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Generalized Hukuhara diamond-alpha derivative of fuzzy valued functions on time scales

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Abstract. In the literature, the delta and nabla derivatives have been considered separately in the study of fuzzy number valued functions on time scales. In this paper, to unify these two derivatives for fuzzy number valued functions, we propose a new dynamic derivative called the diamond-alpha derivative, defined via the generalized Hukuhara difference. We establish several fundamental properties of the diamond-alpha derivative and investigate a particular class of fuzzy initial value problems on time scales with respect to this new derivative. Additionally, we provide numerical examples to illustrate our results.

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1. Introduction

Dynamic equations on time scales theory is a relatively new field of study, and research in this area is expanding considerably in the last 35 years. In order to combine continuous and discrete structures, time scale theory was established. It enables simultaneous treatments to both difference and differential equations and expands the results to dynamic equations. Basics of time-scale calculus and some recent studies can be found in [1, 2, 6, 7, 12–15, 18, 21, 22, 25]. However, it's crucial to consider a lot of uncertain aspects while attempting to fully explore a real-world phenomenon. Zadeh [35] developed fuzzy set theory in order to define these ambiguous or inaccurate concepts. Kaleva [16] and Lakshmikantham and Mohapatra [17] established and explored the theory of fuzzy differential equations (FDEs) and its applications. One drawback of the Hukuhara differentiability-based methods is that the solution to an FDE only exists for longer support lengths. Bede et al. [3] investigated generalized Hukuhara differentiability in order to get over this drawback. And many authors [4, 20, 28] are enthusiastic about this new differentiability concept for fuzzy number valued functions because of this favored benefit. Fard and Bidgoli [10] investigated the calculus of fuzzy functions on time scales. In their study of fuzzy dynamic equations on time scales, Vasavi et al. [31–34], by implementing the Hukuhara difference, introduced the Hukuhara, 2nd type Hukuhara and generalized delta derivatives. The drawback of this derivative is that it only applies to fuzzy number valued functions on time scales where the diameter increases with length.

To the best of our knowledge, the delta and nabla derivatives have been used independently to study the derivatives of fuzzy number valued functions on time scales. The characteristics of generalized nabla differentiability for fuzzy number valued functions on time scales via Hukuhara difference were presented and examined by Leelavathi et al. [19]. Additionally, they acquired some generalized nabla differentiable fuzzy number valued function embedding results. Furthermore, under generalized nabla differentiability, they demonstrated a fundamental principle of a nabla integral calculus for fuzzy functions on time scales. Fuzzy differential equations on time scales under generalized delta derivative were examined by Vasali

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et al [31]. In order to achieve solutions for FDEs with decreasing length of support, they established the generalized delta derivative, which is based on four forms. These types of derivatives, in some cases (such as time scales with discrete points), can only describe the change of functions only on the left or right side of considered points. In order to provide a tool that can catch the change of functions on both sides of points in time scales, dynamic derivatives, called the diamond alpha derivative has been proposed by Sheng et al. [27]. This dynamic derivative is a convex linear combination of the delta and nabla derivatives. Later, Roger et. al [26] redefined the diamond-alpha derivative independently of the standard delta and nabla dynamic derivatives, and further examined its properties. In [30], they introduced a dynamic derivative called diamond-alpha derivative via generalized Hukuhara difference for interval valued functions on time scales. They furthermore studied a particular class of interval differential equations with respect to the diamond-alpha derivative.

In this work, motivated by [30], we introduce a dynamic derivative called as the diamond-alpha derivative, denoted as $\diamond^{\alpha}_{gH}$, for fuzzy number valued functions on time scales via generalized Hukuhara difference and Hausdorff metric for fuzzy sets and investigate its properties under different conditions on time scale $\mathbb{T}$. Through our main results, we establish foundational results concerning the existence and uniqueness of the $\diamond^{\alpha}_{gH}$-derivative for fuzzy functions. Additionally, we explore conditions under which fuzzy functions are $\diamond^{\alpha}_{gH}$-differentiable at both dense and isolated points on the time scale, providing criteria for the existence of limits in these contexts. The final results address the differentiability of the r-level sets of fuzzy functions, particularly under monotonicity "length conditions". These results enhance the understanding of $\diamond^{\alpha}_{gH}$-differentiability in fuzzy functions and its applications within fuzzy differential equations on time scales.

This paper's outline is as follows: We give some basic definitions and results relating to the calculus of time scales and fuzzy sets in Section 2. In Section 3, we present the main results and provide some examples to illustrate some of the results. In Section 4, we consider a particular class of fuzzy initial value problems on time scales and present some numerical examples.

2. Preliminaries

Definition 1. *[6] A nonempty closed subset of the real numbers $\mathbb{R}$ is called a time scale, often denoted by $\mathbb{T}$.*

Definition 2. *[6] The function $\sigma : \mathbb{T} \to \mathbb{R}$ defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

is called the forward jump operator. Additionally, we set $\inf \varnothing := \sup \mathbb{T}$.

