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On left distributive ringoids of groupoids

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Abstract. In this paper, we investigate and study some types of left distributive ringoids over a field which are not rings. For this, we first introduce (quadratic-linear, linear-quadratic, quadratic-quadratic) groupoids over a field, then find their general form, and in the following left distributive ringoids are investigated with respect to these groupoids. Finally, we find the general form of the left distributive quadratic-linear, linear-quadratic, and quadratic-quadratic ringoids over a field K.

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1. Introduction and Preliminaries

A groupoid is an algebraic structure $(G, *)$ consisting of a nonempty set G and a binary operation "$*$" on G. For more information on groupoids the following papers and books are useful: [1]- [7]. The concept of a generalization in mathematics is very important to mathematicians for their research. In 2022, Neggers et al. [8] as the generalization of groupoids, rings, semi-rings and near-rings introduced the notion of a ringoid, and presented several properties of it. They studied several different types of ringoids. They constructed right distributive ringoids over a field and the geometric observations of the (strongly) orthogonal of vectors over a ringoid was investigated. Then, Rezaei et al. [10], discussed the right distributive ringoids in linear-linear groupoids, and investigated several properties in ringoids. By using the notion of a top-row determinant they discussed some properties of matrices over ringoids. Moreover, they discussed the notions of the (strongly, (very-) weak) orthogonality and discussed the notion of incident of vectors and defined the concept of α-K-sphere on a ringoid, where K is a field.

The motivation of this study consists algebraic and logical arguments. The notion of a ring is a generalization of a ring of integers, i.e., $(\mathbf{Z}, +, 0)$ is an abelian group and $(\mathbf{Z}, \cdot)$ is a semigroup, and left- and right-distributive laws. If we consider the multiplication of integers, it can be represented by the addition of integers. Moreover, the distributive laws are not necessary in some cases. If we define a binary operation "$*$" on $\mathbf{Z}$ by $x * y := x \cdot (x - y)$, then we obtain $6 * 4 = 6 \cdot (6 - 4) = (6-4)+(6-4)+(6-4)+(6-4)+(6-4)+(6-4) = 12$. In this calculation we can find that $(\mathbf{Z}, +, 0)$ is an abelian group and $(\mathbf{Z}, *)$ is a groupoid. From this observation, we may construct a notion of a ringoid which can be another generalization of a ring, near-ring, pseudo-ring, semihyperring, etc.. Due to the fact that not enough research has been done on the ringoids, we introduce and study the concept of left distributive (quadratic-linear, linear-quadratic, quadratic-quadratic) ringoids over a field and study its different properties. Also, by finding their general form, we find a family of examples for them that can be used in future researches. Finally, we find the general form of the left distributive quadratic-linear, linear-quadratic and quadratic-quadratic ringoids over a field K.

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A groupoid $(X, *)$ is said to be a *right zero semigroup* if $x * y = y$, for all $x, y \in X$, and a groupoid $(X, *)$ is said to be a *left zero semigroup* if $x * y = x$, for all $x, y \in X$ (see [6]).

Let $(K, \cdot, +, 0, 1)$ be a field with $|K| \geq 3$. Define a binary operation "$*$" on K by

$$x * y := A + Bx + Cy \tag{1}$$

for all $x, y \in K$, where $A, B, C \in K$ are fixed. We call $(K, *)$ a *linear groupoid* [4,5,8] over K.

An algebra $(X, *, +, 0)$ of type $(2, 2, 0)$ is called a *ringoid* if satisfies the following conditions: [8,9]

(I) $(X, +, 0)$ is an abelian group,

(II) $(X, *)$ is a groupoid.

Example. [8] Let $(\mathbb{R}, +, \cdot, 0, 1)$ be the field of real numbers. Define a binary operation "$*$" on $\mathbb{R}$ by $x * y := x \cdot (x - y)$, for all $x, y \in \mathbb{R}$. Hence $(\mathbb{R}, *, +, 0)$ is a ringoid, but neither a ring nor a recognized type of generalization of a ring such as semi-ring, near-ring, etc..

A ringoid $(X, *, +, 0)$ is called *left distributive* if $x * (y + z) = (x * y) + (x * z)$, for all $x, y, z \in X$.

From now on, K denotes the field $(K, \cdot, +, 0, 1)$ with $|K| \geq 3$.

2. Left Distributive Ringoids of Quadratic-Linear Groupoids

In this section, we investigate left distributive ringoids on the quadratic-linear groupoids over a field K.

Definition 1. Let $A, B, C, D, E, F \in K$ are fixed. Define a binary operation "$*$" on K by

$$x * y := A + Bx + Cy + Dx^2 + Exy + Fy^2 \tag{2}$$

for all $x, y \in K$. We call $(K, *)$ a *quadratic groupoid* over K and it is called *proper* if $D \neq 0$ or $E \neq 0$ or $F \neq 0$.

Notice that if $D = E = F = 0$, then the quadratic groupoid $(K, *)$ is a linear groupoid.

Example 1. *Consider an algebra that $(\mathbb{Q}, \cdot, +, 0)$ where $\mathbb{Q}$ is the rational numbers, "$+$" is the usual addition and "$\cdot$" is the usual multiplication. Define a binary operation "$*$" on $\mathbb{Q}$ by $x * y = -1 + 2x + 5y - 5x^2 + 8xy + y^2$. Then $(\mathbb{Q}, *)$ is a quadratic groupoid.*

Beside, assume $\alpha, \beta, \gamma \in K$ are fixed. Consider (2) and define a binary operation "$\oplus$" on K by

$$x \oplus y \ := \ \alpha + \beta x + \gamma y \tag{3}$$

for all $x, y \in K$. We call $(K, *, \oplus)$ a *quadratic-linear groupoid* over K.

If we consider (1) and (3), then $(K, *, \oplus)$ is called *linear-linear groupoid* over K and A ringoid $(K, *, \oplus, 0)$ is called a linear-linear ringoid if $(K, *, \oplus)$ is a linear-linear groupoid over K (see [8]).

A ringoid $(K, *, \oplus, 0)$ is called a *quadratic-linear ringoid* if $(K, *, \oplus)$ is a quadratic-linear groupoid over K, and it is called proper if $(K, *)$ is proper.

Now, we discuss left distributive ringoids related to quadratic-linear groupoids over a field K.

Lemma 1. *Let $(K, *, \oplus)$ be a quadratic-linear groupoid, with the left distributive law. If $F = 0$ and $\beta + \gamma \neq 1$, then*

$$x * y = \frac{\alpha(1 - C)}{1 - \beta - \gamma} - \frac{\alpha E}{1 - \beta - \gamma} x + Cy + Exy,$$

for all $x, y \in K$.

Proof. Assume $(K, *, \oplus)$ is a quadratic-linear groupoid, with the left distributive law. Then for all $x, y, z \in K$, we have

$$\begin{aligned} x * (y \oplus z) &= A + Bx + C(y \oplus z) + Dx^2 + Ex(y \oplus z) + F(y \oplus z)^2 \\ &= A + Bx + C(\alpha + \beta y + \gamma z) + Dx^2 + Ex(\alpha + \beta y + \gamma z) + F(\alpha + \beta y + \gamma z)^2 \\ &= A + Bx + \alpha C + \beta Cy + \gamma Cz + Dx^2 + \alpha Ex + \beta Exy + \gamma Exz + \alpha^2 F + 2F\alpha\beta y \\ &+ \ 2F\alpha\gamma z + 2\beta\gamma Fyz + \beta^2 Fy^2 + \gamma^2 Fz^2 \\ &= (A + \alpha C + \alpha^2 F) + (B + \alpha E)x + (\beta C + 2\alpha\beta)Fy + (\gamma C + 2\alpha\gamma F)z \\ &+ \ Dx^2 + \beta^2 Fy^2 + \gamma^2 Fz^2 + \beta Exy + \gamma Exz + 2\beta\gamma Fyz. \end{aligned}$$

On the other hand,

$$\begin{aligned}(x*y)\oplus(x*z) &= \alpha+\beta(x*y)+\gamma(x*z)\\ &= \alpha+\beta(A+Bx+Cy+Dx^2+Exy+Fy^2)\\ &+ \gamma(A+Bx+Cz+Dx^2+Exz+Fz^2)\\ &= \alpha+A(\beta+\gamma)+B(\beta+\gamma)x+\beta Cy+\gamma Cz+(\beta+\gamma)Dx^2\\ &+ \beta Fy^2+\gamma Fz^2+\beta Exy+\gamma Exz.\end{aligned}$$

Since the left distributive law holds (i.e., $x*(y\oplus z)=(x*y)\oplus(x*z)$), we obtain

(i) $A+\alpha C+\alpha^2F=\alpha+(\beta+\gamma)A$ which implies $(1-\beta-\gamma)A+\alpha(C-1)+\alpha^2F=0$,
(ii) $B+\alpha E=B(\beta+\gamma)$ which implies $(1-\beta-\gamma)B+\alpha E=0$,
(iii) $\beta C+2\alpha\beta F=\beta C$ which implies $\alpha\beta F=0$,
(iv) $\gamma C+2\alpha\gamma F=\gamma C$ which implies $\alpha\gamma F=0$,
(iii) $D=(\beta+\gamma)D$ which implies $(1-\beta-\gamma)D=0$,
(vi) $\beta^2F=\beta F$ which implies $\beta(\beta-1)F=0$,
(vii) $\gamma^2F=\gamma F$ which implies $\gamma(\gamma-1)F=0$,
(viii) $\beta\gamma F=0$.

