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Some Issues on Properties of the Extended IOWA Operators in Cubic Group Decision Making

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Abstract – The concept of this paper to study some IOWA operator to aggregating the individual cubic preference relations (CPR). This paper deal further the study of their properties of group decision problems with the help of CPR, we have proved that the collective preference relation obtained by IOWA operator, then we applied the aggregation operator of individual judgment by using IOWA operators as aggregation procedure by (RAMM) method. Additionally, the result of group Consistency IOWA (C-IOWA) operator is greater than the arithmetic mean of all the individual consistency degree. The numerical application verified the result of this paper.

Keywords – *Cubic preference relation (CPR), induced ordered weighted averaging (IOWA), group decision making*

1. Introduction

The theory of fuzzy sets is developed in 1965 [15] which has been generally used in many area of our present society. Atanassov [1] generalized fuzzy set to intuitionistic fuzzy set (IFS) [2] The IFS categorized by membership and as a non-membership. Atanassov and Gargov further extend the concept of IFS to interval value intuitionistic fuzzy set. IFS the membership and non-membership are the fuzzy number while $IVIFS$ are interval valued intuitionistic fuzzy numbers.

The IFS does not explain the problem when there is some uncertainty. Therefore Jun, defined the new concept so called cubic set [3] In 2012, Jun introduced a new theory which is called cubic set theory. They introduced many concept of cubic set. Cubic deal with uncertainty problem. Jun cubic set explain all the satisfied, unsatisfied and uncertain information, while fuzzy and intuitionistic fuzzy set fail to explain these term. Szmidt and

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Kacprzyk [4] proposed the concept of intuitionistic preference relation (PR) and Xu [5] defined the consistency of intuitionistic fuzzy relation by extending the notion of consistent reciprocal preference relation. Since it is often more difficult for a decision maker to exactly quantify his certainty properties of these $IOWA$ operators.

The application of PR applied to DM [6,7,8,9,10]. Therefore the verification of such preference relation (PR) is some significant to construct worthy DM method. Where the consistency property is most benefit property, in these properties the non existence of consistency in DM must be inconsistent in the conclusions. Therefore this show the important conditions. Its plays a vital role to study the conditions under which consistency is satisfied [11.10]. The obtaing of perfect consistency practice is challenging mostly, when calculting the preference on a classical set with big numbers of choices. There are two problems of consistency

(1) The individually consideration of an expert is called consistent.
(2) when the consideration of consistent in the group.

We define the method of computing consistency in CPR. By using this consistency measure, we verified that if different judgement matrix $(C-IOW)$ have a adequate, then combined judgement matrix $(C-IOWACJM)$ also is of acceptable consistency. Moreover, our result guarantees that the consistency of $(C-IOWACJM)$ is smaller than the arithmetic mean of all the individual consistency. The $(I-IOWA)$ operator also has similar properties.

The paper is consists of the following sections, such that. In Section 2 we review some fundamental concepts such that the $IOWA, (C-IOWA)$ and $(I-IOWA)$ operators. We also defines the concept of consistency degree of (CPR) in Section 3 . In Section 4 , we study the preferred properties of these $(IOWA)$ operators in cubic (GDM) . In Section 5 we provides illustrative examples. This paper is concluded in Section 6 .

2. Preliminaries

$(IOWA), (C-IOWA)$ and $(I-IOWA)$ Operators

In this section we generalized the concept of induced ordered weighted average $(IOWA)$, consistency $IOWA$ $(C-IOWA)$ and individual $(I-IOWA)$ operators, which will be used throughout this paper. [15] Yager and Filev defined an induced OWA $(IOWA)$ operator in which the ordering of the $a_i (i \in n)$ is induced by other variables $u_i (i \in n)$ called the order inducing variables, where a_i and u_i are the factor of OWA set $\langle u_i, a_i \rangle (i \in n)$.

Definition 2.1 [15] An $(IOWA)$ operator of dimension n is a mapping, $\varphi_w^G : R^{+^n} \to R^+$ to which a set of weights or a weighting vector is related,

$$W = (w_1, w_2, ..., w_n)^T, w_j \in [0,1] and \sum_{j=1}^{n} w_j = 1,$$

and it is defined to aggregate the set of $2nd$ arguments of list of two pairs $\left\{ \begin{matrix} \langle u_1, a_1 \rangle, ..., \\ \langle u_n, a_n \rangle \end{matrix} \right\}$, given on the basis of a positive ratio scale, define as following:

$$f_w^G = (\langle u_1, a_1 \rangle, ..., \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_{j,} \qquad (1)$$

where $w =$ $(w_1, w_2, ..., w_n)^T$ is a weighting vector, i.e. $\sum_{j=1}^{n} w_j = 1, w_j \in [0,1], b_j$ is the a_i value of the $IOWA$ pair having the jth largest u_i, and u_i in $\langle u_i, a_i \rangle$ is referred to as the order inducing variable and a_i as the argument variable.

Definition 2.2 [12] If a set of (DMs) $D = \{d_1, d_2, ..., d_m\}$ provides preference about a set of alternatives $X = \{x_1, x_2, ..., x_n\}$ by means of (CPR) $\{M^{(1)}, ..., M^{(l)}, ... , M^{(m)}\}$, and each have an importance degree $\mu(d_k) \in [0,1]$, related to him or her, then an $(I - IOWA)$ operator is an $(IOWA)$ operator in which its order-inducing values is the set of importance degree.

Definition 2.3 If a set of (DMs) $D = \{d_1, d_2, ..., d_m\}$ provides preference about a set of alternatives $X = \{x_1, x_2, ..., x_n\}$ by means of (CPR), $\{M^{(1)}, ..., M^{(l)}, ... , M^{(m)}\}$, $M^{(l)} \in M$, then a $(C - IOWA)$ operator is an $(IOWA)$ operator in which its order-inducing values is the set of consistency index values such that,

$$\{CI(M^{(1)}), ..., CI(M^{(l)}), ..., CI(M^{(m)})\}$$

Definition 2.4 [3] Let X be a fixed non empty set. A cubic set is an object of the form:

$$C = \{\langle a, A(a), \lambda(a) \rangle \ : \ a \in X\},$$

where A is an $(IVFS)$ and λ is a fuzzy set in X. A cubic set $\widetilde{C} = \langle a, A(a), \lambda(a) \rangle$ is simply denoted by $\widetilde{C} = \langle \widetilde{A}, \lambda \rangle$. The collection of all cubic set is denoted by $C(X)$.

(a) if $\lambda \in \widetilde{A}(x)$ $\forall$ $x \in X$ so it is called interval cubic set.

(b) If $\lambda \notin \widetilde{A}(x)$ $\forall$ $x \in X$ so it is called external cubic set.