Definition 3. *[6] The function $\rho : \mathbb{T} \to \mathbb{R}$ defined by*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

is called the backward jump operator. Additionally, we set $\sup \varnothing := \inf \mathbb{T}$.

Definition 4. *[6] If $\sigma(t) > t$, then $t \in \mathbb{T}$ is said to be a right-scattered point.*

Definition 5. *[6] If $\rho(t) < t$, then $t \in \mathbb{T}$ is said to be a left-scattered point.*

Definition 6. *[6] If $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, then $t \in \mathbb{T}$ is said to be a right-dense point.*

Definition 7. *[6] If $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, then $t \in \mathbb{T}$ is said to be a left-dense point.*

Definition 8. *[6] The function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$ is called the (forward) graininess.*

Definition 9. *[6] The function $\nu : \mathbb{T} \to [0, \infty)$ defined by $\nu(t) = t - \rho(t)$ is called the backward graininess.*

Additionally, we define the following notations for simplicity in the definitions and theorems throughout this paper: $\mu_{st} = \sigma(s) - t$ and $\nu_{st} = t - \rho(s)$.
The set $\mathbb{T}^{\kappa}$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} := \mathbb{T} \setminus \{m\}$. If no such maximum exists, then $\mathbb{T}^{\kappa} := \mathbb{T}$. Similarly, the set $\mathbb{T}_{\kappa}$ is defined as follows: if $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} := \mathbb{T} \setminus \{m\}$. If no such minimum exists, then $\mathbb{T}_{\kappa} := \mathbb{T}$.

Definition 10. *[6] Let $h : \mathbb{T} \to \mathbb{R}$ be a function and let $s \in \mathbb{T}^{\kappa}$. We define $(\Delta h)(s)$ as the number (if it exists) that satisfies the following property: for any $\epsilon > 0$, there exists a neighborhood $N_{\mathbb{T}}$ of s given by $N_{\mathbb{T}} := (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|[h(\sigma(s)) - h(t)] - (\Delta h)(s)[\sigma(s) - t]| \leq \epsilon|\sigma(s) - t|$$

for all $t \in N_{\mathbb{T}}$. The value $(\Delta h)(s)$ is called the delta derivative of h at s.

Definition 11. *[6] Let $h : \mathbb{T} \to \mathbb{R}$ be a function, and let $s \in \mathbb{T}_{\kappa}$. We define $(\nabla h)(s)$ as the number (if it exists) that satisfies the following property: for any $\epsilon > 0$, there is a neighborhood $N_{\mathbb{T}}$ of s given by $N_{\mathbb{T}} := (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|[h(\rho(s)) - h(t)] - (\nabla h)(s)[\rho(s) - t]| \leq \epsilon|\rho(s) - t|$$

for all $t \in N_{\mathbb{T}}$. The value $(\nabla h)(s)$ is referred to as the nabla derivative of h at s.

Definition 12. *[26] Let $h : \mathbb{T} \to \mathbb{R}$ be a function and $s \in \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$. Then the $\diamond^{\alpha}$-derivative of h at the point $s \in \mathbb{T}^{\kappa}_{\kappa}$, denoted by $(\diamond^{\alpha} h)(s)$, is the number (provided it exists) that satisfies the following property: for any $\epsilon > 0$, there is a neighborhood $N_{\mathbb{T}}$ of s given by $N_{\mathbb{T}} := (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|\alpha|h(\sigma(s)) - h(t)||\nu_{st}| + (1-\alpha)|h(\rho(s)) - h(t)||\mu_{st}| - (\diamond^{\alpha} h)(s)|\nu_{st}\mu_{st}| \leq \epsilon|\nu_{st}\mu_{st}|,$$

for any $t \in N_{\mathbb{T}}$. Here, $(\diamond^{\alpha} h)(s)$ referred to as the diamond-alpha derivative of h at s.

Definition 13. *[35] A fuzzy set u in a universe of discourse U is represented by a function $u : U \to [0,1]$, where $u(x)$ indicates the membership degree of x to the fuzzy set u.*

We use $F(U)$ to denote the set of all fuzzy subsets of U.

Definition 14. *[23] Let $u : U \to [0,1]$ be a fuzzy set. The r-level sets of u are defined as*

$$u_r = \{x \in U : u(x) \geq r\}$$

for $0 < r \leq 1$. The 0-level set of u

$$u_0 = cl\,\{x \in U : u(x) > 0\}$$

is called the support of the fuzzy set u. Here, cl denotes the closure of the set u.

