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An investigation into the asymptotic stability analysis of delayed *q*-fractional neutral systems

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ABSTRACT

This research analyzes the asymptotic stability of delayed *q*-fractional neutral systems. By developing suitable Lyapunov-Krasovskii functionals (LKFs), certain sufficient conditions for asymptotic stability are derived using linear matrix inequalities (LMIs). The approach used in this paper relies on directly calculating the quantum derivatives of the LKFs. Lastly, we provide two numerical examples to demonstrate how our theoretical findings can be applied.

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1. Introduction

Time delays in information flow within different components of dynamic systems often lead to instability and are commonly observed in various engineering applications, including chemical processes, long-distance transmission lines, and microwave oscillators. Due to their ability to model natural phenomena more effectively than ordinary differential equations, fractional calculus have garnered significant attention from scholars worldwide. Therefore, mathematicians in particular currently appear to have a strong interest in fractional calculus. It is well known that neutral systems, which are of a more general class than those of the delayed type, have been an active area of research in recent years. The stability of neutral systems has proven to be a more complex question, since the system under consideration involves the derivation of the delayed state. In particular, over the past few decades, numerous books and articles have been explored concerning retarded-type or neutral-type differential and fractional differential systems. In this sense, the stability problem, which is one of the important problems in theoretical and practical applications, is considered an index in the study of fractional systems, and numerous articles have addressed different types of stability in such systems without delay and delay. Fractional systems, both with and without time-varying delays, frequently appear in various scientific disciplines, including fields such as engineering, physics, biophysics, polymer rheology, blood flow dynamics, control theory, biology, and signal processing (see [1–37] and references therein).

Based on insights from the relevant literature, fractional calculus often provide a more accurate description of natural phenomena compared to ordinary differential equations. Consequently, this topic has been widely explored by numerous renowned scientists (see [36]). For those interested, several studies in this rapidly growing field are listed below.

The q-fractional calculus, which was brought to the attention of researchers by some q-fractional derivatives and q-fractional integrals, was first discussed by Agarwal [1] and Al-salam [3] in 1969. In [8], Chartbupapan et al. explored the asymptotic stability of nonlinear fractional Riemann-Liouville (RL) differential equations with a fixed delay and included examples to demonstrate the validity of the conditions they established. In [29], presented a simple mathematical approach to explain when and why fractional Brownian motion is appropriate for economic modeling. In [27], various physical implications examined that are pertinent to dynamical processes in complex systems. In [25], three bioengineering research areas bio electrodes, biomechanics, and bioimaging are described, which have been utilized to develop new mathematical models based on fractional calculus. In [11], Jarad et al. examined the conditions for stability, uniform stability, and asymptotic stability within the framework of non-autonomous Caputo fractional derivatives using the Lyapunov direct method. In [24], Lu et al. derived various sufficient conditions for the asymptotic stability and Mittag-Leffler stability of fractional nonlinear neutral singular systems using the Lyapunov direct method. Zhang et al. [37], on the other hand, studied the global asymptotic stability of delayed fractional RL neural networks. Additionally, several books on these topics can be referenced. Examples include works on q-fractional calculus [5, 10, 12], fractional calculus [13, 28], and singular systems [33, 36]. Drawing inspiration from the studies mentioned above and their references, we have focused on nonlinear time delay q-fractional neutral systems.

It is observed that the stability method in the majority of studies concerning system stability in control theory is primarily based on LMIs and Lyapunov stability theory, as seen in references [4, 6, 9, 15–23, 30]. The Lyapunov direct method plays a central role in investigating stability in differential equations. The LKF explicitly incorporates terms that account for time delays in the system (including distributed or discrete delays). This allows it to effectively model the dynamic behavior of systems where the current state depends not only on the current time but also on past states.In contrast, classical Lyapunov functions typically focus on delay-free systems or only handle delays in an approximate or conservative manner. Advantage of LKF, by including integral terms related to the delay, LKFs provide a more flexible structure for capturing the dynamics of time-delay systems. This often leads to less conservative stability conditions, allowing more accurate determination of the system's stability region. In this research, the proof technique relies on fundamental inequalities, LMIs, and the Lyapunov functional approach.

In comparison to the existing literature, the main contributions of this paper can be succinctly outlined as follows:

(i) This study relies on the LMI and the direct calculation of quantum derivatives of LKFs. Therefore, there is no need to compute the fractional-order derivative of the Lyapunov functional. The derived stability criteria are formulated as LMIs, making them both convenient and effective for testing the asymptotically stability of the considered systems.
(ii) In this study, motivated by the above-mentioned findings, the asymptotically stability of time-delayed nonlinear q-fractional neutral systems is investigated by developing appropriate LKFs;
(iii) In this study, several delay dependent sufficient conditions for asymptotic stability are derived.
(iv) In this study believe that the theoretical results obtained are both intriguing and contemporary, providing a significant contribution to the existing literature. When the studies on stability analysis of q-fractional neutral systems are examined, the equation system considered in this research is new and generalizes similar studies in the related literature.