(c) If $\lambda \in \widetilde{A}(x)$ or $\lambda \notin \widetilde{A}(x)$ its called cubic set for all $x \in X$.

Definition 2.5 [3] Let $A = \langle A, \lambda \rangle$ and $B = \langle B, \mu \rangle$ be cubic set in X, then we define

(a) (Equality) $A = B$ if and only if $A = B$ and $\lambda = \mu$.

(b) $(P-$ order) $A \subseteq_A B$ if and only if $A \subseteq B$ and $\lambda \leq \mu$.

(c) $(R-$ order) $A \subseteq_R B$ if and only if $A \subseteq B$ and $\lambda \geq \mu$.

Definition 2.6 [3] The complement of $A = \langle A, \lambda \rangle$ is defined to be the cubic set

$$A^c = \{\langle x, A^c(x), 1 - \lambda(x) \rangle \mid x \in X\}.$$

3. The Measure of Consistency Index of CPR

In GD atmosphere, the problem of consistency itself consist of two problems

(1) The individually consideration of an expert is called consistent.
(2) when the consideration of consistent in the group.

First problem is emphasis in this section. First of all we define the idea's of the additive transitive CPR. Then we define the CI of CPR. In the following section, we will emphasis on the 2nd problem.

Definition 3.1 Suppose $X = \{x_1, x_2, ..., x_n\}$ be a finite set of alternatives. If the DM gives his/her PR information on X by means of a preference relation $M = (C_{ij})_{n\times n}$, where $\widetilde{C}_{ij} = \left\langle \widetilde{A}_{ij}, \lambda_{ij} \right\rangle$ and we have,

$$\widetilde{A}_{ij} + \widetilde{A}_{ji} = 1,\ \widetilde{A}_{ii} = 0.5 \text{ and } \lambda_{ij} + \lambda_{ji} = 1,\ \lambda_{ii} = 0.5\ \forall\, i, j \in N.$$

Where C_{ij} denotes the preference degree or intensity of the alternative X_i over X_j, then M is called a CPR.

Definition 3.2 Suppose $M = (C_{ij})_{n\times n}$ where $\widetilde{C}_{ij} = \left\langle \widetilde{A}_{ij}, \lambda_{ij} \right\rangle$ be a CPR, then M is called an additive transitive CPR, if the following additive transitivity is satisfied:

$$\widetilde{A}_{ij} = \widetilde{A}_{ik} - \widetilde{A}_{jk} + 0.5, \text{ and } \lambda_{ij} = \lambda_{ik} - \lambda_{jk} + 0.5\ \forall\, i, j, k \in N.$$

Definition 3.3 If we utilize the row arithmetic mean method $(RAMM)$, then can get the priority vector $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ of the CPR, $M^{(l)}$, where

$$w_i^{(l)} = \frac{1}{n} \sum_{j=1}^{n} C_{ij}^{(l)}, \quad i = 1, 2, ...n; l = 1, 2, ..., m.$$

Definition 3.4 Suppose $A = (a_{ij})_{n\times n} \in M$ and $b = (b_{ij})_{n\times n} \in M$, then the distance between A and B define as follows:

$$d(A,B) = \frac{1}{3n} \sum_{i=1}^{n} \sum_{j=1}^{n} [\left|\overline{a}_{ij} - \overline{b}_{ij}\right| + | a_{ij}^{+} - b_{ij}^{+} | + | \lambda_{ij} - \lambda_{ij} |] \qquad (2)$$

Clearly, the smaller the value of distance degree $d(A,B)$, the nearer of the CPR, A and B.

Theorem 3.5 Let $A = (a_{ij})_{n\times n} \in M$ and $b = (b_{ij})_{n\times n} \in M$, then

(1) $d(A,B) \geq 0;$

(2) $.d(A,B) = 0 \Leftrightarrow A$ and B are perfectly consistent.

Proof. (1)

$$d(A,B) = \frac{1}{3n} \sum_{i=1}^{n} \sum_{j=1}^{n} [\left|\overline{a}_{ij} - \overline{b}_{ij}\right| + | a_{ij}^{+} - b_{ij}^{+} | + | \lambda_{ij} - \lambda_{ij} |] \geq 0 \qquad (3)$$

(2) Necessity. If $d(A,B)=0$, then $a_{ij}=b_{ij}$ for all $i,j\in N$. Hence, A and B are perfectly consistent.
(3) Sufficiency. If A and B are perfectly consistent, then $a_{ij}=b_{ij}$ $\forall$ $i,j\in N$. Thus, we have $a_{ij}-b_{ij}=0$ $\forall$ $i,j\in N$. Therefore, $d(A,B)=0$.
In (GD) problems based on (CPR) , the study of consistency is related to the transitivity property. And gave a categorization of the consistency property defined by the additive transitivity property of a cubic preference relation

$$M^{K}=(C_{ij}^{k}):\ C_{ij}^{k}+C_{jl}^{k}+C_{il}^{k}=\left\langle\widetilde{\frac{3}{2}},\frac{3}{2}\right\rangle,\forall i,j,l\in\{1,...,n\}.$$

Applying this categorization technique, a method to construct a consistent reciprocal (CPR) M on $X=\{x_1,x_2,...,x_n,\quad n\geq 2\}$ from $n-1$ preference values $\{C_{12}$ $C_{23},...,C_{n-1n}$ $\}$ define as followes:

(1) $M=(C_{ij})$ i.e.

$$C_{ij}=\begin{cases} C_{ij} & \text{if } i\leq j\leq i+1, \\ (C_{ii+1}+C_{i+1i+2}+C_{i+2i+3},...,C_{j-1j})-\frac{j-(i+1)}{2} & \text{if } i+1<j, \\ 1-C_{ij} & \text{if } j<i. \end{cases}$$

But the matrix M could have entries not in the interval $[0,1]$, but in an interval $[-x,1+x]$, being $x=\lfloor\min\{C_{ij};C_{ij}\in M\}\rfloor$ For this case. [13] the alteration function which reserves reciprocity and additive consistency, that is a function $[-x,1+x]\rightarrow[0,1]$ satisfying

$(i) f(-x)=0.$
$(ii) f(1+x)=1.$
$(iii) f(a)+f(1-a)=1,\quad\forall\quad a\in[-x,1+x].$
$(iv)\quad f(a)+f(b)+f(c)=\frac{3}{2},\quad\forall\quad a,b,c\in[-x,1+x].$ i.e. $a+b+c=\frac{3}{2}.$

(**2**) The consistent (CPR) , N is obtained as $N=f(M)$. This (CI) has a certain physical consequence and reflects the deviance degree b/w the (CPR) $M^{(l)}$ and its equivalent consistent matrix $N^{(l)}$. The distance b/w $M^{(l)}$ and its equivalent consistent matrix $N^{(l)}$ define as follows.