Since $F=0$ and $\beta+\gamma\neq 1$, we get $D=0$ by (v). Similarly, we obtain $A=\frac{\alpha(1-C)}{1-\beta-\gamma}$ and $B=\frac{-\alpha E}{1-\beta-\gamma}$ by (i) and (ii). This proves the lemma. □

Theorem 1. *Let $(K,*,\oplus)$ be the quadratic-linear groupoid, $F=0$ and $\beta+\gamma\neq 1$. Then $(K,*,\oplus,\xi)$ is a left distributive ringoid if and only if $\xi=-\alpha$ and for all $x,y\in K$*

$$x*y = \alpha(C-1)+\alpha Ex+Cy+Exy \tag{4}$$
$$x\oplus y = \alpha+x+y. \tag{5}$$

Proof. Assume $(K,*,\oplus,\xi)$ is a left distributive quadratic-linear ringoid, where ξ is the zero element in $(K,\oplus,\xi)$. Then by Lemma 1, $x\oplus\xi=\alpha+\beta x+\gamma\xi=\alpha+\beta\xi+\gamma x=\xi\oplus x=x$, for all $x,y\in K$. Therefore, $\beta=\gamma=1$ and $\alpha+\gamma\xi=\alpha+\beta\xi=0$. So, $\xi=-\alpha$, for all $x,y\in K$. Hence $x*y=\frac{\alpha(1-C)}{1-\beta-\gamma}-\frac{\alpha E}{1-\beta-\gamma}x+Cy+Exy=\alpha(C-1)+\alpha Ex+Cy+Exy$, for all $x,y\in K$. Also, $x\oplus y=\alpha+\beta x+\gamma y=\alpha+x+y$. Conversely, easily the algebra $(K,*,\oplus)$ satisfies the left distributive law and $-\alpha$ is the zero element in $(K,\oplus)$. Now, assume $\hat{x}$ is the inverse of x in $(K,\oplus,-\alpha)$. So, $x\oplus\hat{x}=\alpha+x+\hat{x}=-\alpha$. This shows that $\hat{x}=-2\alpha-x$. Thus, $(K,\oplus,-\alpha)$ is an abelian group and $(K,*,\oplus,-\alpha)$ is a left distributive ringoid. □

Example 2. (*i*) *Every linear-linear groupoid is a quadratic-linear groupoid, as a neutral way, so, every linear-linear ringoid is a quadratic-linear ringoid.*

(*ii*) *Consider an algebra $(\mathbb{Q},\cdot,+,0)$ where $\mathbb{Q}$ is the set of all rational numbers, "+" is the usual addition and "·" is the usual multiplication. Define binary operations "∗" and "⊕" on $\mathbb{Q}$ by $x*y=16x+y+8xy$ and $x\oplus y=2+x+y$. Then $(\mathbb{Q},*,\oplus,-2)$ is a left distributive quadratic-linear ringoid.*

Remark 1. *The left distributive ringoid $(K,*,\oplus,-\alpha)$ discussed in Theorem 1 need not be a ring in general. It is enough to show that $(K,*)$ is not a semigroup. Then for all $x,y,z\in K$, we have:*

$$\begin{aligned}x*(y*z) &= \alpha(C-1)+\alpha Ex+C(y*z)+Ex(y*z)\\ &= \alpha(C-1)+\alpha Ex+C(\alpha(C-1)+\alpha Ey+Cz+Eyz)\\ &+ Ex(\alpha(C-1)+\alpha Ey+Cz+Eyz)\\ &= \alpha(C^2-1)+\alpha CE(x+y)+C^2z+CEyz+\alpha E^2xy+CExz+E^2xyz.\end{aligned}$$

On the other hand,

$$\begin{aligned}(x*y)*z &= \alpha(C-1)+\alpha E(x*y)+Cz+E(x*y)z\\ &= \alpha(C-1)+\alpha E(\alpha(C-1)+\alpha Ex+Cy+Exy)\\ &+ Cz+E(\alpha(C-1)+\alpha Ex+Cy+Exy)z\\ &= \alpha(C-1)+\alpha^2E(C-1)+\alpha^2E^2x+\alpha CEy+\alpha E^2xy\\ &+ (C+\alpha CE-\alpha E)z+\alpha E^2xz+CEzy+E^2xyz.\end{aligned}$$

We can see that $x*(y*z)\neq(x*y)*z$. For example, take $E=0$ and $C\notin\{-1.1\}$.

Lemma 2. *Let* $(K, *, \oplus)$ *be a quadratic-linear groupoid, with the left distributive law. Then*

$$x * y = \begin{cases} A + Bx + y + Dx^2 & \text{if } \beta + \gamma = 1,\ \alpha \neq 0 \text{ and } F = 0, \\ (\alpha(1-C) - \alpha^2 F) + \alpha E x + Cy + Exy + Fy^2 & \text{if } \beta = \gamma = 0 \text{ and } F \neq 0, \end{cases}$$

$$x \oplus y = \begin{cases} y & \text{if } \gamma \neq 0 \text{ and } F \neq 0, \\ x & \text{if } \beta \neq 0 \text{ and } F \neq 0, \end{cases}$$

for all $x, y \in K$.

Proof. Suppose $F = 0$ and $\beta + \gamma = 1$. Then $\alpha(C-1) = 0$ and $E\alpha = 0$ by (i) and (ii).

If $\alpha \neq 0$, then $C = 1$ and $E = 0$. It shows that $x * y = A + Bx + y + Dx^2$.

Assume $F \neq 0$.

If $\beta = \gamma = 0$, then $A = \alpha(1-C) - \alpha^2 F$, $B = \alpha E$ and $D = 0$, by (i), (ii) and (v). Thus, $x * y = ((1-C)\alpha - F\alpha^2) + \alpha E x + Cy + Exy + Fy^2$.

If $\gamma \neq 0$, then $\gamma = 1$, $\alpha = 0$ and $\beta = 0$ by (vii), (iv) and (viii). Therefore, $x \oplus y = y$ i.e., $(X, \oplus)$ is a right zero semigroup.

If $\beta \neq 0$, then $\alpha = 0$, $\beta = 1$ and $\gamma = 0$ by (iii), (vi) and (viii). Therefore, $x \oplus y = x$ i.e., $(X, \oplus)$ is a left zero semigroup. □

Theorem 2. *Let* $(K, *, \oplus)$ *be the quadratic-linear groupoid in Lemma 2. Then there is no left distributive ringoid over* K.

Proof. Assume $\beta + \gamma = 1$, $\alpha \neq 0$ and $F = 0$, claim $(K, \oplus, \xi)$ is not an abelian group. By contrary, let $(K, \oplus, \xi)$ be an abelian group. Then $x \oplus \xi = \xi \oplus x = x$, for all $x \in K$. Hence $\alpha + \beta x + (1-\beta)\xi = \alpha + \beta\xi + (1-\beta)x = x$, it shows that $\beta = 1 - \beta = 1$, which is a contradiction. The proof of the other cases are similar. □

3. Left Distributive Ringoids of Linear-Quadratic Groupoids

In this section, we investigate left distributive ringoids on linear-quadratic groupoids over a field K. Beside, we assume that $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in K$ are fixed. Consider (1) and define a binary operation "$\oplus$" on K by

$$x \oplus y \quad := \quad \alpha + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \eta y^2 \tag{6}$$

for all $x, y \in K$. We call $(K, *, \oplus)$ a *linear-quadratic groupoid* over K and it is called *proper* if $\delta \neq 0$ or $\varepsilon \neq 0$ or $\eta \neq 0$.

Notice that if $\delta = \varepsilon = \eta = 0$, then the quadratic groupoid $(K, \oplus)$ is a linear groupoid.

A ringoid $(K, *, \oplus, 0)$ is called a linear-quadratic ringoid if $(K, *, \oplus)$ is a linear-quadratic groupoid over K and it is called proper if $(K, \oplus)$ is proper.

Now, we discuss left distributive ringoids related to linear-quadratic groupoids over K.