In this study, we describe delayed q-fractional neutral systems. In addition, we introduce fundamental definitions and properties of quantum calculus, along with q-fractional integrals and derivatives. Next, using the Lyapunov method, we derive sufficient conditions to demonstrate the asymptotic stability of the considered system, leveraging foundational information and inequalities.
Consider the following delayed nonlinear q-fractional system:

$$\begin{aligned}\nabla_q^\alpha x(t) &= Ax(t)+Bx(t-\tau)+C\nabla_q^\alpha x(t-\tau)+Df(x(t)) \\ &\quad + Ef(x(t-\tau))+Ff(\nabla_q^\alpha x(t-\tau)), \quad t\geq 0,\end{aligned} \tag{1}$$

with the initial value condition as follows:

$$I_q^{1-\alpha} x(t)=\varphi(t), \quad t\in[-\tau,0], \quad 0<\alpha<1,$$

where the state vector $x(t)\in\mathbb{R}^n, \tau<0$ is a constant time delay, $A,B,B,D,E,F\in\mathbb{R}^{n\times n}$ are constant system matrices, $f(x(t)), f(x(t-\tau))$ and $f\left(\nabla_q^\alpha x(t-\tau)\right)\in\mathbb{R}^n$represent the nonlinear terms of system (1), which satisfy that

$$\begin{aligned}\|f(x(t))\| &\leq \eta_1\|x(t)\| \\ \|f(x(t-\tau))\| &\leq \eta_2\|x(t-\tau)\| \\ \|f(\nabla_q^\alpha x(t-\tau))\| &\leq \eta_3\|\nabla_q^\alpha x(t-\tau)\|\end{aligned} \tag{2}$$

where η_1,η_2 and η_3 are positive real constants. Constraints described by (2) can be rewritten as follows

$$\begin{aligned}f^T(x(t))f(x(t)) &\leq \eta_1^2 x^T(t)x(t) \\ f^T(x(t-\tau))f(x(t-\tau)) &\leq \eta_2^2 x^T(t-\tau)x(t-\tau) \\ f^T(\nabla_q^\alpha x(t-\tau))f(\nabla_q^\alpha x(t-\tau)) &\leq \eta_3^2(\nabla_q^\alpha x(t-\tau))^T(\nabla_q^\alpha x(t-\tau)).\end{aligned} \tag{3}$$

Definition 1 ([28]) The RL fractional integral of order $p>0$ of function g is described as

$${}_{t_0}D_t^{-p}\{g(t)\}=\frac{1}{\Gamma(p)}\int_{t_0}^t (t-s)^{p-1}g(s)ds, \quad t\geq t_0 .$$

Definition 2 ([28]) The RL fractional derivative of order p for a function g is described as

$$ {}_{t_0}^{RL}D_t^p\left\{g(t)\right\}=\frac{1}{\Gamma(n-p)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{g(s)}{(t-s)^{p-n+1}}ds, $$

where $0 \le n-1 \le p < n, n \in Z^{+}$ and Γ denotes the Gamma function.

Definition 3 ([7])**.** For $q \in (0,1)$, $T_q=\left\{q^n : n \in Z\right\}\cup\left\{0\right\}$ is defined as time scale, where Z is the set of all integers.

Definition 4 ([11])**.** The trivial solution $x(t)=0$ of system (1) is said to be

1. Stable, if for each $\varepsilon>0$ and $t_0 \in T_q$, there exists $\delta=\delta(\varepsilon,t_0)>0$ such that for any solution $x(t)=x(t,t_0,\varphi)$ with $\varphi(t)<\delta$, we always have $\left\|x(t)\right\|<\varepsilon$, for all $t\in T_q,\ t\ge t_0$;
2. Uniformly stable, if it is stable and δ depends only ε;
3. Asymptotically stable, if it is stable and for all $t_0 \in T_q$, there exists $\delta=\delta(t_0)>0$ such that if $\varphi(t)<\delta$ implies that $\lim_{t\to\infty} x(t,t_0,\varphi)=0.$

Lemma 1 ([37])**.** Assume that $S \in \mathbb{R}^{n\times n}, S=S^T>0,$ is a constant matrix and $x(t) \in \mathbb{R}^n$ be a vector of q-fractional differentiable function. Therefore, $\forall t \in T_q,\ t>0,$

$$ \nabla_q^{\alpha}(x^{\mathrm{T}}(t)Sx(t)) \le 2x^{\mathrm{T}}(t)S\nabla_q^{\alpha}x(t), \quad 0<\alpha<1, $$

is satisfied.

Lemma 2 ([19])**.** The homogeneous difference operator $\Theta: \mathbb{R}([-\tau,0],\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is defined to be $\Theta:\Theta(x_t)=x(t)-Cx(t-\tau)$. The operator Θ is stable if $\left\|C\right\|<1.$

The notations listed below will be employed throughout this research: $\mathbb{R}$ denotes the set of all real numbers; n represent the dimension of the space; $\mathbb{R}^n$ represents the space of all n-tuples of real numbers: $\mathbb{R}^n=\{(x_1,x_2,\dots,x_n)|x_i\in\mathbb{R}^n, i=1,2,\dots,n\}$; $\|.\|$ represents the Euclidean norm for vectors; A^T means the transpose of the matrix A; B is symmetric if $B=B^T$; C is positive definite (or negative definite) if $\langle Cx,x\rangle>0$ (or $\langle Cx,x\rangle<0$) for all $x\neq 0$; $\|D\|$ represents the spectral norm of matrix D; "*" means conjugate transpose.

