mathscr{C}}$ is an M-fuzzifying convergence structure on X.

Proposition 3.6. *Suppose that $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is a mapping between two M-fuzzifying convex spaces, $(X, lim^{\mathscr{C}_X})$ and $(Y, lim^{\mathscr{C}_Y})$ are induced M-fuzzifying convergence spaces by $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$. Then $Cp(f) \leqslant Con^{lim^{\mathscr{C}}}(f)$.*

Proof. By Proposition 3.5, we know for all $\mathfrak{F} \in \mathfrak{F}_M(X)$ and $x \in X$,

$$\begin{aligned}
& lim^{\mathscr{C}_X}(\mathfrak{F})(x) \to lim^{\mathscr{C}_Y}(f^{\Rightarrow}(\mathfrak{F}))(f(x)) \\
= & \Big(\bigwedge_{x \in A} (\mathscr{C}_X(X - A) \to \mathfrak{F}(A)) \Big) \to \Big(\bigwedge_{f(x) \in B} (\mathscr{C}_Y(Y - B) \to f^{\Rightarrow}(\mathfrak{F})(B)) \Big) \\
= & \bigwedge_{x \in f^{-1}(B)} \left(\bigwedge_{x \in A} \Big(\mathscr{C}_X(X - A) \to \mathfrak{F}(A) \Big) \to \Big(\mathscr{C}_Y(Y - B) \to f^{\Rightarrow}(\mathfrak{F})(B) \Big) \right) \\
\geqslant & \bigwedge_{x \in f^{-1}(B)} \Big((\mathscr{C}_X(X - f^{-1}(B)) \to \mathfrak{F}(f^{-1}(B))) \to (\mathscr{C}_Y(Y - B) \to \mathfrak{F}(f^{-1}(B))) \Big) \\
\geqslant & \bigwedge_{x \in f^{-1}(B)} (\mathscr{C}_Y(Y - B) \to \mathscr{C}_X(X - f^{-1}(B))) \\
= & \bigwedge_{x \in f^{-1}(B)} (\mathscr{C}_Y(Y - B) \to \mathscr{C}_X(f^{-1}(Y - B))) \\
\geqslant & \bigwedge_{D \in 2^Y} (\mathscr{C}_Y(D) \to \mathscr{C}_X(f^{-1}(D))) \\
= & Cp(f).
\end{aligned}$$

This implies

$$\begin{aligned}
Cp(f) & \leqslant \bigwedge_{\mathfrak{F} \in \mathfrak{F}_M(X)} \bigwedge_{x \in X} (lim^{\mathscr{C}_X}(\mathfrak{F})(x) \to lim^{\mathscr{C}_Y}(f^{\Rightarrow}(\mathfrak{F}))(f(x))) \\
& = Con^{lim^{\mathscr{C}}}(f),
\end{aligned}$$

as desired. □

Proposition 3.7 ([9])**.** *Let* $(X, \lim)$ *be an* M*-fuzzifying convergence space and define* $\mathscr{C}^{\lim} : 2^X \longrightarrow M$ *as follows:*

$$\forall A \in 2^X, \mathscr{C}^{\lim}(A) = \bigwedge_{x \notin A} \bigwedge_{\mathfrak{F} \in \mathfrak{F}_M(X)} \Big(\lim(\mathfrak{F})(x) \to \mathfrak{F}(X - A) \Big).$$

Then $\mathscr{C}^{\lim}$ *is an* M*-fuzzifying convex structure on* X*.*

Proposition 3.8. *Suppose that* $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ *is a mapping between two* M*-fuzzifying convergence spaces,* $(X, \mathscr{C}^{lim_X})$ *and* $(Y, \mathscr{C}^{lim_Y})$ *are induced* M*-fuzzifying convex spaces by* (X, lim_X) *and* (Y, lim_Y)*. Then* $Con(f) \leqslant Cp^{\mathscr{C}^{lim}}(f)$*.*