Example 3. (*i*) *Every linear-linear groupoid is a linear-quadratic groupoid, as a neutral way, so, every linear-linear ringoid is a linear-quadratic ringoid.*

(*ii*) *Consider an algebra* $(\mathbb{Q}, \cdot, +, 0)$ *where* $\mathbb{Q}$ *is the set of all rational numbers, "+" is the usual addition and "·" is the usual multiplication. Define binary operations "∗" and "⊕" on* $\mathbb{Q}$ *by* $x * y = 2 + 5x + 7y$ *and* $x \oplus y = 5 + 4x - 3y + 2x^2 + xy + 3y^2$. *Then* $(\mathbb{Q}, *, \oplus)$ *is a linear-quadratic groupoid.*

Lemma 3. *Let* $(K, *, \oplus)$ *be the linear-quadratic groupoid, with the left distributive law and* $B = 0$. *Then*

$$x \oplus y = \begin{cases} (1-\beta-\gamma)A - (\delta+\varepsilon+\eta)A^2 + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \eta y^2 & \text{if } C = 0, \\ \frac{(1-\beta-\gamma)A}{1-C} + \beta x + \gamma y & \text{if } C \notin \{0,1\}, \end{cases}$$

for all $x, y \in K$.

Proof. Assume $(K, *, \oplus)$ be the linear-quadratic groupoid, with the left distributive law and $B = 0$. Then for all $x, y, z \in K$, we have

$$\begin{aligned} x * (y \oplus z) &= x * (\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \eta z^2) \\ &= A + Bx + C(\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \eta z^2) \\ &= A + \alpha C + Bx + \beta C y + \gamma C z + \delta C y^2 + \varepsilon C yz + \eta C z^2. \end{aligned}$$

On the other hand,

$$\begin{aligned}
(x*y)\oplus(x*z) &= (A+Bx+Cy)\oplus(A+Bx+Cz)\\
&= \alpha+\beta(A+Bx+Cy)+\gamma(A+Bx+Cz)+\delta(A+Bx+Cy)^2\\
&+ \varepsilon(A+Bx+Cy)(A+Bx+Cz)+\eta(A+Bx+Cz)^2\\
&= \alpha+\beta A+\beta Bx+\beta Cy+\gamma A+\gamma Bx+\gamma Cz\\
&+ \delta(A^2+2ABx+2ACy+2BCxy+B^2x^2+C^2y^2)\\
&+ \varepsilon(A^2+ABx+ACz+BAx+B^2x^2+BCxz+ACy+BCxy+C^2yz)\\
&+ \eta(A^2+2ABx+2ACz+2BCxz+B^2x^2+C^2z^2)\\
&= \alpha+\beta A+\gamma A+\delta A^2+\varepsilon A^2+\eta A^2\\
&+ (\beta B+\gamma B+2\delta AB+2\varepsilon AB+2\eta AB)x\\
&+ (\beta C+2\delta AC+\varepsilon AC)y+(\gamma C+\varepsilon AC+2\eta AC)z\\
&+ (\delta B^2+\varepsilon B^2+\eta B^2)x^2+(2\delta BC+\varepsilon BC)xy+(\varepsilon BC+2\eta BC)xz\\
&+ \varepsilon C^2yz+\eta C^2z^2+\delta C^2y^2.
\end{aligned}$$

Since the left distributive law holds (i.e., $x*(y\oplus z)=(x*y)\oplus(x*z)$), we obtain

(i) $(1-\beta-\gamma)A+\alpha(C-1)-(\delta+\varepsilon+\eta)A^2=0$,
(ii) $(1-\beta-\gamma)B-2(\delta+\varepsilon+\eta)AB=0$ implies $(1-\beta-\gamma)B=0$ by (v),
(iii) $\beta C=\beta C+2\delta AC+\varepsilon AC$ implies $(2\delta+\varepsilon)AC=0$,
(iv) $\gamma C=\gamma C+\varepsilon AC+2\eta AC$ implies $(2\eta+\varepsilon)AC=0$,
(v) $\delta B^2+\varepsilon B^2+\eta B^2=0$ implies $(\delta+\varepsilon+\eta)B^2=0$,
(vi) $2\delta BC+\varepsilon BC=0$ implies $(2\delta+\varepsilon)BC=0$,
(vii) $\varepsilon BC+2\eta BC=0$ implies $(2\eta+\varepsilon)BC=0$,
(viii) $\delta C(C-1)=0$,
(ix) $\varepsilon C(C-1)=0$,
(x) $\eta C(C-1)=0$.

Assume $B=C=0$, by using (i), we get $\alpha=(1-\beta-\gamma)A-(\delta+\varepsilon+\eta)A^2$ and $x\oplus y=(1-\beta-\gamma)A-(\delta+\varepsilon+\eta)A^2+\beta x+\gamma y+\delta x^2+\varepsilon xy+\eta y^2$.
If $B=0$ and $C\notin\{0,1\}$, then we obtain $\delta=\varepsilon=\eta=0$ by (viii), (ix) and (x). It shows that $(1-\beta-\gamma)A+\alpha(C-1)=0$ and $\alpha=\frac{(1-\beta-\gamma)A}{1-C}$ by (i). Thus, $x\oplus y=\frac{(1-\beta-\gamma)A}{1-C}+\beta x+\gamma y$. □

Theorem 3. *Let $(K,*,\oplus)$ be the linear-quadratic groupoid.*

(i) If $B=C=0$ and $\varepsilon\neq0$, then $(K-\{\frac{-\beta}{\varepsilon}\},*,\oplus,\xi)$ is a left distributive ringoid if and only if $\xi=\frac{1-\beta}{\varepsilon}$, $A\in\{\frac{1-\beta}{\varepsilon},\frac{-\beta}{\varepsilon}\}$ and

$$x\oplus y = \frac{\beta(\beta-1)}{\varepsilon}+\beta(x+y)+\varepsilon xy \tag{7}$$

for all $x,y\in K-\{\frac{-\beta}{\varepsilon}\}$.

(ii) If $B=C=0$ and $\varepsilon=0$, then $(K,*,\oplus,\xi)$ is a left distributive ringoid if and only if $\xi=A$ and

$$x\oplus y = -A+x+y \tag{8}$$

for all $x,y\in K$.

(iii) If $B=0$, $C\notin\{0,1\}$, then $(K,*,\oplus,\frac{A}{1-C})$ is a left distributive ringoid if and only if $\xi=\frac{A}{1-C}$ and

$$x\oplus y = \frac{A}{C-1}+x+y \tag{9}$$

for all $x,y\in K$.

(iv) If $A=B=0$, $C=1$ and $\varepsilon\neq0$, $(K-\{\frac{-\beta}{\varepsilon}\},*,\oplus,\xi)$ is a left distributive ringoid if and only if $\xi=\frac{1-\beta}{\varepsilon}$ and it satisfies in (7).

(v) If $A=B=0$, $C=1$ and $\varepsilon=0$, then $(K,*,\oplus,\xi)$ is a left distributive ringoid if and only $\xi=-\alpha$ and it satisfies in (5).

Proof. (i) Assume $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \xi)$ is a left distributive linear-quadratic ringoid. Then by Lemma 3, $x\oplus\xi=(1-\beta-\gamma)A-(\delta+\varepsilon+\eta)A^2+\beta x+\gamma\xi+\delta x^2+\varepsilon\xi x+\eta\xi^2=(1-\beta-\gamma)A-(\delta+\varepsilon+\eta)A^2+\beta\xi+\gamma x+\delta\xi^2+\varepsilon\xi x+\eta x^2=x$, for all $x\in K-\{\frac{-\beta}{\varepsilon}\}$. Therefore, $\beta=\gamma$, $\delta=\eta=0$, $(1-2\beta)A-\varepsilon A^2+\beta\xi=0$ and $\beta+\varepsilon\xi=1$. Since $\varepsilon\neq 0$, hence $\xi=\frac{1-\beta}{\varepsilon}$, $A\in\{\frac{1-\beta}{\varepsilon}, \frac{-\beta}{\varepsilon}\}$ and

$$\begin{aligned} x\oplus y &= (1-\beta-\gamma)A-(\delta+\varepsilon+\eta)A^2+\beta x+\gamma\xi+\delta x^2+\varepsilon\xi x+\eta\xi^2 \\ &= (1-2\beta)A-\varepsilon A^2+\beta x+\beta y+\varepsilon xy \\ &= -\beta\xi+\beta(x+y)+\varepsilon xy \\ &= \frac{\beta(\beta-1)}{\varepsilon}+\beta(x+y)+\varepsilon xy. \end{aligned}$$