Definition 15. *[23] Let $u : \mathbb{R} \to [0,1]$ be a fuzzy subset of the real numbers. Then, u is said to be a fuzzy number if it fulfills the following criteria:*

1. *u is normal, which means that there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.*
2. *u is quasi-concave, which means that for all $\lambda \in [0,1]$, the inequality $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ holds.*
3. *u is upper semicontinuous on $\mathbb{R}$, which means that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $u(x) - u(x_0) < \epsilon$ whenever $|x - x_0| < \delta$.*
4. *u is compactly supported, which means that the closure $cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.*

We use $F_N(\mathbb{R})$ to denote the set of all fuzzy numbers of $\mathbb{R}$.

Definition 16. *Let $a_1 \leq a_2 \leq a_3$ be real numbers. The fuzzy number denoted by $u = (a_1, a_2, a_3)$ is called a triangular fuzzy number whose membership function is*

$$u(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 17. *[29] Let $u, v \in F_N(\mathbb{R})$. The generalized Hukuhara difference (gH-difference) is the fuzzy number w, if it exists, such that*

$$u \ominus_{gH} v = w \Longleftrightarrow u = v + w \text{ or } v = u + (-1)w.$$

Since level sets of a fuzzy number are closed and bounded intervals, we will denote r-level set of a fuzzy number u by $u_r = [u_r^-, u_r^+]$ and its length by $len(u_r) = u_r^+ - u_r^-$.

Remark 1. *The criteria for the existence of $w = u \ominus_{gH} v$ in $F_N(\mathbb{R})$ are as follows:*

Case (i):

- *$w_r^- = u_r^- - v_r^-$ and $w_r^+ = u_r^+ - v_r^+$*

- *Here, w_r^- must be increasing, w_r^+ must be decreasing, and it must hold that $w_r^- \leq w_r^+$ for all r in $[0,1]$.*

Case (ii):

- $w_r^- = u_r^+ - v_r^+$ *and* $w_r^+ = u_r^- - v_r^-$
- *Similarly, w_r^- must be increasing, w_r^+ must be decreasing, and $w_r^- \leq w_r^+$ must hold for all r in $[0,1]$.*

Theorem 1. *[5, 29] Let $u, v \in F_N(\mathbb{R})$. If gH-difference $u \ominus_{gH} v \in F_N(\mathbb{R})$ exists, then*

$$(u \ominus_{gH} v)_r = [\min\{u_r^- - v_r^-, u_r^+ - v_r^+\}, \max\{u_r^- - v_r^-, u_r^+ - v_r^+\}].$$

Definition 18. *[8] The metric $D_\infty : F_N(\mathbb{R}) \times F_N(\mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$ defined by*

$$D_\infty(u,v) = \sup_{r\in[0,1]} \max\left\{\left|u_r^- - v_r^-\right|, \left|u_r^+ - v_r^+\right|\right\},$$

where $u_r = [u_r^-, u_r^+]$, $v_r = [v_r^-, v_r^+]$, is called Hausdorff metric for fuzzy numbers.

The Hausdorff metric provides a way to measure the distance between two fuzzy sets by considering their level sets. This metric allows researchers to compare the similarity or dissimilarity of fuzzy sets in a rigorous mathematical way. Specifically, it can be used to quantify how far apart two fuzzy sets are based on their support and their membership functions.

Theorem 2. *[8]*
Let a, b, c, $d \in F_N(\mathbb{R})$ and $m \in \mathbb{R}$. The Hausdorff metric satisfies the followings:

(1) $D_\infty(a+c, b+c) = D_\infty(a,b)$.
(2) $D_\infty(ma, mb) = |m| D_\infty(a,b)$.
(3) $D_\infty(a+b, c+d) \leq D_\infty(a,c) + D_\infty(b,d)$.

3. Generalized Hukuhara Diamond-Alpha Derivative of Fuzzy Valued Functions on Time Scales

Definition 19. *[31] Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and let $s \in \mathbb{T}^\kappa$. The generalized Hukuhara delta derivative of f at s, if it exists, is a fuzzy number $(\Delta_{gH} f)(s) \in F_N(\mathbb{R})$ such that for any given $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}(s,\delta) = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that for all $t \in N_\mathbb{T}$, $f(\sigma(s)) \ominus_{gH} f(t)$ exists and we have*

$$D_\infty(f(\sigma(s)) \ominus_{gH} f(t), (\Delta_{gH} f)(s)\mu_{st}) \leq \epsilon\,|\mu_{st}|.$$

Definition 20. *[19] Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and let $s \in \mathbb{T}_\kappa$. The generalized Hukuhara nabla derivative of f at s, if it exists, is a fuzzy number $(\nabla_{gH} f)(s) \in F_N(\mathbb{R})$ such that for any given $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}(s,\delta) = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that for all $t \in N_\mathbb{T}$, $f(t) \ominus_{gH} f(\rho(s))$ exists and we have*

$$D_\infty(f(t) \ominus_{gH} f(\rho(s)), (\nabla_{gH} f)(s)\nu_{st}) \leq \epsilon\,|\nu_{st}|.$$