Definition 3.6 Let $M^{(1)},...,M^{(l)},...,M^{(m)}$ be the (CPR) provided by m decision maker's and $N^{(1)},...,N^{(l)},...,N^{(m)}$ be their equivalent consistent matrix, then we define a measure of (CI) of the (CPR) $M^{(l)}$ as follows:

$$CI(M^{(l)})=1-d(M^{(l)},N^{(l)}). \qquad (4)$$

Clearly, the nearer $CI(M^{(l)})$ is to 1 the ultimate consistent the information provided by the (DM) $d^{(l)}$, and thus more importance should be placed on that information. By using this (CI) , we obtain some preferred properties of $(C-IOWA)$ operator.

4. The Properties Of $IOWA$ Operators In Cubic Group Decision Making

We appliance the $(C-IOWA)$ operator and the ($I-IOWA)$ operator to aggregate individual (CPR) in group decision making problems, and then study their desired properties. in this section.

The Consistency $IOWA$ $(C-IOWA)$ **Operator**

In a standardized group decision making problem, the decision maker's have identical importance. Therefore, every decision maker's continuously can have a (CI) value related with them, which measures the level of consent b/w group preferences and individual preference. Therefore, the (DM) provided further consistency information, the greater weighting value should be placed on that information. We discuss the reciprocity and consistency properties of the $(C-IOWACJM)$, which is found by applying $(C-IOWA)$ operator, in this section.

Definition 4.1 If $M^{(1)},...,M^{(l)},...,M^{(m)}$ are the (CPR) provided by m (DMs), then the $(C-IOWACJM)$ $M=(C_{ij})_{n\times n}$ is difined as follows:

$$\bar{M} = C-IOWA\begin{pmatrix}\langle CI(M^{(1)}),M^{(1)}\rangle,\langle CI(M^{(2)}),M^{(2)}\rangle,\\ ...,\langle CI(M^{(m)}),M^{(m)}\rangle\end{pmatrix}$$
$$= C-IOWA\begin{pmatrix}\langle CI(M^{(\alpha(1))}),M^{(\alpha(1))}\rangle,\langle CI(M^{(\alpha(2))}),M^{(\alpha(2))}\rangle,\\ ...,\langle CI(M^{(\alpha(m))}),M^{(\alpha(m))}\rangle\end{pmatrix}$$
$$= \begin{pmatrix}M^{(\alpha(1))}\times\delta_{(\alpha(1))})+(M^{(\alpha(2))}\times\delta_{(\alpha(2))})+\\ ...+(M^{(\alpha(m))}\times\delta_{(\alpha(m))}\end{pmatrix} \quad (5)$$
$$C_{ij} = \begin{pmatrix}C_{ij}^{(\alpha(1))}\times\delta_{(\alpha(1))})+(C_{ij}^{(\alpha(2))}\times\delta_{(\alpha(2))})+\\ ...+(C_{ij}^{(\alpha(m))}\times\delta_{(\alpha(m))}\end{pmatrix}$$
$$= \prod_{l=1}^{m}(a_{ij}^{(\alpha(l))}\times\delta_{(\alpha(l))}), \quad (6)$$

where $(\alpha(1),\alpha(2),...,$ $\alpha(n)$ is a permutation of $(1,2,...,n)$ such that

$$CI(M^{(\alpha(l-1))})\geq CI(M^{(\alpha(l))}) \text{ and } \delta_{\alpha(l-1)}\geq\delta_{\alpha(l)} \;\forall\, l=2,...,m;$$
$$\langle CI(M^{(\alpha(l))}),M^{(\alpha(l))}\rangle \text{ is two tuple with } CI(M^{(\alpha(l))})$$

the lth largest value in the set $\{CI(M^{(1)}),...,CI(M^{(m)})\}$;

$$\delta=(\delta_{\alpha(1)},\delta_{\alpha(2)},...,\delta_{\alpha(m)})^T \text{ is a weighting vector i.e.}$$
$$\sum_{l=1}^{m}\delta_{\alpha(l)}=1 \text{ and } \delta_{\alpha(l)}\in[0,1].$$

Yager [14] provided a method to define the weighting vector related to an $(IOWA)$ operator. In this case, each remark in the aggregation contains of a triple $(p_{ij}^{(l)}, u_l, v_l)$: $p_{ij}^{(l)}$ is the argument value to aggregate, u_l is the significance weight value related to $p_{ij}^{(l)}$, and v_l is the order inducing value. Therefore, the aggregation is

$$IOWA_Q(p_{ij}^{(1)},...,p_{ij}^{(m)}) = \sum_{l=1}^{m} w_l p_{ij}^{\alpha(l)}, \text{ with}$$

$$w_l = Q\left(\frac{S(l))}{S(n)}\right) - Q\left(\frac{S(l-1)}{S(n)}\right), \tag{7}$$

where $S(l) = \sum_{k=1}^{l} u_{\alpha(k)}$, and α is permutation i.e. $u_{\alpha(l)}$ in $(p_{ij}^{\alpha(l)}, u_{\alpha(l)}, v_{\alpha(l)})$ is the lth largest value in the set of $\{v_1,...,v_n\}$. Q is a function $:[0,1] \to [0,1]$ i.e. $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$. In this case, we suggest to use the consistency values related to each one of the (DM) both as a weight related to the argument and as the order inducing values $u_i = v_i = CI(M^{(i)})$. Therefore the ordering of the preference values is first induced by the ordering of the (DMs) from greatest to smallest consistency one, and the weights of the $(C-IOWA)$ operator is obtained by using the above, E.q. (7) , with decreases to

$$\delta_{\alpha(l)} = Q\left(\frac{S(\alpha(l))}{S(\alpha(n))}\right) - Q\left(\frac{S(\alpha(l-1))}{S(\alpha(n))}\right), \tag{8}$$

where $S(\alpha(l)) = \sum_{k=1}^{l} CI(M^{(\alpha(k))})$, and α is the permutation such that

$$CI(M^{(\alpha(l))}) \text{ in } (C_{ij}^{(\alpha(l))}, CI(M^{(\alpha(l))}), CI(M^{(\alpha(l))}))$$

is the lth largest value in the set $\{CI(M^{(\alpha(l))}),...,CI(M^{(\alpha(n))})\}$. In an aggregation process, we consider that the weighting value of (DMs) should be implemented in such a way that the effect from those (DMs) who are less consistency is reduced, and therefore the above is obtained if the linguistic quantifier Q verifiers that the most the consistency of an (DM) the higher the weighting value of that (DM) in the aggregation, i.e.:

$$CI(M^{(\alpha(1))}) \geq CI(M^{(\alpha(2))}) \geq,...,CI(M^{(\alpha(n))}) \geq 0$$
$$\Rightarrow \delta_{\alpha(1)} \geq \delta_{\alpha(2)},...,\geq \delta_{\alpha(n)} \geq 0.$$