2. Main results

We first present a result for the asymptotically stable of the trivial solution of system (1) with $D=E=F=0,$ as follows

$$ \nabla_q^{\alpha}x(t)=Ax(t)+Bx(t-\tau)+C\nabla_q^{\alpha}x(t-\tau),\ t>0. \tag{4} $$

Theorem 2.1 For given scalar $\tau>0,$ the trivial solution of system (4) is asymptotically stable, if $\left\|C\right\|<1$ and there exist symmetric positive definite matrices P,Q,S and U such that the following LMI holds:

$$ \Pi=\begin{bmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ * & \Pi_{22} & \Pi_{23}\\ * & * & \Pi_{33}\end{bmatrix}<0. \tag{5} $$

where,

$$ \Pi_{11}=PA+A^TP+Q+A^T(S+\tau U)A, $$
$$ \Pi_{12}=PB+A^T(S+\tau U)B, $$
$$ \Pi_{13}=PC+A^T(S+\tau U)C, $$
$$ \Pi_{22}=B^T(S+\tau U)B-Q, $$
$$ \Pi_{23}=B^T(S+\tau U)C, $$
$$ \Pi_{33}=C^T(S+\tau U)C-S. $$

Proof. Let us select the following Lyapunov-Krasovskii functional

$$ \begin{aligned} V(x(t)) &= I_q^{1-\alpha}(x^{\mathrm{T}}(t)Px(t))+\int_{t-\tau}^{t}x^{\mathrm{T}}(s)Qx(s)\nabla_q s \\ &+\int_{t-\tau}^{t}\nabla_q^{\alpha}x^T(s)S\nabla_q^{\alpha}x(s)\nabla_q s \\ &+\int_{t-\tau}^{t}\int_{\theta}^{t}\nabla_q^{\alpha}x^T(s)U\nabla_q^{\alpha}x(s)\nabla_q s\nabla_q\theta. \end{aligned} $$

Clearly, $V(t)>0.$ Let $\Psi(t)=\int_{t-\tau}^{t}\int_{\theta}^{t}\nabla_q^{\alpha}x^T(s)U\nabla_q^{\alpha}x(s)\nabla_q s\nabla_q\theta.$

Based on the preceding results, we obtain

$$ \begin{aligned} \nabla_q\Psi(t)=\nabla_q^{\alpha}\Bigg(&\int_{t-\tau}^{t}\int_{0}^{t}\nabla_q^{\alpha}x^T(s)U\nabla_q^{\alpha}x(s)\nabla_q s\nabla_q\theta \\ &-\int_{t-\tau}^{t}\int_{0}^{\theta}\nabla_q^{\alpha}x^T(s)U\nabla_q^{\alpha}x(s)\nabla_q s\nabla_q\theta\Bigg) \end{aligned} $$

$$= \nabla_q^\alpha \left(\tau \int_0^t \nabla_q^\alpha x^T(s) U \nabla_q^\alpha x(s) \nabla_q s - \int_{t-\tau}^t \int_0^\theta \nabla_q^\alpha x^T(s) U \nabla_q^\alpha x(s) \nabla_q s \nabla_q \theta \right)$$

$$= \tau \nabla_q^\alpha x^T(t) U \nabla_q^\alpha x(t) - \int_{t-\tau}^t \nabla_q^\alpha x^T(s) U \nabla_q^\alpha x(s) \nabla_q s. \tag{6}$$

From $\nabla_q^\alpha I_q^\beta f(t) = \nabla_q^{\alpha-\beta} f(t),\ \alpha > \beta \geq 0$ and (6), we can derive the q-derivate of $V(t)$ along the trajectories of the system (4), we can write the as follows:

$$\begin{aligned} \nabla_q V(x(t)) = {} & \nabla_q^\alpha (x^T(t) P x(t)) + x^T(t) Q x(t) \\ & - x^T(t-\tau) Q x(t-\tau) + \nabla_q^\alpha x^T(t) S \nabla_q^\alpha x(t) \\ & - \nabla_q^\alpha x^T(t-\tau) S \nabla_q^\alpha x(t-\tau) \\ & + \tau \nabla_q^\alpha x^T(t) U \nabla_q^\alpha x(t) \\ & - \int_{t-\tau}^t \nabla_q^\alpha x^T(s) U \nabla_q^\alpha x(s) \nabla_q s. \end{aligned} \tag{7}$$

Since U is positive definite matrix, then

$$\int_{t-\tau}^t \nabla_q^\alpha x^T(s) U \nabla_q^\alpha x(s) \nabla_q s = \int_0^\tau \nabla_q^\alpha x^T(t-s) U \nabla_q^\alpha x(t-s) \nabla_q s \geq 0. \tag{8}$$