Proof. By Proposition 3.7, we can obtain for all $B \in 2^Y$,

$$
\begin{aligned}
& \mathscr{C}^{lim_Y}(B) \to \mathscr{C}^{lim_X}(f^{-1}(B)) \\
= \; & \Big(\bigwedge_{y \notin B} \bigwedge_{\mathcal{G} \in \mathcal{F}_M(Y)} (lim_Y(\mathcal{G})(y) \to \mathcal{G}(Y - B)) \Big) \\
& \to \Big(\bigwedge_{x \notin f^{-1}(B)} \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - f^{-1}(B))) \Big) \\
= \; & \bigwedge_{x \notin f^{-1}(B)} \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \Big((\bigwedge_{y \notin B} \bigwedge_{\mathcal{G} \in \mathcal{F}_M(Y)} (lim_Y(\mathcal{G})(y) \to \mathcal{G}(Y - B))) \\
& \to (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - f^{-1}(B))) \Big) \\
\geqslant \; & \bigwedge_{x \in f^{-1}(B)} \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \Big((lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to f^{\Rightarrow}(\mathcal{F})(Y - B)) \\
& \to (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - f^{-1}(B))) \Big) \\
= \; & \bigwedge_{x \in f^{-1}(B)} \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \Big((lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to \mathcal{F}(f^{-1}(Y - B))) \\
& \to (lim_X(\mathcal{F})(x) \to \mathcal{F}(f^{-1}(Y - B))) \Big) \\
\geqslant \; & \bigwedge_{x \in f^{-1}(B)} \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\
\geqslant \; & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\
= \; & Con(f).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Con(f) \; & \leqslant \; \bigwedge_{B \in 2^Y} (\mathscr{C}^{lim_Y}(B) \to \mathscr{C}^{lim_X}(f^{-1}(B))) \\
& = \; Cp^{\mathscr{C}^{lim}}(f),
\end{aligned}
$$

as desired. □

Definition 3.9. Let $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ be a bijective mapping between two M-fuzzifying convergence spaces. Then the degree $Hom(f)$ to which f is a homeomorphism is defined by $Hom(f) = Con(f) \wedge Con(f^{-1})$.

Proposition 3.10. *Let $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ and $g : (Y, lim_Y) \longrightarrow (Z, lim_Z)$ be bijective mappings between M-fuzzifying convergence spaces. Then $Hom(f) \wedge Hom(g) \leqslant Hom(g \circ f)$.*

Proof. By Proposition 3.4, it is straightforward. □

4. M-fuzzifying closure-preserving mappings in M-fuzzifying preconvex closure spaces

In this section, we introduce M-fuzzifying closure-preserving mappings between M-fuzzifying preconvex closure spaces. Moreover, we establish the relationships between M-fuzzifying preconvex closure operators and M-fuzzifying convexity-preserving mappings, respectively.

Definition 4.1. Let $f:(X,h_X)\longrightarrow(Y,h_Y)$ be a mapping between two M-fuzzifying preconvex closure spaces. Then the degree $Clp(f)$ to which f is M-fuzzifying closure-preserving is defined by

$$Clp(f)=\bigwedge_{A\in 2^X}\bigwedge_{x\in X}(h_X(A)(x)\to h_Y(f(A))(f(x))).$$

Remark 4.2. If $Clp(f)=\top$, by Lemma 2.1, we know that for all $A\in 2^X$ and $x\in X$, $h_X(A)(x)\leqslant h_Y(f(A))(f(x))$, which is exactly the definition of M-fuzzifying closure-preserving mapping in M-fuzzifying preconvex closure spaces in [9].

Example 4.3. Let $X=\{x_1,x_2\}$, $Y=\{y_1,y_2\}$, and $M=[0,1]$. Define $h_X:2^X\longrightarrow M^X$ as follows

$$h_X(\emptyset)(x)=\begin{cases}0, & x=x_1;\\ 0, & x=x_2;\end{cases}\qquad h_X(X)(x)=\begin{cases}1, & x=x_1;\\ 1, & x=x_2;\end{cases}$$

$$h_X(\{x_1\})(x)=\begin{cases}1, & x=x_1;\\ 0.4, & x=x_2;\end{cases}\qquad h_X(\{x_2\})(x)=\begin{cases}0.6, & x=x_1;\\ 1, & x=x_2.\end{cases}$$