Conversely, easily the algebra $(K, *, \oplus)$ satisfies the left distributive law and $\frac{1-\beta}{\varepsilon}$ is the zero element in $(K, \oplus)$. Now assume $\hat{x}$ is the inverse of x in $(K, \oplus, \frac{1-\beta}{\varepsilon})$. So, $x\oplus\hat{x}=\frac{\beta(\beta-1)}{\varepsilon}+\beta(x+\hat{x})+\varepsilon x\hat{x}=\frac{1-\beta}{\varepsilon}$. So, $\hat{x}(\beta+\varepsilon x)=-\beta x+\frac{1-\beta}{\varepsilon}-\frac{\beta(1-\beta)}{\varepsilon}$. If $x\neq\frac{-\beta}{\varepsilon}$, thus, $\hat{x}=\frac{1}{\beta+\varepsilon x}(-\beta x+\frac{1-\beta^2}{\varepsilon})$, for all $x\in K-\{\frac{-\beta}{\varepsilon}\}$. If $x=\frac{-\beta}{\varepsilon}$, then $\frac{\beta(\beta-1)}{\varepsilon}-\frac{\beta^2}{\varepsilon}+\beta\hat{x}-\beta\hat{x}=\frac{1-\beta}{\varepsilon}$, and $-\frac{\beta}{\varepsilon}=\frac{1-\beta}{\varepsilon}$. Which proves that $-\beta=1-\beta$ and $0=1$, a contradiction. Thus there is not an inverse for the element $x=\frac{-\beta}{\varepsilon}$. Then $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \frac{1-\beta}{\varepsilon})$ is an abelian group and $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \frac{1-\beta}{\varepsilon})$ is a left distributive ringoid.
(ii) Assume $(K, *, \oplus, \xi)$ is a left distributive linear-quadratic ringoid. Then by Lemma 3 and the proof of case (i), $\delta=\eta=0$, $\beta=\gamma$, $\beta+\varepsilon\xi=1$ and $(1-2\beta)A-\varepsilon A^2+\beta\xi=0$. Since $\varepsilon=0$, hence $\beta=\gamma=1$, $\xi=A$ and

$$x\oplus y = ((1-\beta-\gamma)A-(\delta+\varepsilon+\eta)A^2)+\beta x+\gamma y+\delta x^2+\varepsilon xy+\eta y^2=-A+x+y.$$

Conversely, easily the algebra $(K, *, \oplus)$ satisfies the left distributive law and A is the zero element in $(K, \oplus)$. Now, assume $\hat{x}$ is the inverse of x in $(K, \oplus, A)$. So, $x\oplus\hat{x}=-A+x+\hat{x}=A$. This shows that $\hat{x}=2A-x$, for all $x\in K$. Thus, $(K, \oplus, A)$ is an abelian group and $(K, *, \oplus, A)$ is a left distributive ringoid.
(iii) Assume $(K, *, \oplus, \xi)$ is a left distributive linear-quadratic ringoid. Then by Lemma 3, $x\oplus\xi=\frac{(1-\beta-\gamma)A}{1-C}+\beta x+\gamma\xi=\frac{(1-\beta-\gamma)A}{1-C}+\beta\xi+\gamma x=x$. Thus, $\beta=\gamma=1$ and $-\frac{A}{1-C}+\xi=0$ so $\xi=\frac{A}{1-C}$ and

$$x\oplus y = \frac{(1-\beta-\gamma)A}{1-C}+\beta x+\gamma y=\frac{A}{C-1}+x+y.$$

Conversely, easily the algebra $(K, *, \oplus)$ satisfies the left distributive law and $\frac{A}{1-C}$ is the zero element in $(K, \oplus)$. Now assume $\hat{x}$ is the inverse of x in $(K, \oplus, \frac{A}{1-C})$. So, $x\oplus\hat{x}=\hat{x}\oplus x=\xi=A$. Hence $\frac{A}{C-1}+\beta x+\hat{x}=\frac{A}{1-C}$. This shows that $\hat{x}=\frac{2A}{1-C}-x$, for all $x\in K$. Thus, $(K, \oplus, \frac{A}{1-C})$ is an abelian group and $(K, *, \oplus, \frac{A}{1-C})$ is a left distributive ringoid.
(iv) Assume $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \xi)$ is a left distributive linear-quadratic ringoid. It follows that $x\oplus\xi=\alpha+\beta x+\gamma\xi+\delta x^2+\varepsilon\xi x+\eta\xi^2=\alpha+\beta\xi+\gamma x+\delta\xi^2+\varepsilon\xi x+\eta x^2=\xi\oplus x=x$, for all $x\in K-\{\frac{-\beta}{\varepsilon}\}$. Thus, $\delta=\eta=0$, $\beta=\gamma$, $\beta+\varepsilon\xi=1$ and $\alpha+\beta\xi=0$. Since $\varepsilon\neq 0$, hence $\xi=\frac{1-\beta}{\varepsilon}$, $\alpha=-\beta\xi=\frac{\beta(\beta-1)}{\varepsilon}$ and

$$x\oplus y = \alpha+\beta x+\gamma y+\delta x^2+\varepsilon xy+\eta y^2=\frac{\beta(\beta-1)}{\varepsilon}+\beta(x+y)+\varepsilon xy.$$

Conversely, it is proved similar to the proof of (i).
(v) Assume $(K, *, \oplus, \xi)$ is a left distributive linear-quadratic ringoid. Then similar to the proof of case (iv), $\delta=\eta=0$, $\beta=\gamma$, $\beta+\varepsilon\xi=1$ and $\alpha+\beta\xi=0$. Since $\varepsilon=0$, hence $\beta=\gamma=1$, $\xi=-\alpha$ and

$$x\oplus y = \alpha+\beta x+\gamma y+\delta x^2+\varepsilon xy+\eta y^2=\alpha+x+y.$$

Conversely, easily the algebra $(K, *, \oplus)$ satisfies the left distributive law and $-\alpha$ is the zero element in $(K, \oplus)$. Now, assume $\hat{x}$ is the inverse of x in $(K, \oplus, -\alpha)$. So, $x\oplus\hat{x}=\alpha+x+\hat{x}=-\alpha$. This shows that $\hat{x}=-2\alpha-x$, for all $x\in K$. Thus, $(K, \oplus, -\alpha)$ is an abelian group and $(K, *, \oplus, -\alpha)$ is a left distributive ringoid. □

Example 4. *Consider an algebra $(\mathbb{Q}, \cdot, +, 0)$ where $\mathbb{Q}$ is the set of all rational numbers, "+" is the usual addition and "$\cdot$" is the usual multiplication. Define binary operations "$*_1$" and "$\oplus_1$" on $\mathbb{Q}$ by $x *_1 y = 2$ and $x\oplus_1 y=-2+x+y$. Then $(\mathbb{Q}, *_1, \oplus_1, 2)$ is a left distributive linear-quadratic ringoid. Beside, if* *define "$\oplus_2$" on K by $x \oplus_2 y = 10 + 5(x + y) + 2xy$, then $(\mathbb{Q} - \{-\frac{5}{2}\}, *_1, \oplus_2, -2)$ is a left distributive linear-quadratic ringoid.*

Remark 2. *Theorem 3 means that the left distributive linear-quadratic ringoids are in the cases (i) and (iv), other cases are the left distributive linear-linear ringoids.*

Lemma 4. *Let $(K, *, \oplus)$ be the linear-quadratic groupoid defined, with the left distributive and $B \neq 0$. Then, for all $x, y \in K$*

$$x \oplus y = \begin{cases} \beta x + (1-\beta)y + \delta x^2 - (\delta+\eta)xy + \eta y^2 & \text{if } C = 0, \\ \alpha + \beta x + (1-\beta)y + \delta(x^2 - 2xy + y^2) & \text{if } C = 1, \\ \beta x + (1-\beta)y & \text{if } C \notin \{0,1\}, \end{cases}$$

Proof. Assume $C = 0$. Then $1 - \beta - \gamma = 0$ by (ii), and so $\gamma = 1 - \beta$. Also, we have $\delta + \varepsilon + \eta = 0$ by (v) and $\alpha = \frac{(1-\beta-\gamma)A}{1-C} = 0$ by (i). Thus, $x \oplus y = \beta x + (1-\beta)y + \delta x^2 - (\delta+\eta)xy + \eta y^2$.

If $C = 1$, then $(1-\beta-\gamma)A - (\delta+\varepsilon+\eta)A^2 = 0$, $(2\delta+\varepsilon)A = 0$, $(2\eta+\varepsilon)A = 0$, $(2\delta+\varepsilon)B = 0$ and $(2\eta+\varepsilon)B = 0$ by (i), (iii), (iv), (vi) and (vii).
Since $B \neq 0$, we have $\delta = \eta$, $\varepsilon = -2\eta$ and by (ii), we get $1 - \beta - \gamma = 0$. It follows that $\gamma = 1 - \beta$. Hence $x \oplus y = \alpha + \beta x + (1-\beta)y + \delta(x^2 - 2xy + y^2)$.