From (7)-(8), then

$$\begin{aligned} \nabla_q V(x(t)) \leq {} & \nabla_q^\alpha (x^T(t) P x(t)) + x^T(t) Q x(t) \\ & - x^T(t-\tau) Q x(t-\tau) + \nabla_q^\alpha x^T(t) S \nabla_q^\alpha x(t) \\ & - \nabla_q^\alpha x^T(t-\tau) S \nabla_q^\alpha x(t-\tau) \\ & + \tau \nabla_q^\alpha x^T(t) U \nabla_q^\alpha x(t) \end{aligned} \tag{9}$$

From Lemma 1, we obtain

$$\begin{aligned} & \nabla_q^\alpha (x^T(t) P x(t)) \leq 2 x^T(t) P \nabla_q^\alpha x(t) \\ & = 2 x^T(t) P \left(Ax(t) + Bx(t-\tau) + C \nabla_q^\alpha x(t-\tau) \right) \\ & = x^T(t)(PA + A^T P) x(t) + 2 x^T(t) P B x(t-\tau) \\ & \quad + 2 x^T(t) P C \nabla_q^\alpha x(t-\tau) \end{aligned} \tag{10}$$

and

$$\begin{aligned} & \nabla_q^\alpha x^T(t)(S+\tau U) \nabla_q^\alpha x(t) = \left(Ax(t) + Bx(t-\tau) + C\nabla_q^\alpha x(t-\tau) \right) \\ & (S+\tau U)\left(Ax(t) + Bx(t-\tau) + C\nabla_q^\alpha x(t-\tau) \right) \\ & = x^T(t) A^T (S+\tau U) A x(t) \\ & \quad + x^T(t) A^T (S+\tau U) B x(t-\tau) \\ & \quad + x^T(t) A^T (S+\tau U) C \nabla_q^\alpha x(t-\tau) \\ & \quad + x^T(t-\tau) B^T (S+\tau U) A x(t) \\ & \quad + x^T(t-\tau) B^T (S+\tau U) B x(t-\tau) \\ & \quad + x^T(t-\tau) B^T (S+\tau U) C \nabla_q^\alpha x(t-\tau) \\ & \quad + (\nabla_q^\alpha x(t-\tau))^T C^T (S+\tau U) A x(t) \\ & \quad + (\nabla_q^\alpha x(t-\tau))^T C^T (S+\tau U) B x(t-\tau) \\ & \quad + (\nabla_q^\alpha x(t-\tau))^T C^T (S+\tau U) C \nabla_q^\alpha x(t-\tau) \end{aligned} \tag{11}$$

Substituting (10) and (11) into (9), we get

$$\nabla_q V(x(t)) \leq \Xi^{\mathrm{T}} \Pi \Xi, \tag{12}$$

where

$$\Xi = \left(x^T(t) \quad x^T(t-\tau) \quad (\nabla_q^\alpha x(t-\tau))^T \right)^T$$

and the matrix Π, is defined with (5).

From (5) and (12), $\nabla_q V(t) < 0.$ Since the the conditions outlined in Theorem 2.1 are met, the trivial solution of the linear q-fractional system (1) is asymptotically stable.

The following theorem presents the asymptotically stability of the trivial solution of the system (1), which is another main result of this study.

Theorem 2.2 For given scalars $\tau, \eta_1, \eta_2, \eta_3 > 0$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0,$ the trivial solution of system (1) is asymptotically stable, if $\|C\| < 1$ and there exist symmetric positive definite matrices P, Q, S, U and W such that the following LMI holds:

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} & \Delta_{16} \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} & \Delta_{25} & \Delta_{26} \\ * & * & \Delta_{33} & \Delta_{34} & \Delta_{35} & \Delta_{36} \\ * & * & * & \Delta_{44} & \Delta_{45} & \Delta_{46} \\ * & * & * & * & \Delta_{55} & \Delta_{56} \\ * & * & * & * & * & \Delta_{66} \end{bmatrix} < 0, \tag{13}$$

where,

$$\Delta_{11} = PA + A^T P + Q + A^T (S+\tau U) A + \varepsilon_1 \eta_1^2 I,$$

$$\Delta_{12} = PB + A^T (S+\tau U) B,$$

$$
\begin{aligned}
\Delta_{13} &= PC + A^T(S+\tau U)C,\\
\Delta_{14} &= PD + A^T(S+\tau U)D,\\
\Delta_{15} &= PE + A^T(S+\tau U)E,\\
\Delta_{16} &= PF + A^T(S+\tau U)F,\\
\Delta_{22} &= B^T(S+\tau U)B + \varepsilon_2\eta_2^2 I - Q,\\
\Delta_{23} &= B^T(S+\tau U)C,\\
\Delta_{24} &= B^T(S+\tau U)D,\\
\Delta_{25} &= B^T(S+\tau U)E,\\
\Delta_{26} &= B^T(S+\tau U)F,\\
\Delta_{33} &= C^T(S+\tau U)C + \varepsilon_3\eta_3^2 I - S,\\
\Delta_{34} &= C^T(S+\tau U)D,\\
\Delta_{35} &= C^T(S+\tau U)E,\\
\Delta_{36} &= C^T(S+\tau U)F,\\
\Delta_{44} &= W + D^T(S+\tau U)D - \varepsilon_1 I,\\
\Delta_{45} &= D^T(S+\tau U)E,\\
\Delta_{46} &= D^T(S+\tau U)F,\\
\Delta_{55} &= E^T(S+\tau U)E - \varepsilon_2 I - W,\\
\Delta_{56} &= E^T(S+\tau U)F,\\
\Delta_{66} &= F^T(S+\tau U)F - \varepsilon_3 I.
\end{aligned}
$$