Then h_X is an M-fuzzifying preconvex closure operator on X. Define $f:X\longrightarrow Y$ as follows: $f(x_1)=y_1$ and $f(x_2)=y_2$. Define h_Y as follows:

$$h_Y(f(\emptyset))(f(x))=\begin{cases}0, & x=x_1;\\ 0, & x=x_2;\end{cases}\qquad h_Y(f(X))(f(x))=\begin{cases}1, & x=x_1;\\ 1, & x=x_2;\end{cases}$$

$$h_Y(f(\{x_1\}))(f(x))=\begin{cases}1, & x=x_1;\\ 0.5, & x=x_2;\end{cases}\qquad h_Y(f(\{x_2\}))(f(x))=\begin{cases}0.2, & x=x_1;\\ 1, & x=x_2.\end{cases}$$

Then h_Y is an M-fuzzifying preconvex closure operator on Y. Therefore, we have

$$\begin{aligned}&Clp(f)\\ =\;&\bigwedge_{A\in 2^X}\bigwedge_{x\in X}(h_X(A)(x)\to h_Y(f(A))(f(x)))\\ =\;&(0\to 0)\wedge(0\to 0)\wedge(1\to 1)\wedge(1\to 1)\wedge(1\to 1)\wedge(0.4\to 0.5)\\ &\wedge(0.6\to 0.2)\wedge(1\to 1)\\ =\;&0.2.\end{aligned}$$

Proposition 4.4. (1) If $id:(X,h_X)\longrightarrow(X,h_X)$ is the identify mapping, then $Clp(id)=\top$.

(2) For all $\alpha\in Y$, let $\overline{\alpha}:(X,h_X)\longrightarrow(Y,h_Y)$ be the constant mapping. Then $Clp(\overline{\alpha})=\top$.

Proof. (1) The proof is straightforward and omitted.

(2) By the Definition 2.4 (MH2), we can obtain for all $x\in A$, $h(A)(x)=\top$. Then

$$\begin{aligned}Clp(\overline{\alpha}) &= \bigwedge_{A\in L^X}\bigwedge_{x\in X}(h_X(A)(x)\\ &\quad\to h_Y(\overline{\alpha}^{\Rightarrow}(A))(\overline{\alpha}(f(x))))\\ &= \bigwedge_{A\in L^X}\bigwedge_{x\in X}(h_X(A)(x)\to h_Y(\{\alpha\})(\alpha))\\ &= \bigwedge_{A\in L^X}\bigwedge_{x\in X}(h_X(A)(x)\to\top)\\ &= \top,\end{aligned}$$

as desired. ∎

Proposition 4.5. *Let $f:(X,h_X)\longrightarrow(Y,h_Y)$ and $g:(Y,h_Y)\longrightarrow(Z,h_Z)$ be mappings between M-fuzzifying preconvex closure spaces. Then $Clp(f)\wedge Clp(g)\leqslant Clp(g\circ f)$.*

Proof. By Definition 4.1, we have

$$\begin{aligned}
& Clp(f)\wedge Clp(g)\\
= & \bigwedge_{A\in 2^X}\bigwedge_{x\in X}(h_X(A)(x)\to h_Y(f(A))(f(x)))\\
& \quad \wedge \bigwedge_{B\in 2^Y}\bigwedge_{y\in Y}(h_Y(B)(y)\to h_Z(g(B))(g(y)))\\
= & \bigwedge_{A\in 2^X}\bigwedge_{x\in X}\bigwedge_{B\in 2^Y}\bigwedge_{y\in Y}\Big((h_X(A)(x)\to h_Y(f(A))(f(x)))\wedge(h_Y(B)(y)\\
& \to h_Z(g(B))(g(y)))\Big)\\
\leqslant & \bigwedge_{A\in 2^X}\bigwedge_{x\in X}\Big((h_X(A)(x)\to h_Y(f(A))(f(x)))\wedge(h_Y(f(A))(f(x))\\
& \to h_Z(g(f(A)))(g(f(x))))\Big)\\
\leqslant & \bigwedge_{A\in 2^X}\bigwedge_{x\in X}(h_X(A)(x)\to h_Z((g\circ f)^{\to}(A))((g\circ f)(x)))\\
= & Clp(g\circ f),
\end{aligned}$$

as desired. □

In [9], the author introduced M-fuzzifying convex closure operator and M-fuzzifying preconvex convergence space as follows.