Theorem 4.2 Let the parameterized family of RIM quantifiers $Q(\lambda) = \lambda^{\alpha}, \alpha \geq 0$, if $a \in [0,1]$ and

$$S(\alpha(l)) = \sum_{k=1}^{l} CI(M^{(\alpha(k))}), then\, \delta_{\alpha(l)} \geq \delta_{\alpha(l+1)}, \forall l = 1,2,...,m$$

Proof If $\alpha \in [0,1]$, then the function $Q(\lambda) = \lambda^{\alpha}$ is concave and, we have $Q(T_l) - Q(T_{l-1}) \geq Q(T_{l+1}) - Q(T_l)$. Suppose

$$T_l = \frac{S(\alpha(l))}{S(\alpha(n))} \text{ and } S(\alpha(l)) = \sum_{k=1}^{l} CI(M^{(\alpha(k))}), \text{ then}$$

$$\delta_{\alpha(l)} = Q\left(\frac{S(\alpha(l))}{S(\alpha(n))}\right) - Q\left(\frac{S(\alpha(l-1))}{S(\alpha(n))}\right) = Q(T_l) - Q(T_{l-1}) \text{ and}$$

$$\delta_{\alpha(l+1)} = Q\left(\frac{S(\alpha(l+1))}{S(\alpha(n))}\right) - Q\left(\frac{S(\alpha(l))}{S(\alpha(n))}\right) = Q(T_{l+1}) - Q(T_l)$$

Thus, we can obtain $\delta_{\alpha(l)} \geq \delta_{\alpha(l+1)}$.

In group decision making models with (CP) assessments, it is frequently supposed that the (CPR) , to express the judgments are reciprocal. The $(C-IOWA)$ operator is able to maintain both the reciprocity and the consistency properties in the collective (CPR) . In order to study these properties, we construct the next theorem.

Theorem 4.3 Let $M^{(1)}, M^{(2)}, ..., M^{(m)}$ be (CPR) provided by m decision maker's where $M^{(l)} = (C_{ij}^{(l)})_{n\times n}$, $l = 1,2,...,m;$ $i, j = 1,2,...,n,$ then their $(CI-IOWACJM)$ $\bar{M} = (C_{ij}^{(l)})_{n\times n}$ is also a (CPR), where

$$\begin{aligned} C_{ij} &= CI-IOWA\begin{pmatrix} \langle CI(M^{(1)}, C_{ij}^{(1)})\rangle, \langle CI(M^{(2)}), C_{ij}^{(2)}\rangle \\ ,..., \langle CI(M^{(m)}), C_{ij}^{(m)}\rangle \end{pmatrix} \\ &= CI-IOWA\begin{pmatrix} \langle CI(M^{(\alpha(1))}), C_{ij}^{(\alpha(1))}\rangle, \langle CI(M^{(\alpha(2))}), C_{ij}^{(\alpha(2))},\rangle \\ ,..., \langle CI(M^{(\alpha(m))}), C_{ij}^{(\alpha(m))}\rangle \end{pmatrix} \\ &= (C_{ij}^{(\alpha(1))} \times \delta_{(\alpha(1))}) + (C_{ij}^{(\alpha(2))} \times \delta_{(\alpha(2))}) + ... + (C_{ij}^{(\alpha(m))} \times \delta_{(\alpha(m))}) \end{aligned}$$

and

$$C_{ij} \geq 0,\ \widetilde{A}_{ij} + \widetilde{A}_{ji} = 1,\ A_{ii} = 0.5 \text{ and}$$
$$\lambda_{ij} + \lambda_{ji} = 1,\ \lambda_{ii} = 0.5\ \forall\, i, j \in N.$$

Also $\bar{M}$ is also consistent, subject to $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are consistent.

Proof Since $M^{(1)}, M^{(2)}, ..., M^{(m)}$ are (CPR) , we have then

$$C_{ij} = \begin{pmatrix} (C_{ij}^{(\alpha(1))} \times \delta_{(\alpha(1))}) + (C_{ij}^{(\alpha(2))} \times \delta_{(\alpha(2))}) \\ + ... + (C_{ij}^{(\alpha(m))} \times \delta_{(\alpha(m))}) \end{pmatrix}$$
$$\geq (0 \times \delta_{(\alpha(1))}) + (0 \times \delta_{(\alpha(2))}) + ... + (0 \times \delta_{(\alpha(m))}) = 0.$$
$$C_{ij} + C_{ji} = \begin{pmatrix} (C_{ij}^{(\alpha(1))} + C_{ji}^{(\alpha(1))})\delta_{(\alpha(1))} + (C_{ij}^{(\alpha(2))} + C_{ji}^{(\alpha(2))})\delta_{(\alpha(2))} \\ + ... + (C_{ij}^{(\alpha(m))} + C_{ji}^{(\alpha(m))})\delta_{(\alpha(m))} \end{pmatrix}$$
$$= \delta_{\alpha(1)} + \delta_{\alpha(2)} + ... + \delta_{\alpha(m)} = 1,$$
$$C_{ij} = (C_{ii}^{(\alpha(1))} \times \gamma_{\alpha(1)}) + (C_{ii}^{(\alpha(2))} \times \gamma_{\alpha(2)}) + ... + (C_{ii}^{(\alpha(m))} \times \gamma_{\alpha(m)})$$
$$= (\frac{1}{2}\delta_{\alpha(1)}) + (\frac{1}{2}\delta_{\alpha(2)}) + ... + (\frac{1}{2}\delta_{\alpha(m)}) = \frac{1}{2}.$$

Thus, $M = (C_{ij})_{n\times n}$ is also a (CPR).

(ii) Since all the $M^{(1)}, M^{(2)}, ..., M^{(m)}$ are consistent, *i.e.*, then

$$\widetilde{A}_{ij}^{l} = \widetilde{A}_{ik}^{l} + \widetilde{A}_{kj}^{l}, -0.5 \text{ and}$$
$$\lambda_{ij} = \lambda_{ik}^{l} + \lambda_{kj}^{l} - 0.5 \; \forall \, l = 1, 2, ..., m \;\; i, j \in N.$$

Thus

$$C_{ik} + C_{kj} = \sum_{l=1}^{m} C_{ik}^{(\alpha(l))}\delta_{(\alpha(l))} + \sum_{l=1}^{m} C_{kj}^{(\alpha(l))}\delta_{(\alpha(l))}$$
$$= \sum_{l=1}^{m} (C_{ik}^{(\alpha(l))} + C_{kj}^{(\alpha(l))})\delta_{(\alpha(l))}$$
$$= \sum_{l=1}^{m} (C_{ij}^{(\alpha(l))} + \langle 0.\widetilde{5}, 0.5 \rangle)\delta_{(\alpha(l))}$$
$$= C_{ij} + \langle 0.\widetilde{5}, 0.5 \rangle$$

and thus, $\bar{M}$ is also consistent.