Proof. Let us select the following Lyapunov-Krasovskii functional

$$
\begin{aligned}
V(x(t)) = I_q^{1-\alpha}(x^{\mathrm{T}}(t)Px(t)) &+ \int_{t-\tau}^{t} x^{\mathrm{T}}(s)Qx(s)\nabla_q s\\
&+ \int_{t-\tau}^{t} \nabla_q^\alpha x^T(s)S\nabla_q^\alpha x(s)\nabla_q s\\
&+ \int_{t-\tau}^{t}\int_{\theta}^{t} \nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s\nabla_q\theta\\
&+ \int_{t-\tau}^{t} f^{\mathrm{T}}(x(s))Wf(x(s))\nabla_q s.
\end{aligned}
$$

Clearly, $V(t) > 0$. Let $\Psi(t) = \int_{t-\tau}^{t}\int_{\theta}^{t} \nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s\nabla_q\theta.$

Based on the preceding results, we obtain

$$
\begin{aligned}
\nabla_q\Psi(t) &= \nabla_q^\alpha\left(\int_{t-\tau}^{t}\int_{0}^{t}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s\nabla_q\theta - \int_{t-\tau}^{t}\int_{0}^{\theta}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s\nabla_q\theta\right)\\
&= \nabla_q^\alpha\left(\tau\int_{0}^{t}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s - \int_{t-\tau}^{t}\int_{0}^{\theta}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s\nabla_q\theta\right.\\
&= \tau\nabla_q^\alpha x^T(t)U\nabla_q^\alpha x(t)\\
&\quad - \int_{t-\tau}^{t}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s. \qquad (14)
\end{aligned}
$$

From $\nabla_q^\alpha I_q^\beta f(t) = \nabla_q^{\alpha-\beta} f(t),\ \alpha > \beta \ge 0$ and (14), we can derive the q-derivate of $V(t)$ along the trajectories of the system (1), we can write the as follows:

$$
\begin{aligned}
\nabla_q V(x(t)) = \nabla_q^\alpha(x^T(t)Px(t)) &+ x^T(t)Qx(t)\\
&- x^T(t-\tau)Qx(t-\tau) + \nabla_q^\alpha x^T(t)S\nabla_q^\alpha x(t)\\
&- \nabla_q^\alpha x^T(t-\tau)S\nabla_q^\alpha x(t-\tau)\\
&+ \tau\nabla_q^\alpha x^T(t)U\nabla_q^\alpha x(t)\\
&- \int_{t-\tau}^{t}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s\\
&+ f^T(x(t))Wf(x(t))\\
&- f^T(x(t-\tau))Wf(x(t-\tau)) \qquad (15)
\end{aligned}
$$

Since U is positive definite matrix, then

$$
\int_{t-\tau}^{t}\nabla_q^\alpha x^T(s)U\nabla_q^\alpha x(s)\nabla_q s = \int_{0}^{\tau}\nabla_q^\alpha x^T(t-s)U\nabla_q^\alpha x(t-s)\nabla_q s \ge 0. \qquad (16)
$$

From (15)-(16), then

$$
\begin{aligned}
\nabla_q V(x(t)) \le \nabla_q^\alpha(x^T(t)Px(t)) &+ x^T(t)Qx(t)\\
&- x^T(t-\tau)Qx(t-\tau) + \nabla_q^\alpha x^T(t)S\nabla_q^\alpha x(t)\\
&- \nabla_q^\alpha x^T(t-\tau)S\nabla_q^\alpha x(t-\tau)\\
&+ \tau\nabla_q^\alpha x^T(t)U\nabla_q^\alpha x(t)\\
&+ f^T(x(t))Wf(x(t))\\
&- f^T(x(t-\tau))Wf(x(t-\tau)) \qquad (17)
\end{aligned}
$$