Definition 4.6 ([9])**.** For an M-fuzzifying convergence space $(X,\lim)$, define $\mathfrak{c}^{\lim}:2^X\longrightarrow M^X$ as follows:

$$\forall A\in 2^X,\ \forall x\in X,\ \mathfrak{c}^{\lim}(A)(x)=\bigvee_{\mathcal{F}\in\mathcal{F}_M(X)}\Big(\lim(\mathcal{F})(x)\to\mathcal{F}(X-A)\Big)'.$$

Then $\mathfrak{c}^{\lim}$ is called the M-fuzzifying convex closure operator of $(X,\lim)$.

Definition 4.7 ([9])**.** An M-fuzzifying convergence space $(X,\lim)$ is called preconvex if it satisfies
(MCP) $\lim(\mathcal{F})(x)=\bigwedge_{A\in 2^X}(\mathfrak{c}^{\lim}(X-A)(x)'\to\mathcal{F}(A))$.
Further, it will be called convex if it satisfies moreover,
(MCT) $\mathfrak{c}^{\lim}(A)(x)=\bigwedge_{x\notin B\supseteq A}\bigvee_{y\notin B}\mathfrak{c}^{\lim}(B)(y)$.

Proposition 4.8 ([9])**.** *Let $(X,\lim)$ be an M-fuzzifying preconvex convergence space and define $h^{\lim}=\mathfrak{c}^{\lim}$. Then $h^{\lim}$ is an M-fuzzifying preconvex closure operator on X. Moreover, if $(X,\lim)$ is convex, then $h^{\lim}$ is an M-fuzzifying hull operator on X.*

Next, we disscuss the relationships between M-fuzzifying preconvex closure operators and M-fuzzifying convexity-preserving mappings, respectively.

Proposition 4.9. *Suppose that $f:(X,lim_X)\longrightarrow(Y,lim_Y)$ is a mapping between two M-fuzzifying preconvex convergence spaces, (X,h^{lim_X}) and (Y,h^{lim_Y}) are induced M-fuzzifying preconvex closure spaces by (X,lim_X) and (Y,lim_Y). Then*

$$Con(f)\leqslant\bigwedge_{A\in 2^X}\bigwedge_{x\in X}(h^{lim_Y}(f(A))(f(x))'\to h^{lim_X}(A)(x)').$$

Proof. By Proposition 4.6 and 4.8, we have for all $A \in 2^X$ and $x \in X$,

$$\begin{aligned}
& h^{lim_Y}(f(A))(f(x))' \to h^{lim_X}(A)(x)' \\
= & \Big(\bigwedge_{\mathcal{G} \in \mathcal{F}_M(Y)} (lim_Y(\mathcal{G})(f(x)) \to \mathcal{G}(Y - f(A)))\Big) \\
& \to \Big(\bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - A))\Big) \\
\geqslant & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \Big((lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to f^{\Rightarrow}(\mathcal{F})(Y - f(A))) \\
& \to (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - A))\Big) \\
= & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \Big((lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to \mathcal{F}(X - f^{-1}(f(A)))) \\
& \to (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - A))\Big) \\
\geqslant & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \Big((lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \to \mathcal{F}(X - A)) \\
& \to (lim_X(\mathcal{F})(x) \to \mathcal{F}(X - A))\Big) \\
\geqslant & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\
\geqslant & \bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))) \\
= & Con(f).
\end{aligned}$$

This implies

$$Con(f) \leqslant \bigwedge_{A \in 2^X} \bigwedge_{x \in X} (h^{lim_Y}(f(A))(f(x))' \to h^{lim_X}(A)(x)'),$$

as desired. □

Proposition 4.10 ([9]). *Let (X, h) be an M-fuzzifying preconvex closure space. Define $\lim^h : \mathcal{F}_M(X) \longrightarrow M^X$ as follows:*

$$\forall \mathcal{F} \in \mathcal{F}_M(X),\ \forall x \in X,\ \lim^h(\mathcal{F})(x) = \bigwedge_{B \in 2^X} \Big(h(X - B)(x)' \to \mathcal{F}(B)\Big).$$

Then $\lim^h$ is an M-fuzzifying preconvex convergence structure on X. Moreover, if (X, h) is an M-fuzzifying hull space, then $\lim^h$ is convex.