If $C \notin \{0,1\}$, then $\delta = \varepsilon = \eta = 0$ by (viii), (ix) and (x). Also, $1 - \beta - \gamma = 0$ by (ii) and $\alpha = 0$ by (i). Thus, $x \oplus y = \beta x + (1-\beta)$. ☐

Lemma 5. *Let $(K, *, \oplus)$ be the linear-quadratic groupoid, with the left distributive, $A \neq 0$, $B = 0$ and $C = 1$. Then $x \oplus y = \alpha + \beta x + (1-\beta)y + \delta(x^2 - 2xy + y^2)$, for all $x, y \in K$.*

Proof. Since $A \neq 0$, $B = 0$ and $C = 1$, we have $\delta = \eta$, $\varepsilon = -2\eta$. By (i), we get $1 - \beta - \gamma = 0$. Hence $x \oplus y = \alpha + \beta x + (1-\beta)y + \delta(x^2 - 2xy + y^2)$. ☐

Theorem 4. *Let $(K, *, \oplus)$ be the linear-quadratic groupoid in Lemmas 4 and 5. Then there is no left distributive ringoid over K.*

Proof. By contrary, let $(K, \oplus, \xi)$ be an abelian group in Lemma 4, $B \neq 0$ and $C = 0$, where ξ is zero element in $(K, \oplus, \xi)$. Then $x \oplus \xi = \xi \oplus x = x$, for all $x \in K$. This shows that $\beta x + (1-\beta)\xi + \delta x^2 - (\delta+\eta)\xi x - \eta\xi^2 = \beta\xi + (1-\beta)x + \delta\xi^2 - (\delta+\eta)\xi x - \eta x^2 = x$, which proves that $\delta = \eta = 0$, $\beta - (\delta+\eta)\xi = (1-\beta) - (\delta+\eta)\xi = 1$, $(1-\beta)\xi - \eta\xi^2 = \beta\xi + \delta\xi^2 = 0$. Therefore, $1 = 1 - \beta$ and $\beta = 1$, which is a contradiction. Thus, there is no left distributive ringoid over K.
Similarly, other cases are proven. ☐

4. Left Distributive Ringoids of Quadratic-Quadratic Groupoids

In this section, we define quadratic-quadratic groupoids and investigate left distributive ringoids on the quadratic-quadratic groupoids over a field K.

Definition 2. Consider (2) and (6), we call $(K, *, \oplus)$ a *quadratic-quadratic groupoid* over K. Also, is called proper if the binary operations $*$ and $\oplus$ are not linear (i.e., $(K, *)$ and $(K, \oplus)$ are proper).

Notice that, if $D = E = F = \delta = \varepsilon = \eta = 0$, the quadratic-quadratic groupoid $(K, *, \oplus)$ is a linear-linear groupoid.

Example 5. *Let $(\mathbb{R}, +, \cdot, 0, 1)$ be the field of real numbers. Define a binary operation "$*$" on $\mathbb{R}$ by $x * y = 2 + x + y - 5x^2 + 3y^2$ and $x \oplus y = -1 + 2y - 8xy + y^2$, for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}, *, \oplus)$ is a proper quadratic-quadratic groupoid over $\mathbb{R}$.*

Lemma 6. *Let $(K, *, \oplus)$ be the quadratic-quadratic groupoid, with the left distributive law. Then, for all $x, y \in K$*

$$x * y = \begin{cases} \frac{\alpha(1-C)}{1-\beta-\gamma} - \frac{\alpha E}{1-\beta-\gamma}x + Cy + Exy & \text{if } F = D = 0,\ E \neq 0 \text{ and } \beta+\gamma \neq 1, \\ \frac{\beta(\beta-1)}{\varepsilon} + 2\beta y + \varepsilon y^2 & \text{if } F \neq 0 \text{ and } \varepsilon \neq 0, \end{cases}$$

$$x \oplus y = \begin{cases} \alpha + \beta x + \gamma y & \text{if } F = D = 0,\ E \neq 0 \text{ and } \beta+\gamma \neq 1, \\ \frac{\beta(\beta-1)}{\varepsilon} + \beta(x+y) + \varepsilon xy & \text{if } F \neq 0 \text{ and } \varepsilon \neq 0. \end{cases}$$

Proof. Assume $(K, *, \oplus)$ is the quadratic-quadratic groupoid, with the left distributive law. Then for all $x, y \in K$, we have

$$\begin{aligned}
x*(y\oplus z) &= A+Bx+C(y\oplus z)+Dx^2+Ex(y\oplus z)+F(y\oplus z)^2\\
&= A+Bx+C(\alpha+\beta y+\gamma z+\delta y^2+\varepsilon yz+\eta z^2)+Dx^2\\
&+ Ex(\alpha+\beta y+\gamma z+\delta y^2+\varepsilon yz+\eta z^2)+F(\alpha+\beta y+\gamma z+\delta y^2+\varepsilon yz+\eta z^2)^2\\
&= A+Bx+\alpha C+\beta Cy+\gamma Cz+\delta Cy^2+\varepsilon Cyz+\eta Cz^2+Dx^2+\alpha Ex+\beta Exy\\
&+ \gamma Exz+\delta Exy^2+\varepsilon Exyz+\eta Exz^2+\alpha^2F+\beta^2Fy^2+\gamma^2Fz^2+\delta^2Fy^4+\varepsilon^2Fy^2z^2\\
&+ \eta^2Fz^4+2\alpha\beta Fy+2\alpha\gamma Fz+2\alpha\delta Fy^2+2\alpha\varepsilon Fyz+2\alpha\eta Fz^2+2\beta\gamma Fzy+2\beta\delta Fy^3\\
&+ 2\beta\varepsilon Fy^2z+2\beta\eta Fyz^2+2\gamma\delta Fy^2z+2\varepsilon\gamma Fyz^2+2\gamma\eta Fz^3+2\varepsilon\delta Fy^3z+2\delta\eta Fy^2z^2\\
&+ 2\varepsilon\eta Fyz^3=(A+\alpha C+\alpha^2F)+(B+\alpha E)x+(\beta C+2\alpha\beta F)y+(\gamma C+2\alpha\gamma F)z\\
&+ Dx^2+\beta Exy+(\delta C+\beta^2F+2\alpha\delta F)y^2+(\eta C+\gamma^2F+2\alpha\eta F)z^2\\
&+ (\varepsilon C+2\alpha\varepsilon F+2\beta\gamma F)yz+\gamma Exz+\delta Exy^2+\varepsilon Exyz+\eta Exz^2+\delta^2Fy^4\\
&+ (\varepsilon^2F+2\delta\eta F)y^2z^2+\eta^2Fz^4+2\beta\delta Fy^3+(2\beta\varepsilon F+2\gamma\delta F)y^2z+(2\beta\eta F+2\varepsilon\gamma F)yz^2\\
&+ 2\gamma\eta Fz^3+2\varepsilon\delta Fy^3z+2\varepsilon\eta Fyz^3.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(x*y)\oplus(x*z) &= (A+Bx+Cy+Dx^2+Exy+Fy^2)\oplus(A+Bx+Cz+Dx^2+Exz+Fz^2)\\
&= \alpha+\beta(A+Bx+Cy+Dx^2+Exy+Fy^2)\\
&+ \gamma(A+Bx+Cz+Dx^2+Exz+Fz^2)\\
&+ \delta(A+Bx+Cy+Dx^2+Exy+Fy^2)^2\\
&+ \varepsilon(A+Bx+Cy+Dx^2+Exy+Fy^2)(A+Bx+Cz+Dx^2+Exz+Fz^2)\\
&+ \eta(A+Bx+Cz+Dx^2+Exz+Fz^2)^2\\
&= \alpha+\beta A+\beta Bx+\beta Cy+\beta Dx^2+\beta Exy+\beta Fy^2+\gamma A+\gamma Bx+\gamma Cz+\gamma Dx^2\\
&+ \gamma Exz+\gamma Fz^2+\delta A^2+2\delta ABx+2\delta ACy+2\delta ADx^2+2\delta AExy+2\delta AFy^2\\
&+ \delta B^2x^2+2\delta BCxy+2\delta BDx^3+2\delta BEx^2y+2\delta BFxy^2+\delta C^2y^2+2\delta CDx^2y\\
&+ 2\delta ECxy^2+2\delta CFy^3+\delta D^2x^4+2\delta EDx^3y+2\delta DFx^2y^2+\delta E^2x^2y^2+2\delta EFxy^3\\
&+ \delta F^2y^4+\varepsilon A^2+2\varepsilon ABx+\varepsilon ACz+\varepsilon ADx^2+\varepsilon AExz+\varepsilon AFz^2+\varepsilon B^2x^2\\
&+ \varepsilon BCxz+\varepsilon BDx^3+\varepsilon BEx^2z+\varepsilon BFxz^2+\varepsilon ACy+\varepsilon CByx+\varepsilon C^2yz+\varepsilon CDyx^2\\
&+ \varepsilon CEyxz+\varepsilon CFyz^2+\varepsilon ADx^2+\varepsilon BDx^3+\varepsilon CDx^2z+\varepsilon D^2x^4+\varepsilon EDx^3z\\
&+ \varepsilon DFx^2z^2+\varepsilon AExy+\varepsilon EBx^2y+\varepsilon ECxyz+\varepsilon EDx^3y+\varepsilon E^2x^2yz\\
&+ \varepsilon EFxyz^2+\varepsilon AFy^2+\varepsilon BFy^2x+\varepsilon CFy^2z+\varepsilon DFy^2x^2+\varepsilon FExy^2z+\varepsilon F^2y^2z^2\\
&+ \eta A^2+2\eta ABx+2\eta ACz+2\eta ADx^2+2\eta AExz+2\eta AFz^2+\eta B^2x^2+2\eta BCxz\\
&+ 2\eta BDx^3+2\eta BEx^2z+2\eta BFxz^2+\eta C^2z^2+2\eta CDx^2z+2\eta ECxz^2+2\eta CFz^3\\
&+ \eta D^2x^4+2\eta EDx^3z+2\eta DFx^2z^2+\eta E^2x^2z^2+2\eta EFxz^3+\eta F^2z^4\\
&= (\alpha+\beta A+\gamma A+\delta A^2+\varepsilon A^2+\eta A^2)+(\beta B+\gamma B+2\delta AB+2\varepsilon AB+2\eta AB)x\\
&+ (\beta C+2\delta AC+\varepsilon AC)y+(\gamma C+\varepsilon AC+2\eta AC)z+\varepsilon C^2yz\\
&+ (\beta D+\gamma D+2\delta AD+\varepsilon B^2+2\varepsilon AD+2\eta AD+\eta B^2+\delta B^2)x^2\\
&+ (\beta E+2\delta BC+\varepsilon CB+\varepsilon AE+2\delta AE)xy+(\beta F+2\delta AF+\delta C^2+\varepsilon AF)y^2\\
&+ (\gamma E+\varepsilon AE+\varepsilon BC+2\eta BC+2\eta AE)xz+(\gamma F+\varepsilon AF+\eta C^2+2\eta AF)z^2\\
&+ (2\delta BF+2\delta EC+\varepsilon BF)xy^2+2\varepsilon CEyxz+(\varepsilon BF+2\eta BF+2\eta EC)xz^2\\
&+ \delta F^2y^4+\varepsilon F^2y^2z^2+\eta F^2z^4+(\delta+\varepsilon+\eta)D^2x^4+2\delta CFy^3+\varepsilon CFy^2z\\
&+ \varepsilon CFyz^2+2\eta CFz^3+(2\delta BD+2\varepsilon BD+2\eta BD)x^3\\
&+ (2\delta BE+2\delta CD+\varepsilon CD+\varepsilon EB)x^2y\\
&+ (2\delta DF+\delta E^2+\varepsilon DF)x^2y^2+2\delta EFxy^3+(2\delta ED+\varepsilon ED)x^3y
\end{aligned}$$