From Lemma 1, we get

$$
\begin{aligned}
\nabla_q^\alpha(x^T(t)Px(t)) &\le 2x^T(t)P\nabla_q^\alpha x(t)\\
&= 2x^T(t)P\left(Ax(t) + Bx(t-\tau)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+C\nabla_q^\alpha x(t-\tau)+Df(x(t))\\
&+Ef(x(t-\tau))+Ff(\nabla_q^\alpha x(t-\tau))\big)\\
&=x^T(t)(PA+A^TP)x(t)\\
&+2x^T(t)PBx(t-\tau)\\
&+2x^T(t)PC\nabla_q^\alpha x(t-\tau)\\
&+2x^T(t)PDf(x(t))\\
&+2x^T(t)PEf(x(t-\tau))\\
&+2x^T(t)PFf(\nabla_q^\alpha x(t-\tau)) \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
&\nabla_q^\alpha x^T(t)(S+\tau U)\nabla_q^\alpha x(t)=(Ax(t)+Bx(t-\tau)\\
&+C\nabla_q^\alpha x(t-\tau)+Df(x(t))+Ef(x(t-\tau))\\
&+Ff(\nabla_q^\alpha x(t-\tau)))(S+\tau U)\big(Ax(t)+Bx(t-\tau)\\
&+C\nabla_q^\alpha x(t-\tau)+Df(x(t))+Ef(x(t-\tau))\\
&+Ff(\nabla_q^\alpha x(t-\tau)))\\
&=x^T(t)A^T(S+\tau U)Ax(t)+x^T(t)A^T(S+\tau U)\\
&Bx(t-\tau)+x^T(t)A^T(S+\tau U)C\nabla_q^\alpha x(t-\tau)\\
&+x^T(t)A^T(S+\tau U)Df(x(t))+x^T(t)A^T(S+\tau U)E\\
&f(x(t-\tau))+x^T(t)A^T(S+\tau U)Ff(\nabla_q^\alpha x(t-\tau)))\\
&+x^T(t-\tau)B^T(S+\tau U)Ax(t)+x^T(t-\tau)B^T\\
&(S+\tau U)Bx(t-\tau)+x^T(t-\tau)B^T(S+\tau U)C\\
&\nabla_q^\alpha x(t-\tau)+x^T(t-\tau)B^T(S+\tau U)Df(x(t))\\
&+x^T(t-\tau)B^T(S+\tau U)Ef(x(t-\tau))+x^T(t-\tau)B^T\\
&(S+\tau U)Ff(\nabla_q^\alpha x(t-\tau))+(\nabla_q^\alpha x(t-\tau))^TC^T\\
&(S+\tau U)Ax(t)+(\nabla_q^\alpha x(t-\tau))^TC^T(S+\tau U)\\
&Bx(t-\tau)+(\nabla_q^\alpha x(t-\tau))^TC^T(S+\tau U)C\nabla_q^\alpha x(t-\tau)\\
&+(\nabla_q^\alpha x(t-\tau))^TC^T(S+\tau U)Df(x(t))\\
&+(\nabla_q^\alpha x(t-\tau))^TC^T(S+\tau U)Ef(x(t-\tau))+(\nabla_q^\alpha x(t-\tau))^T\\
&C^T(S+\tau U)Ff(\nabla_q^\alpha x(t-\tau))+f^T(x(t))D^T(S+\tau U)\\
&Ax(t)+f^T(x(t))D^T(S+\tau U)Bx(t-\tau)\\
&+f^T(x(t))D^T(S+\tau U)C\nabla_q^\alpha x(t-\tau)+f^T(x(t))\\
&D^T(S+\tau U)Df(x(t))+f^T(x(t))D^T(S+\tau U)E\\
&f(x(t-\tau))+f^T(x(t))D^T(S+\tau U)Ff(\nabla_q^\alpha x(t-\tau))\\
&+f^T(x(t-\tau))E^T(S+\tau U)Ax(t)+f^T(x(t-\tau))\\
&E^T(S+\tau U)Bx(t-\tau)+f^T(x(t-\tau))E^T(S+\tau U)\\
&C\nabla_q^\alpha x(t-\tau)+f^T(x(t-\tau))E^T(S+\tau U)Df(x(t))\\
&+f^T(x(t-\tau))E^T(S+\tau U)Ef(x(t-\tau))\\
&+f^T(x(t-\tau))E^T(S+\tau U)Ff(\nabla_q^\alpha x(t-\tau))\\
&+f^T(\nabla_q^\alpha x(t-\tau))F^T(S+\tau U)Ax(t)+f^T(\nabla_q^\alpha x(t-\tau))\\
&F^T(S+\tau U)Bx(t-\tau)+f^T(\nabla_q^\alpha x(t-\tau))F^T(S+\tau U)\\
&C\nabla_q^\alpha x(t-\tau)+f^T(\nabla_q^\alpha x(t-\tau))F^T(S+\tau U)Df(x(t))\\
&+f^T(\nabla_q^\alpha x(t-\tau))F^T(S+\tau U)Ef(x(t-\tau))\\
&+f^T(\nabla_q^\alpha x(t-\tau))F^T(S+\tau U)Ff(\nabla_q^\alpha x(t-\tau)) \qquad (19)
\end{aligned}
$$

Note that for any $\varepsilon_1,\varepsilon_2,\varepsilon_3\geq 0,$ it follows from (2) and (3) that

$$\varepsilon_1[\eta_1^2x^T(t)x(t)-f^T(x(t))f(x(t))]\geq 0,$$

$$\varepsilon_2[\eta_2^2x^T(t-\tau)x(t-\tau)-f^T(x(t-\tau))f(x(t-\tau))]\geq 0,$$

and

$$\varepsilon_3[\eta_3^2(\nabla_q^\alpha x(t-\tau))^T(\nabla_q^\alpha x(t-\tau))-f^T(\nabla_q^\alpha x(t-\tau))f(\nabla_q^\alpha x(t-\tau))]\geq 0.$$

Substituting (18) and (19) into (17), we have

$$\nabla_qV(x(t))\leq\varphi^T\Delta\varphi, \qquad (20)$$

where the matrix Δ is defined with (13) and

$\varphi=\big(x(t)\quad x(t-\tau)\quad \nabla_q^\alpha x(t-\tau)\quad f(x(t))\quad f(x(t-\tau))\quad f(\nabla_q^\alpha x(t-\tau))\big).$

From (13) and (20), $\nabla_qV(t)<0.$ Since the conditions outlined in Theorem 2.2 are satisfied, the trivial solution of nonlinear q-fractional delay system (1) is asymptotically stable.