Definition 4.4 Denote $M^{(l)} \in M$ be the cubic judgement matrix provided by the lth (DM) when comparing n alternatives, $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ as its priority vector, $P^{(l)} = (P_{ij}^{(l)})_{n\times n}$ as the equivalent consistent matrix; $\bar{w} = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_n)^T$ as the priority vector of $(C-IOWACJM)$ $\bar{M}$, and $\bar{N} = (p_{ij})_{n\times n}$ as the equivalent consistent matrix of $\bar{M}$.

Theorem 4.5 Applying the $(C-IOWACJM)$ as the aggregation method, the weighting vector

$$\delta = (\delta_{\alpha(1)}, \delta_{\alpha(2)} + ... + \delta_{\alpha(m)})^T, \; \delta_{\alpha(l-1)} \geq \delta_{\alpha(l)}, \; \sum_{l=1}^{m} \delta_{\alpha(l)} = 1,$$

and the $(RAMM)$ as the prioritization method, such that the (AIJ) and the (AIP) offers the same priorities of alternatives.

Proof Let $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ be the priority of the individual judgement matrix $M^{(l)}$ and $\bar{w} = (w_1, w_2, ..., w_n)^T$ be the group priorities, so we define as following,

$$\begin{aligned} w_i(AIP) &= C - IOWA\begin{pmatrix} \langle CI(M^{(1)}), w^{(1)} \rangle, \langle CI(M^{(2)}), w^{(2)} \rangle \\ ,..., \langle CI(M^{(m)}), w^{(m)} \rangle \end{pmatrix} \\ &= C - IOWA\begin{pmatrix} \langle CI(M^{(\alpha(1))}), w^{(\alpha(1))} \rangle, \langle CI(M^{(\alpha(2))}), w^{(\alpha(2))} \rangle \\ ,..., \langle CI(M^{(\alpha(m))}), w^{(\alpha(m))} \rangle \end{pmatrix} \\ &= (w^{(\alpha(1))}\delta_{\alpha(1)}) + (w^{(\alpha(2))}\delta_{\alpha(2)}) + ... + (w^{(\alpha(m))}\delta_{\alpha(m)}) \\ w_i(AIP) &= \sum_{l=1}^{m} w_i^{\alpha(l)} \times \delta_{\alpha(l)} \\ w_i(AIJ) &= \frac{1}{n}\sum_{j=1}^{n} C_{ij} = \frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{m} C_{ij}^{(\alpha(l))} \times \delta_{\alpha(l)} \\ &= \sum_{l=1}^{m} \delta_{\alpha(l)} \left(\sum_{j=1}^{n} \frac{1}{n} C_{ij}^{(\alpha(l))} \right) = \sum_{l=1}^{m} w_i^{\alpha(l)} \times \delta_{\alpha(l)}. \end{aligned}$$

Thus $w_i(AIP) = w_i(AIJ)$.

Definition 4.6 Let $CI(\bar{M})$ be a measure of the consistency of the collective matrix $\bar{M}$, and $CI(M^{(l)})$ be a measure of the consistency of matrix $M^{(l)}$.

Theorem 4.7 Suppose $M^{(1)}, M^{(2)}, ..., M^{(m)}$ be the (CPR) provided by m decision maker's when comparing n alternatives with the corresponding weighting vector

$$\delta = (\delta_{\alpha(1)}, \delta_{\alpha(2)} + ... + \delta_{\alpha(m)})^T,\ \delta_{\alpha(l-1)} \geq \delta_{\alpha(l)},\ \sum_{l=1}^{n} \delta_{\alpha(l)} = 1.$$

Using the $(C - IOWACJM)$ as the aggregation procedure and the row arithmetic mean method as the prioritization method i.e.

$$CI(\bar{M}) \geq \frac{1}{m}\sum_{l=1}^{m} CI(M^{(l)}) \qquad (9)$$

Proof By Definition 16 and E.g. (4), we have

$$CI(\bar{M}) = 1 - d(\bar{M}, \bar{N}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left| C_{ij} - \bar{p}_{ij} \right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left| \begin{array}{l} \sum_{l=1}^{m} C_{ij}^{(\alpha(l))} \times \delta_{\alpha(l)} \\ -\sum_{l=1}^{q} p_{ij}^{(\alpha(l))} \times \delta_{\alpha(l)} \end{array} \right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{l=1}^{m}(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)})\right|.$$

Since $\left|\sum_{l=1}^{m}(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)}\right| \le \sum_{l=1}^{m}\left|(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)}\right|.$

Then $CI(\bar{M}) \ge 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{m}\left|(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\delta_{\alpha(l)}\right|$

$$= 1 - \sum_{l=1}^{m}\delta_{\alpha(l)}(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(C_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\right|$$

$$= \sum_{l=1}^{m}\delta_{\alpha(l)}\left(1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(\lambda_{ij}^{(\alpha(l))} - p_{ij}^{(\alpha(l))})\right|\right)$$

$$= \sum_{l=1}^{m}\delta_{\alpha(l)} CI(M^{(l)})$$

$$CI(M^{(\alpha(l))}) \ge CI(M^{(\alpha(l+1))}) \text{ and } \delta_{\alpha(1)} \ge \delta_{\alpha(2)} \ldots \ge \delta_{\alpha(m)}$$

Then we have,

$$\sum_{l=1}^{m}\delta_{\alpha(l)} CI(M^{(\alpha(l)}) \ge \frac{1}{m}\sum_{l=1}^{n} CI(M^{(\alpha(l)}) = \frac{1}{m}\sum_{l=1}^{n} CI(M^{(l)}).$$

Thus $CI(\bar{M}) \ge \frac{1}{m}\sum_{l=1}^{m} CI(M^{(l)}).$

The importance $IOWA$ $(I - IOWA)$ **operator**

In a heterogeneous group decision making problem every expert has an importance degree related with the $(I - IOWA)$ operator, which used this importance degree variable as the order-inducing variable to induce the ordering of the argument values before their aggregation. In this section, we study the reciprocity and consistency properties of the $(I - IOWACJM)$, which is obtained by using $(I - IOWA)$ operator.