$$+ \quad (\varepsilon CD + 2\eta BE + 2\eta CD + \varepsilon BE)x^2z + (\varepsilon ED + 2\eta ED)x^3z$$
$$+ \quad (\varepsilon DF + \eta E^2 + 2\eta DF)x^2z^2 + \varepsilon E^2x^2yz + \varepsilon EFxyz^2 + \varepsilon FExy^2z + 2\eta EFxz^3.$$

Since the left distributive law holds, we get

(i) $A + \alpha C + \alpha^2 F = \alpha + \beta A + \gamma A + \delta A^2 + \varepsilon A^2 + \eta A^2$ which implies $(1-\beta-\gamma)A + \alpha(C-1) + \alpha^2 F - (\delta+\varepsilon+\eta)A^2 = 0$,
(ii) $B + \alpha E = \beta B + \gamma B + 2\delta AB + 2\varepsilon AB + 2\eta AB$ which implies $(1-\beta-\gamma)B + \alpha E - 2(\delta+\varepsilon+\eta)AB = 0$,
(iii) $\beta C + 2\alpha\beta F = \beta C + 2\delta AC + \varepsilon AC$ which implies $2\alpha\beta F = (2\delta+\varepsilon)AC$,
(iv) $\gamma C + 2\alpha\gamma F = \gamma C + \varepsilon AC + 2\eta AC$ which implies $2\alpha\gamma F = (2\eta+\varepsilon)AC$,
(v) $D = \beta D + \gamma D + 2\delta AD + \varepsilon B^2 + 2\varepsilon AD + 2\eta AD + \eta B^2 + \delta B^2$ which implies $(1-\beta-\gamma)D - (\delta+\varepsilon+\eta)(B^2+2AD) = 0$,
(vi) $\beta E = \beta E + 2\delta BC + \varepsilon CB + \varepsilon AE + 2\delta AE$ which implies $(2\delta+\varepsilon)(BC+AE) = 0$,
(vii) $\delta C + \beta^2 F + 2\alpha\delta F = \beta F + 2\delta AF + \delta C^2 + \varepsilon AF$,
(viii) $\eta C + \gamma^2 F + 2\alpha\eta F = \gamma F + \varepsilon AF + \eta C^2 + 2\eta AF$,
(ix) $\varepsilon C + 2\alpha\varepsilon F + 2\beta\gamma F = \varepsilon C^2$,
(x) $\gamma E = \gamma E + \varepsilon AE + \varepsilon BC + 2\eta BC + 2\eta AE$ which implies $(\varepsilon+2\eta)(AE+BC) = 0$,
(xi) $\delta E = 2\delta BF + 2\delta EC + \varepsilon BF$,
(xii) $\varepsilon E = 2\varepsilon CE$,
(xiii) $\eta E = \varepsilon BF + 2\eta BF + 2\eta EC$,
(xiv) $\delta^2 F = \delta F^2$,
(xv) $\varepsilon^2 F + 2\delta\eta F = \varepsilon F^2$,
(xvi) $\eta^2 F = \eta F^2$,
(xvii) $2\beta\delta F = 2\delta CF$,
(xviii) $2\beta\varepsilon F + 2\gamma\delta F = \varepsilon CF$,
(xix) $2\beta\eta F + 2\varepsilon\gamma F = \varepsilon CF$,
(xx) $2\gamma\eta F = 2\eta CF$,
(xxi) $2\varepsilon\delta F = 0$ which implies $\varepsilon\delta F = 0$,
(xxii) $2\varepsilon\eta F = 0$ which implies $\varepsilon\eta F = 0$,
(xxiii) $2\delta BD + 2\varepsilon BD + 2\eta BD = 0$ which implies $BD(\delta+\varepsilon+\eta) = 0$,
(xxiv) $2\delta BE + 2\delta CD + \varepsilon CD + \varepsilon EB = 0$ which implies $(2\delta+\varepsilon)(BE+CD) = 0$,
(xxv) $2\delta DF + \delta E^2 + \varepsilon DF = 0$ which implies $(2\delta+\varepsilon)DF + \delta E^2 = 0$,
(xxvi) $2\delta EF = 0$ which implies $\delta EF = 0$,
(xxvii) $2\delta ED + \varepsilon ED = 0$ which implies $(2\delta+\varepsilon)ED = 0$,
(xxviii) $\varepsilon CD + 2\eta BE + 2\eta CD + \varepsilon BE = 0$ which implies $(\varepsilon+2\eta)(BE+CD) = 0$,
(xxix) $\varepsilon ED + 2\eta ED = 0$ which implies $(2\eta+\varepsilon)ED = 0$,
(xxx) $\varepsilon DF + \eta E^2 + 2\eta DF = 0$ which implies $(2\eta+\varepsilon)DF + \eta E^2 = 0$,
(xxxi) $\varepsilon E^2 = 0$ which implies $\varepsilon E = 0$,
(xxxii) $\varepsilon FE = 0$,
(xxxiii) $2\eta EF = 0$ which implies $\eta EF = 0$,
(xxxiv) $(\delta+\eta+\varepsilon)D^2 = 0$ which implies $(\delta+\eta+\varepsilon)D = 0$.