3. Numerical applications

Two numerical examples are presented below to illustrate the effectiveness of the obtained theoretical results.

Example 3.1 Let us define the below linear q-fractional neutral delay system as:

$$
\begin{aligned}
\nabla_q^\alpha x(t)&=\begin{pmatrix}-3.8 & -1.5\\ -1.8 & -2.4\end{pmatrix}x(t)+\begin{pmatrix}0.3 & -0.01\\ 0 & 0.2\end{pmatrix}x(t-\tau)\\
&+\begin{pmatrix}0.4 & 0\\ 0 & 0.3\end{pmatrix}\nabla_q^\alpha x(t-\tau),\quad t>0.
\end{aligned} \qquad (21)
$$

where $0<\alpha<1,\ 0<q<1,\ x(t)=\big(x_1(t)\quad x_2(t)\big)^T,\ \tau=1.5.$

Now, we choose

$$P=\begin{pmatrix}28 & 1\\ 1 & 20\end{pmatrix},\ Q=\begin{pmatrix}8 & 1\\ 1 & 4\end{pmatrix},\ S=\begin{pmatrix}4 & 1\\ 1 & 3\end{pmatrix},\ U=\begin{pmatrix}1.2 & 0\\ 0 & 2\end{pmatrix}.$$

Under the above assumptions, by making a straightforward calculation with the help of MATLAB software, we can show that:

$$\Pi=\begin{bmatrix} -91.5280 & -12.4000 & 1.2480 & -2.7616 & 1.6640 & -4.0800\\ -12.4000 & -40.1900 & -3.0300 & 0.9210 & -4.0400 & 1.2300\\ 1.2480 & -3.0300 & -7.4780 & -0.9574 & 0.6960 & 0.0900\\ -2.7616 & 0.9210 & -0.9574 & -3.7634 & 0.0568 & 0.3570\\ 1.6640 & -4.0400 & 0.6960 & 0.0568 & -3.0720 & -0.8800\\ -4.0800 & 1.2300 & 0.0900 & 0.3570 & -0.8800 & -2.4600 \end{bmatrix}.$$

In this case, $\Pi<0$, since all eigenvalues of matrix Π are -94.5910, -38.3651, -7.6909, -3.5436, -3.3680, and -0.9328, respectively. Thus, all the conditions of Theorem 2.1 are fulfilled. From Theorem 2.1, the trivial solution of the system (21) is asymptotically stable.

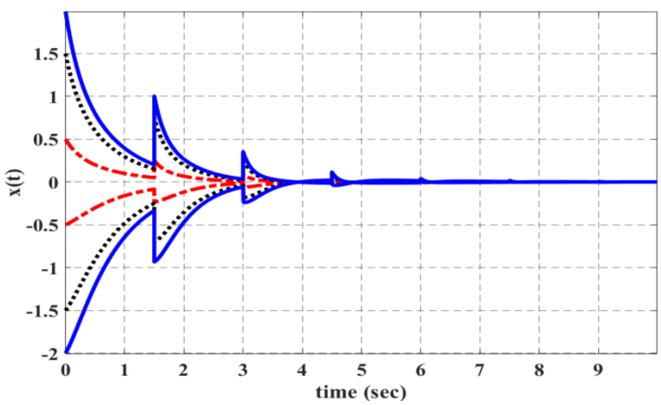

Figure 1. *The simulation of the Example 3.1 for* $\tau=1.5$.

Example 3.2 Let us define the below delayed nonlinear q-fractional system as:

$$\begin{aligned}\nabla_q^{\alpha}x(t) &= \begin{pmatrix}-4.8 & -1.2\\ -1.8 & -3.2\end{pmatrix}x(t)+\begin{pmatrix}0.3 & -0.02\\ 0 & 0.2\end{pmatrix}x(t-\tau)\\ &+\begin{pmatrix}0.5 & 0\\ 0 & 0.2\end{pmatrix}\nabla_q^{\alpha}x(t-\tau)+\begin{pmatrix}1 & -0.1\\ 0 & -2\end{pmatrix}f(x(t))\\ &+\begin{pmatrix}0.3 & -0.04\\ 0 & 0.2\end{pmatrix}f(x(t-\tau))+\begin{pmatrix}0.3 & 0\\ -0.1 & 0.2\end{pmatrix}f(\nabla_q^{\alpha}x(t-\tau)).\end{aligned} \tag{22}$$

where $0<\alpha<1,\ 0<q<1,\ x(t)=(x_1(t)\quad x_2(t))^T, \tau=1.5$.