Definition 4.8 If a set of (DMs) $D = \{d_1, d_2, ..., d_m\}$ provides preference about a set of alternatives $X = \{x_1, x_2, ..., x_n\}$ by means of (CPR) $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$, whose associated importance degree $\mu = (\mu_1, \mu_2, ..., \mu_m)$, $\sum_{l=1}^{m}\mu_1 = 1, 0 \le \mu_1 \le 1$, then the $(I - IOWACJM)$ $\bar{M} = (C_{ij})_{n \times n}$ is defined as follows:

$$\bar{M} = I - IOWA(\langle \mu_1, M^{(1)} \rangle, \langle \mu_2, M^{(2)} \rangle, ..., \langle \mu_m, M^{(m)} \rangle)$$
$$= (M^{(1)} \times \mu_1) + (M^{(2)} \times \mu_2) + ... + (M^{(m)} + \mu_m) \qquad (10)$$

$$C_{ij} = \sum_{l=1}^{m} M_{ij}^{(l)} \times \mu_1 \qquad (11)$$

In group decision making models with (CP) calculations, it usually is supposed that the (CPR) to express the judgments are reciprocal. The $(I - ILOWA)$ operator also is able to maintain both the reciprocity and consistency properties in the collective (CPR). therefore we define the following theorems.

Theorem 4.9 Consider $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ be (CPR) provided by m (DMs), where $M^{(l)} = (C_{ij}^{(l)})_{n\times n}$ $(l = 1,2,...,m;$ $i,$ $J = 1,2,...,n)$, then their $(I - IOWACJM)$ $\bar{M} = (C_{ij})_{n\times n}$ is also a (CPR) ,

$$C_{ij} = \sum_{l=1}^{m} ((C_{ij}^{(l)}) \times \mu_1)$$

where $C_{ij} \geq 0, \widetilde{A}_{ij} + \widetilde{A}_{ji} = 1, A_{ii} = 0.5$ and $\lambda_{ij} + \lambda_{ji} = 1, \lambda_{ii} = 0.5$

Also $\bar{M}$ is also consistent, subject to $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are consistent.

Proof (i). Since $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are (CPR), we have

$$C_{ij} \geq 0,\ \widetilde{A}_{ij} + \widetilde{A}_{ji} = 1,\ A_{ii} = 0.5 \text{ and } \lambda_{ij} + \lambda_{ji} = 1,\ \lambda_{ii} = 0.5\ \forall\, i, j \in N.$$

$$C_{ij} = \sum_{l=1}^{m} ((C_{ij}^{(l)}) \times \mu_1) \geq \sum_{l=1}^{m} (0 \times \mu_1) = 0,$$

$$C_{ij} + C_{ji} = \sum_{l=1}^{m} (C_{ij}^{(l)}) \times \mu_1 + \sum_{l=1}^{m} (C_{ji}^{(l)}) \times \mu_1$$
$$= \sum_{l=1}^{m} (C_{ij}^{(l)} + C_{ji}^{(l)}) \times \mu_1 = \sum_{l=1}^{m} \mu_1 = 1.$$
$$C_{ii} = \sum_{l=1}^{m} C_{ii}^{(l)} \times \mu_1 = \sum_{l=1}^{m} \frac{1}{2} \times \mu_1 = \frac{1}{2}.$$

Thus, $M = (C_{ij})_{n\times n}$ is also a (CPR) .

(**ii**) Since the $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are consistent such that,

$$\widetilde{A}_{ij}^{l} = \widetilde{A}_{ik}^{l} + \widetilde{A}_{kj}^{l}, - 0.5 \text{ and}$$
$$\lambda_{ij} = \lambda_{ik}^{l} + \lambda_{kj}^{l} - 0.5, \forall\, l = 1,2,...,m\ \ i, j \in N.$$

Then

$$\bar{C}_{ik} + \bar{C}_{kj} = \sum_{l=1}^{m} (C_{ik}^{(l)} + C_{kj}^{(l)}) \times \mu_1 = \sum_{l=1}^{m} (C_{ij}^{(l)} + \langle 0.\tilde{5}, 0.5 \rangle) \mu_1 = \bar{C}_{ij} + \langle 0.\tilde{5}, 0.5 \rangle$$

Hence, $\bar{M}$ is also consistent.

Definition 4.10 Denote $M^{(l)} \in M$ be the cubic judgement matrix provided by the $l-th$ DM when comparing n alternatives, $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ as its priority vector, $P^{(l)} =$ $(P_{ij}^{(l)})_{n\times n}$ as equivalent consistent matrix; $\bar{w} = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_n)^T$ as the priority vector of $(I-IOWACJM)$ $\bar{M}$, and $\bar{N} = (\bar{p}_{ij})_{n\times n}$ as the equavelent consistent matrix of $\bar{M}$.

Theorem 4.11 Applying the $(I-IOWACJM)$ as the aggregation technique, the weighting vector

$$\lambda = (\mu_1, \mu_2, ..., \mu_n)^T, \sum_{l=1}^{m} \mu_1 = 1,$$

The row arithmetic mean method as the prioritization method, such that the (AIP) and the (AIJ) provides the same priorities of alternatives.

Proof. Let $w^{(l)} = (w_1^{(l)}, w_2^{(l)}, ..., w_n^{(l)})^T$ be the priority of the individual judgement matrix $M^{(l)}$ and $\bar{w} = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_n)^T$ be the group priorities, then we get, i.e.

$$w_i(AIP) = I-IOWA(\langle \mu_1, w^{(1)} \rangle, \langle \mu_2, w^{(2)} \rangle, ..., \langle \mu_m, w^{(m)} \rangle)$$

$$= \sum_{l=1}^{m} w_i^{(l)} \mu_1$$

$$w_i(AIP) = \sum_{l=1}^{m} w_i^{(l)} \times \mu_1 \text{ and}$$

$$w_i(AIJ) = \frac{1}{n}\sum_{j=1}^{n} C_{ij} = \frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{m} C_{ij}^{l} \times \mu_1$$

$$= \sum_{l=1}^{m} \mu_1 (\sum_{j=1}^{n} \frac{1}{n} C_{ij}^{(l)}) = \sum_{l=1}^{m} w_i^{(l)} \mu_1$$

Thus $w_i(AIP) = w_i(AIJ)$.

Theorem 4.12 Suppose $M^{(1)}, M^{(2)}, ..., M^{(m)}$ be the (CPR) provided by m decision maker's when comparing n alternatives with the corresponding weighting vector

$$\delta = (\delta_{\alpha(1)}, \delta_{\alpha(2)}, ..., \delta_{\alpha(m)})^T, \delta_{\alpha(l-1)} \geq \delta_{\alpha(l)}, \sum_{l=1}^{m} \delta_{\alpha(l)} = 1.$$

Applying the $(I-IOWACJM)$ as the aggregation procedure and the $(RAMM)$ as the prioritization procedure, it holds that:

$$CI(\bar{M}) \geq \sum_{l=1}^{m} \mu_1 CI(M^{(l)}) \qquad (11)$$

Proof. Definition 22 and Eq. (4), we have

$$CI(\bar{M}) = 1 - d(\bar{M}, \bar{N}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|C_{ij} - \bar{p}_{ij}\right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\begin{matrix}\sum_{l=1}^{m} C_{ij}^{(l)} \times \mu_1 \\ -\sum_{l=1}^{m} p_{ij}^{(l)} \times \mu_1\end{matrix}\right|$$

$$= 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\sum_{l=1}^{m}(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right|.$$