Definition 21. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and let $s \in \mathbb{T}_\kappa^\kappa$. The generalized Hukuhara diamond-alpha derivative of f at s, if it exists, is a fuzzy number $\left(\diamond_{gH}^\alpha f\right)(s) \in F_N(\mathbb{R})$ such that for any given $\epsilon > 0$, there exists a neighborhood $N_\mathbb{T}(s,\delta) = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that for all $t \in N_\mathbb{T}$ $f(\sigma(s)) \ominus_{gH} f(t)$ and $f(t) \ominus_{gH} f(\rho(s))$ exist and we have*

$$D_\infty(\alpha[f(\sigma(s)) \ominus_{gH} f(t)]\nu_{st} + (1-\alpha)[f(t) \ominus_{gH} f(\rho(s))]\mu_{st}, (\diamond_{gH}^\alpha f)(s)\mu_{st}\nu_{st}) \leq \epsilon\,|\mu_{st}\nu_{st}|.$$

Theorem 3. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a fuzzy function and $s \in \mathbb{T}_\kappa^\kappa$. $\left(\diamond_{gH}^\alpha f\right)(s) \in F_N(\mathbb{R})$ is unique, if it exists.*

Proof. Let $s \in \mathbb{T}_\kappa^\kappa$. Assume $\left(\diamond_{gH}^\alpha f\right)_1(s)$ and $\left(\diamond_{gH}^\alpha f\right)_2(s)$ are $\diamond_{gH}^\alpha$-derivative of f at s. Let $\epsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that for any $t \in N_\mathbb{T}(s,\delta)$ we have

$$D_\infty\big(\alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st} + (1-\alpha)\big[f(t) \ominus_{gH} f(\rho(s))\big]\,\mu_{st},$$
$$\left(\diamond_{gH}^\alpha f\right)_1(s)\,\mu_{st}\,\nu_{st}\big) \leq \frac{\epsilon}{2}\,|\mu_{st}\,\nu_{st}|,$$
$$D_\infty\big(\alpha\big[f(\sigma(s)) \ominus_{gH} f(t)\big]\,\nu_{st} + (1-\alpha)\big[f(t) \ominus_{gH} f(\rho(s))\big]\,\mu_{st},$$

$$\left(\diamond_{gH}^{\alpha} f\right)_2 (s)\, \mu_{st}\, \nu_{st}\right) \leq \frac{\epsilon}{2} \left|\mu_{st}\, \nu_{st}\right|.$$

$$\begin{aligned}
&D_\infty\left(\left(\diamond_{gH}^{\alpha} f\right)_1 (s), \left(\diamond_{gH}^{\alpha} f\right)_2 (s)\right) \\
&= \frac{1}{|\mu_{st}\, \nu_{st}|} D_\infty\left(\left(\diamond_{gH}^{\alpha} f\right)_1 (s)\, \mu_{st}\, \nu_{st}, \left(\diamond_{gH}^{\alpha} f\right)_2 (s)\, \mu_{st}\, \nu_{st}\right) \\
&= \frac{1}{|\mu_{st}\, \nu_{st}|} D_\infty \begin{pmatrix} \left(\diamond_{gH}^{\alpha} f\right)_1 (s)\, \mu_{st}\, \nu_{st} + \alpha\left[f(\sigma(s)) \ominus_{gH} f(t)\right] \nu_{st} \\ +(1-\alpha)\left[f(t) \ominus_{gH} f(\rho(s))\right] \mu_{st}, \\ \left(\diamond_{gH}^{\alpha} f\right)_2 (s)\, \mu_{st}\, \nu_{st} + \alpha\left[f(\sigma(s)) \ominus_{gH} f(t)\right] \nu_{st} \\ +(1-\alpha)\left[f(t) \ominus_{gH} f(\rho(s))\right] \mu_{st} \end{pmatrix} \\
&\leq \frac{1}{|\mu_{st}\, \nu_{st}|} D_\infty\left(\left(\diamond_{gH}^{\alpha} f\right)_1 (s)\, \mu_{st}\, \nu_{st}, \alpha\left[f(\sigma(s)) \ominus_{gH} f(t)\right] \nu_{st} \right. \\
&\quad \left. + (1-\alpha)\left[f(t) \ominus_{gH} f(\rho(s))\right] \mu_{st}\right) \\
&\quad + \frac{1}{|\mu_{st}\, \nu_{st}|} D_\infty\left(\left(\diamond_{gH}^{\alpha} f\right)_2 (s)\, \mu_{ts}\, \nu_{ts}, \alpha\left[f(\sigma(s)) \ominus_{gH} f(t)\right] \nu_{st} \right. \\
&\quad \left. + (1-\alpha)\left[f(t) \ominus_{gH} f(\rho(s))\right] \mu_{st}\right) \\
&\leq \frac{1}{|\mu_{st}\, \nu_{st}|} \frac{\epsilon}{2} \left|\mu_{st}\, \nu_{st}\right| + \frac{1}{|\mu_{st}\, \nu_{st}|} \frac{\epsilon}{2} \left|\mu_{st}\, \nu_{st}\right| \\
&\leq \epsilon.
\end{aligned}$$