Now, we choose $\varepsilon_1=42, \varepsilon_2=16, \varepsilon_3=20, \eta_1=0.02, \eta_2=0.03, \eta_3=0.04$,

$$P=\begin{pmatrix}24 & 2\\ 2 & 20\end{pmatrix}, Q=\begin{pmatrix}10 & 1\\ 1 & 4\end{pmatrix}, S=\begin{pmatrix}4 & 1\\ 1 & 3\end{pmatrix},$$
$$U=\begin{pmatrix}1.2 & 0\\ 0 & 0.8\end{pmatrix}, W=\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}.$$

Under the above assumptions, by making a straightforward calculation with the help of MATLAB software, we can show that:

$$\Delta = \begin{bmatrix} -63.0632 & -4.6800 & -1.6920 & -1.9592 & -2.8200 & -2.0720 & -5.6400 & 21.2840 & -1.6920 & -1.8464 & -0.6560 & -2.0720 \\ -4.6800 & -69.7432 & -2.4480 & 1.2352 & -4.0800 & 1.0720 & -8.1600 & -9.9040 & -2.4480 & 1.3984 & -2.9840 & 1.0720 \\ -1.6920 & -2.4480 & -9.4636 & -0.9748 & 0.8700 & 0.0600 & 1.7400 & -0.7740 & 0.5220 & -0.0096 & 0.4920 & 0.0600 \\ -1.9592 & 1.2352 & -0.9748 & -3.8233 & 0.0420 & 0.1640 & 0.0840 & -1.6484 & 0.0252 & 0.1606 & -0.0568 & 0.1640 \\ -2.8200 & -4.0800 & 0.8700 & 0.0420 & -2.5180 & -0.9000 & 2.9000 & -1.2900 & 0.8700 & -0.0160 & 0.8200 & 0.1000 \\ -2.0720 & 1.0720 & 0.0600 & 0.1640 & -0.9000 & -2.8000 & 0.2000 & -1.7000 & 0.0600 & 0.1600 & -0.0240 & 0.1680 \\ -5.6400 & -8.1600 & 1.7400 & 0.0840 & 2.9000 & 0.2000 & -32.2000 & -2.5800 & 1.7400 & -0.0320 & 1.6400 & 0.2000 \\ 21.2840 & -9.9040 & -0.7740 & -1.6484 & -1.2900 & -1.7000 & -2.5800 & -23.7420 & -0.7740 & -1.5968 & 0.0760 & -1.7000 \\ -1.6920 & -2.4480 & 0.5220 & 0.0252 & 0.8700 & 0.0600 & 1.7400 & -0.7740 & -19.4780 & -0.0096 & 0.5220 & 0.0252 \\ -1.8464 & 1.3984 & -0.0096 & 0.1606 & -0.0160 & 0.1600 & -0.0320 & -1.5968 & -0.0096 & -16.8387 & -0.0096 & 0.1606 \\ -0.6560 & -2.9840 & 0.4920 & -0.0568 & 0.8200 & -0.0240 & 1.6400 & 0.0760 & 0.4920 & -0.0896 & -19.4960 & -0.0240 \\ -2.0720 & 1.0720 & 0.0600 & 0.1640 & 0.1000 & 0.1680 & 0.2000 & -1.7000 & 0.0600 & 0.1600 & -0.0240 & -19.8320 \end{bmatrix}.$$

In this case, $\Delta < 0$, since all eigenvalues of matrix Δ are -75.0337, -72.1298, -31.7295, -20.7979, -20.0050, -17.5758, -18.4755, -12.1922, -9.1788, -3.5531, -1.5474 and -0.7794, respectively. Thus, all conditions of Theorem 2.2 are fulfilled. From Theorem 2.2, the trivial solution of the system (22) is asymptotically stable.

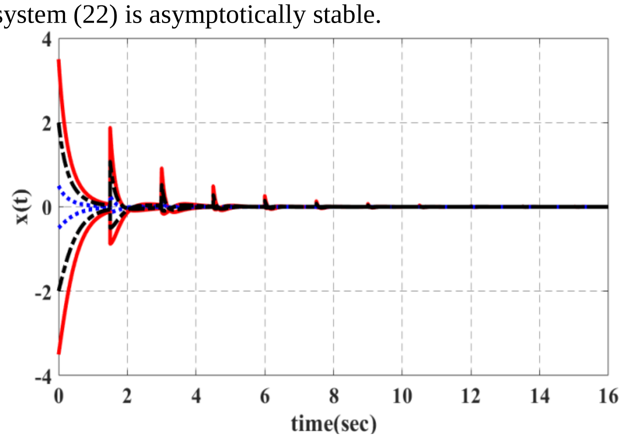

Figure 2. *The simulation of the Example* 3.2 *for* $\tau = 1.5$.

An examination of the theoretical solutions for the above examples (Examples 3.1 and 3.2) indicates that the trivial solution of the systems becomes stable after a certain time interval under different initial conditions. This stability is supported by the corresponding simulation results (Figures 1 and 2).

4. Conclusions

In this paper, we derive sufficient conditions for the asymptotic stability of certain kinds of *q*-fractional neutral type systems using LMIs and based on the direct computation of quantum derivatives of LKFs. Two examples are provided to highlight the validity of the proposed methods. In this study believe that the theoretical results obtained are both intriguing and contemporary, providing a significant contribution to the existing literature. Our future research will focus on the stability and synchronization of q-fractional systems with time-varying delays and q-fractional coupled complex networks.

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