Since $\left|\sum_{l=1}^{m}(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right| \le \sum_{l=1}^{m}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right|.$

Then $CI(\bar{M}) \ge 1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{m}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\mu_1\right|$

$$= 1 - \frac{1}{n}\sum_{l=1}^{m}\mu_1\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\right|\right.$$

$$= \sum_{l=1}^{m}\mu_1\left(1 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(C_{ij}^{(l)} - p_{ij}^{(l)})\right|\right)$$

$$= \sum_{l=1}^{m}\mu_1 CI(M^{(l)}).$$

Corollary If the individual cubic judegements $\{M^{(1)}, M^{(2)}, ..., M^{(m)}\}$ are of acceptable consistency, then the $(I-IOWACJM)$ M is also acceptable consistency, that is to say,

$$CI(M^{(l)}) \ge \tau, \text{ for all } l = 1,...,m \Rightarrow CI(\bar{M}) \ge \tau, \qquad (12)$$

where τ is for acceptable consistency.

Corollary The consistency degree of $\overline{M}$ is more than the minimum of the consistency degree between $M^{(l)}$, i.e.

$$CI(\bar{M}) \ge Min_{l=1,...,m}\{CI(M^{(l)})\} \qquad (13)$$

4. Numerical Example

Consider there are the set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$, and four (DMs), $D = \{d_1, d_2, d_3, d_4\}$. Suppose that these decision maker's provide the following (CPR) on the set of alternative.

$$M^{(1)} = \begin{bmatrix} \langle[0.5,0.5],0.5\rangle\langle[0.3,0.4],0.6\rangle\langle[0.6,0.7],0.3\rangle\langle[0.7,0.8],0.3\rangle \\ \langle[0.6,0.7],0.4\rangle\langle[0.5,0.5],0.5\rangle\langle[0.6,0.7],0.8\rangle\langle[0.3,0.5],0.4\rangle \\ \langle[0.3,0.4],0.7\rangle\langle[0.3,0.4],0.2\rangle\langle[0.5,0.5],0.5\rangle\langle[0.6,0.7],0.5\rangle \\ \langle[0.2,0.3],0.7\rangle\langle[0.5,0.7],0.6\rangle\langle[0.3,0.4],0.5\rangle\langle[0.5,0.5],0.5\rangle \end{bmatrix}$$

$$M^{(2)} = \begin{bmatrix} \langle[0.5,0.5],0.5\rangle \langle[0.3,0.4],0.2\rangle \langle[0.4,0.5],0.4\rangle \langle[0.2,0.3],0.6\rangle \\ \langle[0.6,0.7],0.8\rangle \langle[0.5,0.5],0.5\rangle \langle[0.4,0.5],0.6\rangle \langle[0.5,0.6],0.4\rangle \\ \langle[0.5,0.6],0.6\rangle \langle[0.5,0.6],0.4\rangle \langle[0.5,0.5],0.5\rangle \langle[0.3,0.5],0.6\rangle \\ \langle[0.7,0.8],0.4\rangle \langle[0.4,0.5],0.6\rangle \langle[0.5,0.7],0.4\rangle \langle[0.5,0.5],0.5\rangle \end{bmatrix}$$

$$M^{(3)} = \begin{bmatrix} \langle[0.5,0.5],0.5\rangle \langle[0.3,0.4],0.6\rangle \langle[0.6,0.7],0.2\rangle \langle[0.4,0.5],0.3\rangle \\ \langle[0.6,0.7],0.4\rangle \langle[0.5,0.5],0.5\rangle \langle[0.5,0.6],0.1\rangle \langle[0.7,0.8],0.2\rangle \\ \langle[0.3,0.4],0.8\rangle \langle[0.4,0.5],0.9\rangle \langle[0.5,0.5],0.5\rangle \langle[0.3,0.6],0.5\rangle \\ \langle[0.5,0.6],0.7\rangle \langle[0.2,0.3],0.8\rangle \langle[0.4,0.7],0.5\rangle \langle[0.5,0.5],0.5\rangle \end{bmatrix}$$

$$M^{(4)} = \begin{bmatrix} \langle[0.5,0.5],0.5\rangle \langle[0.4,0.5],0.6\rangle \langle[0.6,0.7],0.2\rangle \langle[0.6,0.7],0.3\rangle \\ \langle[0.5,0.6],0.4\rangle \langle[0.5,0.5],0.5\rangle \langle[0.3,0.4],0.5\rangle \langle[0.4,0.5],0.4\rangle \\ \langle[0.3,0.4],0.8\rangle \langle[0.6,0.7],0.5\rangle \langle[0.5,0.5],0.5\rangle \langle[0.3,0.4],0.8\rangle \\ \langle[0.3,0.4],0.7\rangle \langle[0.5,0.6],0.6\rangle \langle[0.6,0.7],0.2\rangle \langle[0.5,0.5],0.5\rangle \end{bmatrix}$$

By using the above procedure, we can obtain four consistent matrices as follows:

$N^{(1)} =$

$\langle[0.5000, 0.5000], 0.5000\rangle$	$\langle[0.3753, 0.4444], 0.5357\rangle$
$\langle[0.5556, 0.6251], 0.4643\rangle$	$\langle[0.5000, 0.5000], 0.5000\rangle$
$\langle[0.4445, 0.5630], 0.3581\rangle$	$\langle[0.3889, 0.4375], 0.3929\rangle$
$\langle[0.3334, 0.5000], 0.3929\rangle$	$\langle[0.2778, 0.3751], 0.3929\rangle$
$\langle[0.4371, 0.5505], 0.4221\rangle$	$\langle[0.5000, 0.6666], 0.6071\rangle$
$\langle[0.5625, 0.6111], 0.6071\rangle$	$\langle[0.6253, 0.7222], 0.6071\rangle$
$\langle[0.5000, 0.5000], 0.5000\rangle$	$\langle[0.5625, 0.6111], 0.5000\rangle$
$\langle[0.3889, 0.4375], 0.5000\rangle$	$\langle[0.5000, 0.5000], 0.5000\rangle$