Proposition 4.11. *Suppose that $f : (X, h_X) \longrightarrow (Y, h_Y)$ is a mapping between two M-fuzzifying preconvex closure spaces. (X, lim^{h_X}) and (Y, lim^{h_Y}) are induced M-fuzzifying preconvex convergence spaces by (X, h_X) and (Y, h_Y). Then*

$$Con^{lim^h}(f) \geqslant \bigwedge_{B \in 2^Y} \bigwedge_{x \in X} (h_Y(Y - B)(f(x))' \to h_X(f^{-1}(Y - B))(x)').$$

Proof. By Proposition 4.10, we can obtain for all $\mathcal{F} \in \mathcal{F}_M(X)$ and $x \in X$,

$$\begin{aligned}
& lim^{h_X}(\mathcal{F})(x) \to lim^{h_Y}(f^{\Rightarrow}(\mathcal{F}))(f(x)) \\
= & \Big(\bigwedge_{A \in 2^X} (h_X(X - A)(x)' \to \mathcal{F}(A))\Big) \\
& \to \Big(\bigwedge_{B \in 2^Y} (h_Y(Y - B)(f(x))' \to f^{\Rightarrow}(\mathcal{F})(B))\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{B\in 2^Y}\Big(\Big(\bigwedge_{A\in 2^X}(h_X(X-A)(x)'\to \mathcal{F}(A))\Big)\\
&\quad \to (h_Y(Y-B)(f(x))'\to f^{\Rightarrow}(\mathcal{F})(B))\Big)\\
&\geqslant \bigwedge_{B\in 2^Y}\Big((h_X(X-f^{-1}(B))(x)'\to \mathcal{F}(f^{-1}(B)))\\
&\quad \to (h_Y(Y-B)(f(x))'\to f^{\Rightarrow}(\mathcal{F})(B))\Big)\\
&= \bigwedge_{B\in 2^Y}\Big((h_X(f^{-1}(Y-B))(x)'\to f^{\Rightarrow}(\mathcal{F})(B))\\
&\quad \to (h_Y(Y-B)(f(x))'\to f^{\Rightarrow}(\mathcal{F})(B))\Big)\\
&\geqslant \bigwedge_{B\in 2^Y}(h_Y(Y-B)(f(x))'\to h_X(f^{-1}(Y-B))(x)').
\end{aligned}
$$

This implies

$$
\begin{aligned}
Con^{lim^h}(f) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)}\bigwedge_{x\in X}(lim^{h_X}(\mathcal{F})(x)\to lim^{h_Y}(f^{\Rightarrow}(\mathcal{F}))(f(x)))\\
&\geqslant \bigwedge_{B\in 2^Y}\bigwedge_{x\in X}(h_Y(Y-B)(f(x))'\to h_X(f^{-1}(Y-B))(x)'),
\end{aligned}
$$

as desired. □

5. Relationships among M-fuzzifying convexity-preserving mappings and separation properties

In this section, we discuss the relationships among M-fuzzifying convexity-preserving mappings and separation properties in M-fuzzifying convergence spaces. Moreover, it is proved that separation properties S_0, S_1 and S_2 are preserved by homeomorphisms in M-fuzzifying convergence spaces. For convenience, we denote $a\to\perp$ by $\neg a$ for each $a\in M$. In [9], the author introduced separation properties in M-fuzzifying convergence spaces as follows.

Definition 5.1 ([9])**.** Let $(X,\lim)$ be an M-fuzzifying convergence space.

(1) The degree $S_0(X,\lim)$ to which $(X,\lim)$ is S_0-separated is defined by

$$S_0(X,\lim)=\bigwedge_{x\neq y}(\neg\lim([x])(y)\vee\neg\lim([y])(x)).$$

(2) The degree $S_1(X,\lim)$ to which $(X,\lim)$ is S_1-separated is defined by

$$S_1(X,\lim)=\bigwedge_{x\neq y}(\neg\lim([x])(y)\wedge\neg\lim([y])(x)).$$

(3) The degree $S_2(X,\lim)$ to which $(X,\lim)$ is S_2-separated is defined by

$$S_2(X,\lim)=\bigwedge_{x\neq y}\bigwedge_{\mathcal{F}\in\mathcal{F}_M(X)}(\neg\lim(\mathcal{F})(x)\vee\neg\lim(\mathcal{F})(y)).$$