Therefore, $\left(\diamond_{gH}^{\alpha} f\right)_1 (s) = \left(\diamond_{gH}^{\alpha} f\right)_2 (s)$. ☐

Theorem 4. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and $s \in \mathbb{T}_\kappa^\kappa$ a dense point. Then f is $\diamond_{gH}^{\alpha}$-differentiable at s if and only if the limit*

$$\lim_{t \to s} \frac{f(s) \ominus_{gH} f(t)}{s - t}$$

exists and

$$\left(\diamond_{gH}^{\alpha} f\right)(s) = \lim_{t \to s} \frac{f(s) \ominus_{gH} f(t)}{s - t}.$$

Proof. Since s is dense, $\sigma(s) = \rho(s) = s$. Hence, we obtain

$$\begin{aligned}
\alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}} &= \alpha \frac{f(s) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(t) \ominus_{gH} f(s)}{\nu_{st}} \\
&= \alpha \frac{f(s) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(s) \ominus_{gH} f(t)}{\mu_{st}} \\
&= (\alpha + 1 - \alpha) \frac{f(s) \ominus_{gH} f(t)}{\mu_{st}} \\
&= \frac{f(s) \ominus_{gH} f(t)}{\mu_{st}}.
\end{aligned}$$

So, we have

$$D_\infty\left(\frac{f(s) \ominus_{gH} f(t)}{\mu_{st}}, \left(\diamond_{gH}^{\alpha} f\right)(s)\right) < \epsilon.$$

Therefore,

$$\left(\diamond_{gH}^{\alpha} f\right)(s) = \lim_{t \to s} \frac{f(s) \ominus_{gH} f(t)}{s - t}.$$

☐

Theorem 5. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and $s \in \mathbb{T}_\kappa^\kappa$ be an isolated point. Then f is $\diamond_{gH}^{\alpha}$-differentiable at s and*

$$\left(\diamond_{gH}^{\alpha} f\right)(s) = \alpha \frac{f(\sigma(s)) \ominus_{gH} f(s)}{\mu(s)} + (1-\alpha) \frac{f(s) \ominus_{gH} f(\rho(s))}{\nu(s)}.$$

Proof. Since s is an isolated point, we have

$$\lim_{t \to s} \left[\alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha) \frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}} \right]$$

$$= \quad \alpha\frac{f(\sigma(s))\ominus_{gH} f(s)}{\mu(s)}+(1-\alpha)\frac{f(s)\ominus_{gH} f(\rho(s))}{\nu(s)}.$$

Hence, we obtain that f is $\diamond_{gH}^{\alpha}$-differentiable at s and

$$\left(\diamond_{gH}^{\alpha} f\right)(s)=\alpha\frac{f(\sigma(s))\ominus_{gH} f(s)}{\mu(s)}+(1-\alpha)\frac{f(s)\ominus_{gH} f(\rho(s))}{\nu(s)}.$$

□

Theorem 6. *Let $f:\mathbb{T}\to F_N(\mathbb{R})$ be a fuzzy function and $s\in\mathbb{T}_\kappa^\kappa$. Assume f is Δ_{gH} and ∇_{gH} differentiable at s. Then, f is $\diamond_{gH}^{\alpha}$-differentiable at s and*

$$(\diamond_{gH}^{\alpha} f)(s)=\alpha(\Delta_{gH} f)(s)+(1-\alpha)(\nabla_{gH} f)(s).$$

Proof. Let $\epsilon>0$ be given. Since f is Δ_{gH} and ∇_{gH} differentiable at s, there exists $\delta>0$ such that for any $t\in N_{\mathbb{T}}(s,\delta)=(s-\delta,s+\delta)\cap\mathbb{T}$, we have

$$\begin{aligned} D_\infty(f(\sigma(s))\ominus_{gH} f(t),(\Delta_{gH} f)(s)\mu_{st}) &\le \frac{\epsilon}{2}\left|\mu_{st}\right|,\\ D_\infty(f(t)\ominus_{gH} f(\rho(s)),(\nabla_{gH} f)(s)\nu_{st}) &\le \frac{\epsilon}{2}\left|\nu_{st}\right|. \end{aligned}$$