$N^{(2)} =$

$\langle[0.5000, 0.5000], 0.5000\rangle$	$\langle[0.3571, 0.4375], 0.3750\rangle$
$\langle[0.5625, 0.6429], 0.6250\rangle$	$\langle[0.5000, 0.5000], 0.5000\rangle$
$\langle[0.5625, 0.7143], 0.5417\rangle$	$\langle[0.4375, 0.5000], 0.5417\rangle$
$\langle[0.5625, 0.5872], 0.5417\rangle$	$\langle[0.5625, 0.7143], 0.4167\rangle$
$\langle[0.2857, 0.4375], 0.6166\rangle$	$\langle[0.4128, 0.4375], 0.4583\rangle$
$\langle[0.5000, 0.5625], 0.4583\rangle$	$\langle[0.2857, 0.4375], 0.5833\rangle$
$\langle[0.5000, 0.5000], 0.5000\rangle$	$\langle[0.3571, 0.5000], 0.5416\rangle$
$\langle[0.5000, 0.6429], 0.5484\rangle$	$\langle[0.5000, 0.5000], 0.5000\rangle$

$$N^{(3)}=\begin{bmatrix}\langle[0.5000,0.5000],0.5000\rangle & \langle[0.3333,0.4444],0.5416\rangle\\ \langle[0.5555,0.6667],0.4584\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \langle[0.5000,0.6667],0.6253\rangle & \langle[0.4445,0.5000],0.6667\rangle\\ \langle[0.4445,0.3344],0.4167\rangle & \langle[0.3889,0.6667],0.6667\rangle\\ \langle[0.3333,0.5000],0.3750\rangle & \langle[0.1666,0.5555],0.5833\rangle\\ \langle[0.5000,0.5555],0.3333\rangle & \langle[0.3333,0.6111],0.3330\rangle\\ \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3333,0.5555],0.5000\rangle\\ \langle[0.4445,0.6667],0.5000\rangle & \langle[0.5000,0.5000],0.5000\rangle\end{bmatrix}$$

$$N^{(4)}=\begin{bmatrix}\langle[0.5000,0.5000],0.5000\rangle & \langle[0.4164,0.5000],0.5384\rangle\\ \langle[0.5000,0.5834],0.4616\rangle & \langle[0.5000,0.5000],0.5000\rangle\\ \langle[0.5625,0.7510],0.4616\rangle & \langle[0.6251,0.7500],0.5000\rangle\\ \langle[0.6235,0.8334],0.3834\rangle & \langle[0.6253,0.3847],0.3847\rangle\\ \langle[0.2502,0.7375],0.5384\rangle & \langle[0.1666,0.3750],0.6153\rangle\\ \langle[0.2500,0.3750],0.5000\rangle & \langle[0.1666,0.3759],0.6153\rangle\\ \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3333,0.4375],0.6135\rangle\\ \langle[0.5625,0.6667],0.6667\rangle & \langle[0.5000,0.5000],0.5000\rangle\end{bmatrix}$$

According to E.q.(4), we can calculate the consistency degree $CI(M^l)$, $l=1,2,3,4$:

$$CI(M^1)=0.5481,\ CI(M^2)=0.6701,\ CI(M^3)=0.5984,\ CI(M^4)=0.499$$

and the judgment matrices $M^{(1)},M^{(2)},M^{(3)},M^{(4)}$ and having equivalent consistent matrices $N^{(1)},N^{(2)},N^{(3)},N^{(4)}$ are reordered as follows respectively:

$$M^{(\alpha(1))}=M^{(2)};\ M^{(\alpha(2))}=M^{(3)};\ M^{(\alpha(3))}=M^{(1)};\ M^{(\alpha(4))}=M^{(4)};$$
$$N^{(\alpha(1))}=N^{(2)};\ N^{(\alpha(2))}=N^{(3)};\ N^{(\alpha(3))}=N^{(1)};\ N^{(\alpha(4))}=N^{(4)};$$

Using E.q. (8) with $Q(r)=r^{\frac{1}{2}}$, we obtain the weight as followes:

$$\delta_{\alpha(1)}=0.51;\ \delta_{\alpha(2)}=0.19;\ \delta_{\alpha(3)}=0.23;\ \delta_{\alpha(4)}=0.07.$$

Then, the $(C-IOWACJM)$ $\overline{M}_1$ and its equivalent consistent matrix $\overline{P}_1$ are calculated as;

$$\overline{M}_1 = \begin{bmatrix} \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3075,0.4077],0.3426\rangle \\ \langle[0.3937,0.6939],0.5696\rangle & \langle[0.5000,0.5000],0.5000\rangle \\ \langle[0.4014,0.5121],0.6421\rangle & \langle[0.4494,0.5510],0.4041\rangle \\ \langle[0.5604,0.6713],0.5262\rangle & \langle[0.4001,0.5334],0.6336\rangle \\ \langle[0.5081,0.6107],0.3126\rangle & \langle[0.4242,0.5361],0.4227\rangle \\ \langle[0.4664,0.5684],0.4503\rangle & \langle[0.5035,0.6225],0.3507\rangle \\ \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3845,0.5684],0.5670\rangle \\ \langle[0.4494,0.6481],0.5000\rangle & \langle[0.5000,0.5000],0.5000\rangle \end{bmatrix}$$

$$\overline{P}_1 = \begin{bmatrix} \langle[0.5000,0.5000],0.5000\rangle & \langle[0.3675,0.4449],0.4479\rangle \\ \langle[0.5554,0.6369],0.5388\rangle & \langle[0.5000,0.5000],0.5000\rangle \\ \langle[0.5259,0.6786],0.5197\rangle & \langle[0.4441,0.5106],0.5204\rangle \\ \langle[0.5011,0.5567],0.4673\rangle & \langle[0.4825,0.6608],0.4470\rangle \\ \langle[0.3257,0.4764],0.3126\rangle & \langle[0.4874,0.5137],0.5216\rangle \\ \langle[0.5012,0.5021],0.4630\rangle & \langle[0.3855,0.5509],0.5312\rangle \\ \langle[0.5000,0.5000],0.5000\rangle & \langle[0.4059,0.5347],0.5284\rangle \\ \langle[0.4708,0.8024],0.5146\rangle & \langle[0.5000,0.5000],0.5000\rangle \end{bmatrix}$$

A/to Definition 16 and, E.q. (4) we get such that

$$CI(\overline{M}_1) = 0.7487 > \frac{1}{4}\sum_{l=1}^{4} CI(M^l) = \frac{0.5481+0.6701+0.5984+0.499}{4} = 0.5789.$$

This result is in accordance with Theorem 5.

5. Conclusion

We have discussed the properties of $IOWA$ operators in the aggregation of CPR in group decision making problems in this paper. We have also defined that the collective preference get by these cases of $IOWA$ operators which shown the reciprocity and consistency conditions. Then, it is verified that the aggregation of individual judgments and the aggregation of individual properties define the same properties of the alternatives by applying $RAMM$ as prioritization technique and $IOWA$ operators as aggregation technique. By using the distance between $M^{(l)}$ and its corresponding consistent matrix N $^{(l)}$, we present the consistency index of CPR. Using this consistency measure, we proved that the $C-IOWA$ and the $I-IOWA$ operator can improve consistency degree in the collective CPR. In a future we plan that we will extend this work.

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