Let $F = D = 0$ and $E \neq 0$ and $\beta + \gamma \neq 1$. Then $\delta = 0$, $\eta = 0$ and $\varepsilon = 0$ by (xxv), (xxx) and (xxxi). Hence $(1-\beta-\gamma)A + \alpha(C-1) = 0$ and $(1-\beta+\gamma)B + \alpha E = 0$ by (i) and (ii). Since $\beta+\gamma \neq 1$, so $A = \frac{\alpha(1-C)}{1-\beta-\gamma}$ and $B = \frac{-\alpha E}{1-\beta-\gamma}$. Thus,

$$x * y \quad = \quad \frac{\alpha(1-C)}{1-\beta-\gamma} - \frac{\alpha E}{1-\beta-\gamma}x + Cy + Exy.$$

Now, let $F \neq 0$ and $\varepsilon \neq 0$. then $\delta = \eta = E = 0$ by (xxi), (xxii) and (xxxi). So by (xxxiv), $(\delta+\eta+\varepsilon)D = \varepsilon D = 0$ and $D = 0$. By (xv), $\varepsilon^2 F = \varepsilon F^2$ and $\varepsilon = F$. By (v) $(1-\beta-\gamma)D - (\delta+\varepsilon+\eta)(B^2+2AD) = \varepsilon B^2 = 0$ and $B = 0$. By (xviii) and (xix) $C = 2\beta = 2\gamma$. So, $\gamma = \beta$ and $C = 2\beta$. By (viii) we get $\eta C + \gamma^2 F + 2\alpha\eta F = \gamma F + \varepsilon AF + \eta C^2 + 2\eta AF$. It shows that $\beta^2\varepsilon = \beta\varepsilon + \varepsilon^2 A$, and so $\beta^2 = \beta + \varepsilon^2 A$. On the other hand, by (ix), we have $2\varepsilon\beta + 2\alpha\varepsilon^2 + 2\beta^2\varepsilon = 4\varepsilon\beta^2$, and so $\beta + \alpha\varepsilon + \beta^2 = 2\beta^2$. This shows that

$\beta^2 = \beta + \alpha\varepsilon$. Therefore, $\varepsilon\alpha = A\varepsilon$, and so $\alpha = A$. Hence $\alpha = \frac{\beta^2-\beta}{\varepsilon} = \frac{\beta(\beta-1)}{\varepsilon}$. Thus,

$$\begin{aligned} x * y &= \frac{\beta(\beta-1)}{\varepsilon} + 2\beta y + \varepsilon y^2, \\ x \oplus y &= \frac{\beta(\beta-1)}{\varepsilon} + \beta(x+y) + \varepsilon xy. \end{aligned}$$

□

Lemma 7. *Let $(K, *, \oplus)$ be the quadratic-quadratic groupoid, with the left distributive law and $F = D = E = B = 0$. Then*

$$x \oplus y = \begin{cases} (1-\beta-\gamma)A - (\delta+\varepsilon+\eta)A^2 + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \eta y^2 & \text{if } C = 0, \\ \frac{(1-\beta-\gamma)A}{1-C} + \beta x + \gamma y & \text{if } C \notin \{0,1\}, \end{cases}$$

for all $x, y \in K$.

Proof. Since $F = D = E = B = 0$, then $(1-\beta-\gamma)A + \alpha(C-1) - (\delta+\varepsilon+\eta)A^2 = 0$, $(2\delta+\varepsilon)AC = 0$, $(2\eta+\varepsilon)AC = 0$, $\delta C(1-C) = 0$, $\eta C(1-C) = 0$ and $\varepsilon C(1-C) = 0$ by (i), (iii), (iv), (vii), (viii), (ix).

If $C = 0$, $\alpha = (1-\beta-\gamma)A - (\delta+\varepsilon+\eta)A^2$, then $x * y = A$ and

$$x \oplus y = (1-\beta-\gamma)A - (\delta+\varepsilon+\eta)A^2 + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \eta y^2.$$

If $C \notin \{0,1\}$, then $\delta = \varepsilon = \eta = 0$ and $(1-\beta-\gamma)A + \alpha(C-1) = 0$. Thus, $\alpha = \frac{(1-\beta-\gamma)A}{1-C}$ and

$$\begin{aligned} x * y &= A + Cy, \\ x \oplus y &= \frac{(1-\beta-\gamma)A}{1-C} + \beta x + \gamma y. \end{aligned}$$

□

Theorem 5. *Let $(K, *, \oplus)$ be the quadratic-quadratic groupoid.*

(*i*) *If $F = D = 0$, $E \neq 0$ and $\beta + \gamma \neq 1$, then $(K, *, \oplus, \xi)$ is a left distributive ringoid if and only if $\xi = -\alpha$ and it satisfies* (4) and (5).

(*ii*) If $F \neq 0$, $\varepsilon \neq 0$, then $(K - \{\frac{-\beta}{\varepsilon}\}, *, \oplus, \xi)$ is a left distributive ringoid if and only if $\xi = \frac{1-\beta}{\varepsilon}$ and it satisfies (7) and

$$x * y = \frac{\beta(\beta-1)}{\varepsilon} + 2\beta x + \varepsilon y^2, \tag{10}$$

for all $x, y \in K - \{\frac{-\beta}{\varepsilon}\}$.

(*iii*) If $F = D = E = B = C = 0$ and $\varepsilon \neq 0$, then $(K - \{\frac{-\beta}{\varepsilon}\}, *, \oplus, \xi)$ is a left distributive ringoid if and only if $\xi = \frac{1-\beta}{\varepsilon}$, $A \in \{\frac{-\beta}{\varepsilon}, \frac{1-\beta}{\varepsilon}\}$ and it satisfies (7).

(*iv*) If $F = D = E = B = C = \varepsilon = 0$, then $(K, *, \oplus, \xi)$ is a left distributive ringoid if and only if $\xi = A$ and it satisfies (8).

(*v*) If $F = D = E = B = 0$ and $C \notin \{0,1\}$, then $(K, *, \oplus, \xi)$ is a left distributive ringoid if and only if $\xi = \frac{A}{1-C}$ and it satisfies in (9).

Proof. (*i*) Assume $(K, *, \oplus, \xi)$ is a left distributive quadratic-quadratic ringoid. Then by Lemma 6, $\delta = \varepsilon = \eta = 0$ and $x \oplus \xi = \alpha + \beta x + \gamma\xi = \alpha + \beta\xi + \gamma x = \xi \oplus x = x$, for all $x, y \in K$. Therefore, $\beta = \gamma = 1$ and $\alpha + \gamma\xi = \alpha + \beta\xi = 0$. So, $\xi = -\alpha$. Hence $x \oplus y = \alpha + \beta x + \gamma y = \alpha + x + y$, for all $x, y \in K$. Also, $x * y = \frac{\alpha(1-C)}{1-\beta-\gamma} - \frac{\alpha E}{1-\beta-\gamma}x + Cy + Exy = \alpha(C-1) + \alpha Ex + Cy + Exy$. Conversely, easily the algebra $(K, *, \oplus)$ satisfies the left distributive law and $-\alpha$ is the zero element in $(K, \oplus)$. Now, assume $\hat{x}$ is the inverse of x in $(K, \oplus, -\alpha)$. So, $x \oplus \hat{x} = \alpha + x + \hat{x} = -\alpha$. This shows that $\hat{x} = -2\alpha - x$. Thus, $(K, \oplus, -\alpha)$ is an abelian group and $(K, *, \oplus, -\alpha)$ is a left distributive ringoid.

(*ii*) Assume $(K, *, \oplus, \xi)$ is a left distributive quadratic-quadratic ringoid. Then by Lemma 6 we get (8), (10) and $x \oplus \xi = \frac{\beta(\beta-1)}{\varepsilon} + \beta(x+\xi) + \varepsilon x\xi = x$, for all $x \in K$. Therefore, $\beta + \varepsilon\xi = 1$ and $\beta\xi + \frac{\beta(\beta-1)}{\varepsilon} = 0$. Hence $\xi = \frac{1-\beta}{\varepsilon}$.