It follows that

$$\begin{aligned} D\left(\alpha\left[f(\sigma(s))\ominus_{gH} f(t)\right]\nu_{st},\alpha(\Delta_{gH} f)(s)\mu_{st}\nu_{st}\right) &\le \frac{\epsilon\alpha}{2}\left|\mu_{st}\nu_{st}\right|,\\ D\left((1-\alpha)\left[f(t)\ominus_{gH} f(\rho(s))\right]\mu_{st},(1-\alpha)(\nabla_{gH} f)(s)\mu_{st}\nu_{st}\right) &\le \frac{\epsilon(1-\alpha)}{2}\left|\mu_{st}\nu_{st}\right|. \end{aligned}$$

We get

$$\begin{aligned} &D\Big(\alpha\big[f(\sigma(s))\ominus_{gH} f(t)\big]\,\nu_{st}+(1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st},(\alpha(\Delta_{gH} f)(s)+(1-\alpha)(\nabla_{gH} f)(s))\;\mu_{st}\,\nu_{st}\Big)\\ &\quad\le D\big(\alpha\big[f(\sigma(s))\ominus_{gH} f(t)\big]\,\nu_{st},\alpha(\Delta_{gH} f)(s)\,\mu_{st}\,\nu_{st}\big)\\ &\qquad+D\big((1-\alpha)\big[f(t)\ominus_{gH} f(\rho(s))\big]\,\mu_{st},(1-\alpha)(\nabla_{gH} f)(s)\,\mu_{st}\,\nu_{st}\big)\\ &\quad\le\frac{\epsilon\alpha}{2}\left|\mu_{st}\,\nu_{st}\right|+\frac{\epsilon(1-\alpha)}{2}\left|\mu_{st}\,\nu_{st}\right|\\ &\quad\le\epsilon\left|\mu_{st}\,\nu_{st}\right|. \end{aligned}$$

Therefore, f is $\diamond_{gH}^{\alpha}$- differentiable at s and

$$(\diamond_{gH}^{\alpha} f)(s)=\alpha(\Delta_{gH} f)(s)+(1-\alpha)(\nabla_{gH} f)(s).$$

□

Theorem 7. *Let $f:\mathbb{T}\to F_N(\mathbb{R})$ be a function and the r-level sets of f be*

$$f_r(t)=\left[f_r^-(t),f_r^+(t)\right]$$

for any $t\in\mathbb{T}$ and $r\in[0,1]$, where $f_r^-:\mathbb{T}\to\mathbb{R}$ and $f_r^+:\mathbb{T}\to\mathbb{R}$ are the left and right end-points of the r-level sets. Assume $len(f_r(t)):=f_r^+(t)-f_r^-(t)$ is monotone on a neigborhood $N_{\mathbb{T}}(s,\delta)=(s-\delta,s+\delta)\cap\mathbb{T}$ for some $\delta>0$ and f is $\diamond_{gH}^{\alpha}$- differentiable at $s\in\mathbb{T}_\kappa^\kappa$. Then, f_r^- and f_r^+ are $\diamond^\alpha$-differentiable at s as well. Moreover,

(1) *if $len(f_r(t))$ is increasing on a neighborhood of $s\in\mathbb{T}_\kappa^\kappa$, then*

$$\left(\diamond_{gH}^{\alpha} f\right)_r(s)=\left[\left(\diamond^\alpha f_r^-\right)(s),\left(\diamond^\alpha f_r^+\right)(s)\right],$$

(2) *if $len(f_r(t))$ is decreasing on a neighborhood of $s\in\mathbb{T}_\kappa^\kappa$, then*

$$\left(\diamond_{gH}^{\alpha} f\right)_r(s)=\left[\left(\diamond^\alpha f_r^+\right)(s),\left(\diamond^\alpha f_r^-\right)(s)\right].$$

Proof. (1) Assume that $len(f_r(t)$ is increasing on $N_{\mathbb{T}}(s,\delta)=(s-\delta,s+\delta)\cap\mathbb{T}$ for some $\delta>0$, $\sigma(s)\neq t$ and $\rho(s)\neq t$ for any fixed $r\in[0,1]$. Let us consider the following cases.

Case 1: Let $\rho(s) < t < \sigma(s)$. Hence, $\mu_{st} = \sigma(s) - t > 0$ and $\nu_{st} = t - \rho(s) > 0$. Since $len(f_r(t))$ is increasing on $N_{\mathbb{T}}(s,\delta)$ we have $len(f(\sigma(s))) > len(f(t))$ and $len(f(t)) > len(f(\rho(s)))$ for any $t \in N_{\mathbb{T}}(s,\delta)$. Let

$$g(t) := \alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}.$$

By Remark 1 and some interval arithmetics, we obtain

$$\left[g_r^-(t), g_r^+(t)\right] = \left[\alpha \frac{f_r^-(\sigma(s)) - f_r^-(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^-(t) - f_r^-(\rho(s))}{\nu_{st}},\right.$$
$$\left.\alpha \frac{f_r^+(\sigma(s)) - f_r^+(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^+(t) - f_r^+(\rho(s))}{\nu_{st}}\right].$$