Proposition 5.2. *Suppose that $f:(X,lim_X)\longrightarrow(Y,lim_Y)$ is an injective mapping between two M-fuzzifying convergence spaces. Then*

(1) $S_0(Y,lim_Y)\wedge Con(f)\leqslant S_0(X,lim_X)$;
(2) $S_1(Y,lim_Y)\wedge Con(f)\leqslant S_1(X,lim_X)$;
(3) $S_2(Y,lim_Y)\wedge Con(f)\leqslant S_2(X,lim_X)$.

Proof. We only prove (1), the proof of (2) and (3) are similar.

(1) For all $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have

$$\begin{aligned}
& (\neg lim_X([x_1])(x_2)) \vee (\neg lim_X([x_2])(x_1)) \\
= \;& (lim_X([x_1])(x_2) \to \bot) \vee (lim_X([x_2])(x_1) \to \bot) \\
\geqslant \;& \Big((lim_X([x_1])(x_2) \to lim_Y([f(x_1)])(f(x_2))) \wedge (lim_Y([f(x_1)])(f(x_2)) \to \bot)\Big) \\
& \vee\Big((lim_X([x_2])(x_1) \to lim_Y([f(x_2)])(f(x_1))) \wedge (lim_Y([f(x_2)])(f(x_1)) \to \bot)\Big) \\
\geqslant \;& \Big((lim_Y([f(x_1)])(f(x_2)) \to \bot) \vee (lim_Y([f(x_2)])(f(x_1)) \to \bot)\Big) \\
& \wedge\Big((lim_X([x_1])(x_2) \to lim_Y([f(x_1)])(f(x_2))) \wedge (lim_X([x_2])(x_1) \\
& \to lim_Y([f(x_2)])(f(x_1)))\Big) \\
= \;& \Big((\neg lim_Y([f(x_1)])(f(x_2))) \vee (\neg lim_Y([f(x_2)])(f(x_1)))\Big) \wedge \Big((lim_X([x_1])(x_2) \\
& \to lim_Y(f^{\Rightarrow}([x_1])])(f(x_2))) \wedge (lim_X([x_2])(x_1) \to lim_Y(f^{\Rightarrow}([x_2]))(f(x_1)))\Big) \\
\geqslant \;& \Big(\bigwedge_{y_1 \neq y_2} \Big((\neg lim_Y([y_1])(y_2)) \vee (\neg lim_Y([y_2])(y_1))\Big)\Big) \\
& \wedge\Big(\bigwedge_{\mathcal{F} \in \mathcal{F}_M(X)} \bigwedge_{x \in X} (lim_X(\mathcal{F})(x) \to lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)))\Big) \\
= \;& S_0(Y, lim_Y) \wedge Con(f).
\end{aligned}$$

This implies

$$\begin{aligned}
S_0(X, lim_X) &= \bigwedge_{x_1 \neq x_2} (\neg \lim([x_1])(x_2) \vee \neg \lim([x_2])(x_1)) \\
&\geqslant S_0(Y, lim_Y) \wedge Con(f),
\end{aligned}$$

as desired. □

Proposition 5.3. *Suppose that $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ is a bijective mapping between two M-fuzzifying convergence spaces. Then*

(1) $S_0(X, lim_X) \wedge Con(f^{-1}) \leqslant S_0(Y, lim_Y)$;
(2) $S_1(X, lim_X) \wedge Con(f^{-1}) \leqslant S_1(Y, lim_Y)$;
(3) $S_2(X, lim_X) \wedge Con(f^{-1}) \leqslant S_2(Y, lim_Y)$.

Proof. We only prove (1), the proof of (2) and (3) are similar.