Conversely, easily the algebra $(K, *, \oplus)$ satisfies the left distributive law and $\frac{1-\beta}{\varepsilon}$ is the zero element in $(K, \oplus)$. Now assume $\hat{x}$ is the inverse of x in $(K, \oplus, \frac{1-\beta}{\varepsilon})$. So, $x \oplus \hat{x} = \frac{\beta(\beta-1)}{\varepsilon} + \beta(x+\hat{x}) + \varepsilon x\hat{x} = \frac{1-\beta}{\varepsilon}$. If $x \neq \frac{-\beta}{\varepsilon}$, then $\hat{x} = \frac{1}{\beta+\varepsilon x}(-\beta x + \frac{1-\beta^2}{\varepsilon})$, for all $x \in K - \{\frac{-\beta}{\varepsilon}\}$. If $x = \frac{-\beta}{\varepsilon}$, then $\frac{-\beta}{\varepsilon} \oplus \hat{x} = \frac{\beta(\beta-1)}{\varepsilon} + \beta(\frac{-\beta}{\varepsilon} +$

$\hat{x})+\varepsilon\frac{-\beta}{\varepsilon}\hat{x}=\frac{1-\beta}{\varepsilon}$. Which proves that $-\beta=1-\beta$ and $0=1$, a contradiction. Thus there is not an inverse for the element $x=\frac{-\beta}{\varepsilon}$. Then $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \frac{1-\beta}{\varepsilon})$ is an abelian group and $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \frac{1-\beta}{\varepsilon})$ is a left distributive ringoid.
(iii) The proof is similar to the proof of Theorem 3(i).
(iv) The proof is similar to the proof of Theorem 3(ii).
(v) The proof is similar to the proof of Theorem 3(iii). □

Remark 3. *Theorem 5 means that the left distributive quadratic-quadratic ringoid in the case* (i) *is a left distributive quadratic-linear ringoid* (iii) *is the left distributive linear-quadratic ringoid, and* (iv), (v) *are the left distributive linear-linear ringoids.*

Example 6. (i) *Every (linear-linear, quadratic-linear, linear-quadratic) groupoid is a quadratic-quadratic groupoid, as a neutral way, so, every linear-linear, quadratic-linear, linear-quadratic ringoid is a quadratic-linear ringoid.*

(ii) *Consider an algebra* $(\mathbb{Q}, \cdot, +, 0)$ *where* $\mathbb{Q}$ *is the set of all rational numbers, "+" is the usual addition and "·" is the usual multiplication. Define binary operations "∗" and "⊕" on* $\mathbb{Q}$ *by* $x*y=3+6y+2y^2$ *and* $x\oplus y=3+3x+3y+2xy$. *Then* $(\mathbb{Q}, *, \oplus, -1)$ *is a quadratic-quadratic ringoid.*

(iii) *Consider an algebra* $(\mathbb{C}, \cdot, +, 0)$ *where* $\mathbb{C}$ *is the set of all complex numbers, "+" is the usual addition and "·" is the usual multiplication. Define binary operations "∗" and "⊕" on* $\mathbb{C}$ *by* $x*y=2i+5ix+3y+5xy$ *and* $x\oplus y=i+x+y$. *Then* $(\mathbb{C}, *, \oplus, -i)$ *is a distributive quadratic-quadratic ringoid.*

Remark 4. *Notice that let* $(K, *, \oplus)$ *be the quadratic-quadratic groupoid. If* $A=B=D=E=F=0$ *and* $C=1$, *then*

(i) *if* $\varepsilon\neq 0$, *then* $(K-\{\frac{-\beta}{\varepsilon}\}, *, \oplus, \xi)$ *is a left distributive ringoid if and only if* $\xi=\frac{1-\beta}{\varepsilon}$ *and it satisfies in* (7),

(ii) *if* $\varepsilon=0$, *then* $(K, *, \oplus, \xi)$ *is a left distributive ringoid if and only if* $\xi=-\alpha$ *and it satisfies in* (5).

Example 7. *Consider an algebra* $(\mathbb{Q}, \cdot, +, 0)$ *where* $\mathbb{Q}$ *is the set of all rational numbers, "+" is the usual addition and "·" is the usual multiplication. Define binary operations "∗" and "⊕" on* $\mathbb{Q}$ *by* $x*y=-8+12x-3y+6xy$ *and* $x\oplus y=2+x+y$. *Then* $(\mathbb{Q}, *, \oplus, -2)$ *is a quadratic-quadratic ringoid, but not a ring.*

Lemma 8. *Let* $(K, *, \oplus)$ *be the quadratic-quadratic groupoid, with the left distributive law. Then, for all* $x, y\in K$

$$x*y=\begin{cases} A+Bx+Cy+Dx^2 & \text{if } F=0, C\in\{0,1\},\ D\neq 0 \text{ and } \eta\neq 0 \text{ or}\\ & \quad F=\eta=0,\ D\neq 0 \text{ and } \varepsilon\neq 0 \text{ or}\\ & \quad F=\varepsilon=\eta=0,\ D\neq 0 \text{ and } \alpha\neq 0,\\ -\alpha(C-1)-\alpha^2F-\alpha Ex+Cy+Exy+Fy^2 & \text{if } F\neq 0,\ \varepsilon=\delta=\eta=\gamma=\beta=0,\\ A+Cy+Fy^2 & \text{if } F\neq 0,\ \varepsilon=\delta=0 \text{ and } \eta\neq 0 \text{ or}\\ & \quad F\neq 0,\ \varepsilon=0 \text{ and } \delta\neq 0 \end{cases}$$

and

$$x\oplus y=\begin{cases} \beta x+(1-\beta)y+\delta x^2-(\delta+\eta)xy+\eta y^2 & \text{if } F=C=0,\ D\neq 0 \text{ and } \eta\neq 0 \text{ or}\\ & \quad C=F=D=E=0 \text{ and } B\neq 0,\\ \alpha+\beta x+(1-\beta)y+\eta x^2-2\eta xy+\eta y^2 & \text{if } F=0, C=1\ D\neq 0 \text{ and } \eta\neq 0,\\ \beta x+(1-\beta)y-\varepsilon x^2+\varepsilon xy & \text{if } F=\eta=0,\ D\neq 0 \text{ and } \varepsilon\neq 0;\\ \alpha+\beta x+(1-\beta)y & \text{if } F=\varepsilon=\eta=0,\ D\neq 0 \text{ and } \alpha\neq 0,\\ \beta x+(1-\beta)y & \text{if } F=\varepsilon=\eta=\alpha=0 \text{ and } D\neq 0 \text{ or}\\ & \quad F=D=0,\ E\neq 0 \text{ and } \beta+\gamma=1 \text{ or}\\ & \quad \text{if } F=D=E=0,\ B\neq 0 \text{ and } C\notin\{0,1\},\\ \alpha+\beta x+(1-\beta)y+\delta x^2-2\delta xy+\delta y^2 & \text{if } F=D=E=B=0,\ C=1 \text{ and } A\neq 0 \text{ or}\\ & \quad F=D=E=0,\ B\neq 0 \text{ and } C=1,\\ x & \text{if } F\neq 0,\ \varepsilon=\delta=\eta=\gamma=0 \text{ and } \beta\neq 0,\\ y & \text{if } F\neq 0,\ \varepsilon=\delta=\eta=0 \text{ and } \gamma\neq 0,\\ A+Cy+Fy^2 & \text{if } F\neq 0,\ \varepsilon=\delta=0 \text{ and } \eta\neq 0,\\ A+Cx+Fx^2 & \text{if } F\neq 0,\ \varepsilon=0 \text{ and } \delta\neq 0. \end{cases}$$

Theorem 6. *Let $(K, *, \oplus)$ be the quadratic-quadratic groupoid in Lemma 8. Then there is no left distributive ringoid over K.*

Proof. By contrary, let $(K, \oplus, \xi)$ be an abelian group in Lemma 8, $F = C = 0, D \neq 0$ and $\eta \neq 0$, where ξ is the zero element in $(K, \oplus, \xi)$. Then by Lemma 8, we have $x \oplus \xi = \xi \oplus x = x$, for all $x \in K$. Hence $\beta x + (1-\beta)\xi + \delta x^2 - (\delta+\eta)\xi x + \eta\xi^2 = \beta\xi + (1-\beta)x + \delta\xi^2 - (\delta+\eta)\xi x + \eta x^2 = x$, which proves that $\eta = 0$, which is a contradiction.
Similarly, other cases are proven. □

Conclusions

By considering the notions of (quadratic-linear, linear-quadratic, quadratic-quadratic) groupoids, we investigate the left distributive ringoids over a field. Denote by LLG/LLR the set of all linear-linear groupoids/rigoids, LQG/LQR the set of all linear-quadratic groupoids/rigoids, QLG/QLR the set of all quadratic-linear groupoids/rigoids and QQG/QQR the set of all quadratic-quadratic groupoids/rigoids over a field K. Then we have the following diagrams:

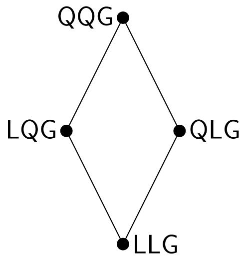

Diagram 1. Relation between LLG, LQG, QLG and QQG

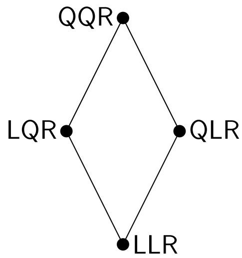

Diagram 2. Relation between LLR, LQR, QLR and QQR

A direction of research, one could investigate another polynomially defined ringoids and extend these results to module theory.

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