Case 2: Let $t > \sigma(s)$. Hence, $\mu_{st} = \sigma(s) - t < 0$ and $\nu_{st} = t - \rho(s) > 0$. Since $len(f_r(t))$ is increasing on $N_{\mathbb{T}}(s,\delta)$ we have $len(f(\sigma(s))) < len(f(t))$ and $len(f(t)) > len(f(\rho(s)))$ for any $t \in N_{\mathbb{T}}(s,\delta)$. Let

$$g(t) := \alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}.$$

By Remark 1 and some interval arithmetics, we obtain

$$\left[g_r^-(t), g_r^+(t)\right] = \left[\alpha \frac{f_r^-(\sigma(s)) - f_r^-(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^-(t) - f_r^-(\rho(s))}{\nu_{st}},\right.$$
$$\left.\alpha \frac{f_r^+(\sigma(s)) - f_r^+(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^+(t) - f_r^+(\rho(s))}{\nu_{st}}\right].$$

Case 3: Let $t < \rho(s)$. Hence, $\mu_{st} = \sigma(s) - t > 0$ and $\nu_{st} = t - \rho(s) < 0$. Since $len(f_r(t))$ is increasing on $N_{\mathbb{T}}(s,\delta)$ we have $len(f(\sigma(s))) > len(f(t))$ and $len(f(t)) < len(f(\rho(s)))$ for any $t \in N_{\mathbb{T}}(s,\delta)$. Let

$$g(t) := \alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}.$$

Similarly, by Remark 1 and some interval arithmetics, we obtain

$$\left[g_r^-(t), g_r^+(t)\right] = \left[\sigma \frac{f_r^-(\sigma(s)) - f_r^-(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^-(t) - f_r^-(\rho(s))}{\nu_{st}},\right.$$
$$\left.\alpha \frac{f_r^+(\sigma(s)) - f_r^+(t)}{\mu_{st}} + (1-\alpha)\frac{f_r^+(t) - f_r^+(\rho(s))}{\nu_{st}}\right].$$

Furthermore, since f is $\diamond_{gH}^{\alpha}$- differentiable at s, we derive

$$\lim_{t\to s}\left[\alpha \frac{f(\sigma(s)) \ominus_{gH} f(t)}{\mu_{st}} + (1-\alpha)\frac{f(t) \ominus_{gH} f(\rho(s))}{\nu_{st}}\right] = \left(\diamond_{gH}^{\alpha} f\right)(s) \in F_N(\mathbb{R}).$$

The proof of (2) can be done similarly. □

Definition 22. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and the r-level sets of f be*

$$f_r(t) = \left[f_r^-(t), f_r^+(t)\right]$$

for any $t \in \mathbb{T}$ and $r \in [0,1]$, where $f_r^- : \mathbb{T} \to \mathbb{R}$ and $f_r^+ : \mathbb{T} \to \mathbb{R}$ are the left and right end-points of the r-level sets. Assume that f is $\diamond_{gH}^{\alpha}$- differentiable at $s \in \mathbb{T}_{\kappa}^{\kappa}$. Then, f is said to be

(1) *$\diamond_{gH1}^{\alpha}$- differentiable at s if*

$$\left(\diamond_{gH}^{\alpha} f\right)_r(s) = \left[\left(\diamond^{\alpha} f_r^-\right)(s), \left(\diamond^{\alpha} f_r^+\right)(s)\right],$$

(2) *$\diamond_{gH2}^{\alpha}$- differentiable at s if*

$$\left(\diamond_{gH}^{\alpha} f\right)_r(s) = \left[\left(\diamond^{\alpha} f_r^+\right)(s), \left(\diamond^{\alpha} f_r^-\right)(s)\right].$$

Theorem 8. *Let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function and the r-level sets of f be*

$$f_r(t) = \left[f_r^-(t), f_r^+(t)\right]$$

for any $t \in \mathbb{T}$ and $r \in [0,1]$, where $f_r^- : \mathbb{T} \to \mathbb{R}$ and $f_r^+ : \mathbb{T} \to \mathbb{R}$ are left and right end-points of the r-level sets. Assume that f is $\diamond_{gH}^{\alpha}$- differentiable at $s \in \mathbb{T}_{\kappa}^{\kappa}$.