(1) Since f is a bijective mapping, we know for all $y \in Y$, $[f^{-1}(y)] = (f^{-1})^{\Rightarrow}([y])$. Therefore, for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$,

$$\begin{aligned}
& (\neg lim_Y([y_1])(y_2)) \vee (\neg lim_Y([y_2])(y_1)) \\
\geqslant \;& \Big((lim_Y([y_1])(y_2) \to lim_X([f^{-1}(y_1)])(f^{-1}(y_2))) \wedge (lim_X([f^{-1}(y_1)])(f^{-1}(y_2)) \\
& \to \bot)\Big) \vee \Big((lim_Y([y_2])(y_1) \to lim_X([f^{-1}(y_2)])(f^{-1}(y_1))) \\
& \wedge(lim_X([f^{-1}(y_2)])(f^{-1}(y_1)) \to \bot)\Big) \\
= \;& \Big((lim_Y([y_1])(y_2) \to lim_X([f^{-1}(y_1)])(f^{-1}(y_2))) \wedge (\neg lim_X([f^{-1}(y_1)])(f^{-1}(y_2)))\Big) \\
& \vee\Big((lim_Y([y_2])(y_1) \to lim_X([f^{-1}(y_2)])(f^{-1}(y_1)))
\end{aligned}$$

$$
\begin{aligned}
& \wedge(\neg lim_X([f^{-1}(y_2)])(f^{-1}(y_1)))\Big) \\
\geqslant\ & \Big((\neg lim_X([f^{-1}(y_1)])(f^{-1}(y_2))) \vee (\neg lim_X([f^{-1}(y_2)])(f^{-1}(y_1)))\Big) \\
& \wedge\Big((lim_Y([y_1])(y_2) \to lim_X([f^{-1}(y_1)])(f^{-1}(y_2))) \wedge (lim_Y([y_2])(y_1) \\
& \to lim_X([f^{-1}(y_2)])(f^{-1}(y_1)))\Big) \\
=\ & \Big((\neg lim_X([f^{-1}(y_1)])(f^{-1}(y_2))) \vee (\neg lim_X([f^{-1}(y_2)])(f^{-1}(y_1)))\Big) \\
& \wedge\Big((lim_Y([y_1])(y_2) \to lim_X((f^{-1})^{\Rightarrow}([y_1]))(f^{-1}(y_2))) \wedge (lim_Y([y_2])(y_1) \\
& \to lim_X((f^{-1})^{\Rightarrow}([y_2]))(f^{-1}(y_1)))\Big) \\
\geqslant\ & \Big(\bigwedge_{x_1 \neq x_2} \Big((\neg lim_X([x_1])(x_2)) \vee (\neg lim_X([x_2])(x_1))\Big)\Big) \\
& \wedge\Big(\bigwedge_{\mathcal{G} \in \mathcal{F}_M(Y)} \bigwedge_{y \in Y} (lim_Y(\mathcal{G})(y) \to lim_X(f^{-1})^{\Rightarrow}(\mathcal{G})(f^{-1}(y)))\Big) \\
=\ & S_0(X, lim_X) \wedge Con(f^{-1}).
\end{aligned}
$$

This implies

$$
\begin{aligned}
S_0(Y, lim_Y) &= \bigwedge_{y_1 \neq y_2} (\neg lim_Y([y_1])(y_2) \vee \neg lim_Y([y_2])(y_1)) \\
&\geqslant S_0(X, lim_X) \wedge Con(f^{-1}),
\end{aligned}
$$

as desired. □

By Definition 3.9, Propositions 5.2 and 5.3, we can obtain the following theorem.

Theorem 5.4. *Suppose that $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ is a bijective mapping between two M-fuzzifying convergence spaces. Then*
(1) $S_0(X, lim_X) \wedge Hom(f) = S_0(Y, lim_Y) \wedge Hom(f)$;
(2) $S_1(X, lim_X) \wedge Hom(f) = S_1(Y, lim_Y) \wedge Hom(f)$;
(3) $S_2(X, lim_X) \wedge Hom(f) = S_2(Y, lim_Y) \wedge Hom(f)$.

6. Conclusions

In this paper, we endow the concepts of convexity-preserving mappings and closure-preserving mappings with some degrees by using implication operation. Moreover, the relationships of M-fuzzifying convexity-preserving mappings with M-CP mappings, M-fuzzifying preconvex closure operators, and separation axioms in M-fuzzifying convergence spaces are discussed.

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This research was funded by the National Natural Science Foundation of China (Nos.12371462) and the Fundamental Research Funds for the Central Universities (No.2019zy20).

Data availability. No data was used for the research described in the article.

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