(1) *If f is $\diamond^{\alpha}_{gH1}$- differentiable on $N_T(s,\delta)$, then f has non-decreasing length of the closure of its support.*
(2) *If f is $\diamond^{\alpha}_{gH2}$- differentiable on $N_T(s,\delta)$, then f has non-increasing length of the closure of its support.*

Proof. (1) Assume f is $\diamond^{\alpha}_{gH1}$- differentiable on $N_{\mathbb{T}}(s,\delta)$ and $len(f_r(t))$ is decreasing length of the closure of its support for some $t \in N_{\mathbb{T}}(s,\delta)$ for any fixed $r \in [0,1]$. Then, by Theorem 7 we have

$$\left(\diamond^{\alpha}_{gH} f\right)_0 (t) = \left[\left(\diamond^{\alpha} f_0^{+}\right)(t), \left(\diamond^{\alpha} f_0^{-}\right)(t)\right],$$

which contradicts with the fact that f is $\diamond^{\alpha}_{gH1}$- differentiable on $N_{\mathbb{T}}(s,\delta)$. Hence, f has non-decreasing length of the closure of its support.

(2) Assume f is $\diamond^{\alpha}_{gH2}$- differentiable on $N_{\mathbb{T}}(s,\delta)$ and $len(f_r(t))$ is increasing length of the closure of its support for some $t \in N_{\mathbb{T}}(s,\delta)$ for any fixed $r \in [0,1]$. Then, by Theorem 7 we have

$$\left(\diamond^{\alpha}_{gH} f\right)_0 (t) = \left[\left(\diamond^{\alpha} f_0^{-}\right)(t), \left(\diamond^{\alpha} f_0^{+}\right)(t)\right].$$

which contradicts with the fact that f is $\diamond^{\alpha}_{gH2}$- differentiable on $N_{\mathbb{T}}(s,\delta)$. Hence, f has non-decreasing length of the closure of its support.

□

3.1. Examples.

Example 1. *Consider the time scale $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z}, h > 0\}$ and let $f : [0,\infty)_{\mathbb{T}} \to F_N(\mathbb{R})$ be a function such that $f(t) = (1,2,3)t$. The r-level sets of f are $f_r(t) = [1+r, 3-r]\,t$. By Theorem 5 , f is $\diamond^{\alpha}_{gH}$-differentiable at any $s \in [h,\infty)_{\mathbb{T}}$ such that*

$$\left(\diamond^{\alpha}_{gH} f\right)(s) = \alpha \frac{f(\sigma(s)) \ominus_{gH} f(s)}{\mu(s)} + (1-\alpha)\frac{f(s) \ominus_{gH} f(\rho(s))}{\nu(s)}.$$

Since $len(f_r(t)) = 2t(1-r)$, which is increasing for any fixed $r \in [0,1]$, we have

$$\left(\diamond^{\alpha}_{gH} f\right)_r (s) = \left[\left(\diamond^{\alpha} f_r^{-}\right)(s), \left(\diamond^{\alpha} f_r^{+}\right)(s)\right].$$

Let $\alpha = \frac{1}{2}$, then we have

$$\begin{aligned}
\left(\diamond^{\frac{1}{2}} f_r^{-}\right)(s) &= \frac{1}{2}\frac{f_r^{-}(\sigma(s)) - f_r^{-}(s)}{\mu(s)} + \frac{1}{2}\frac{f_r^{-}(s) - f_r^{-}(\rho(s))}{\nu(s)} \\
&= \frac{1}{2}\left[\frac{f_r^{-}(s+h) - f_r^{-}(s)}{h} + \frac{f_r^{-}(s) - f_r^{-}(s-h)}{h}\right] \\
&= \frac{1}{2}\left[\frac{(1+r)(s+h) - (1+r)s}{h} + \frac{(1+r)s - (1+r)(s-h)}{h}\right] \\
&= 1+r.
\end{aligned}$$

Similarly we can obtain $\left(\diamond^{\frac{1}{2}} f_r^{+}\right)(s) = 3-r$. Therefore, $\left(\diamond^{\frac{1}{2}}_{gH} f\right)_r (s) = [1+r, 3-r]$ and $\left(\diamond^{\frac{1}{2}}_{gH} f\right)(s) = (1,2,3)$.

Example 2. *Consider the time scale $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}\}$ and let $f : \mathbb{T} \to F_N(\mathbb{R})$ be a function such that $f(t) = (1,2,3)\frac{1}{t^2}$. The r-level sets of f are $f_r(t) = [1+r, 3-r]\,\frac{1}{t^2}$ and*

$$\begin{aligned}
len(f_r(t)) &= \frac{1}{t^2}(3-r-1-r) \\
&= \frac{1}{t^2}(2-2r) \\
&= \frac{2}{t^2}(1-r),
\end{aligned}$$

which is decreasing for any fixed $r \in [0,1]$. Hence, f is $\diamond^{\alpha}_{gH}$-differentiable at $s \in [\sqrt{2},\infty)_T$ with

$$\left(\diamond^{\alpha}_{gH} f\right)_r (s) = \left[\left(\diamond^{\alpha} f_r^{+}\right)(s), \left(\diamond^{\alpha} f_r^{-}\right)(s)\right],$$

where

$$\begin{aligned}
\left(\diamond^{\alpha} f_r^{-}\right)(s) &= \alpha \frac{f_r^{-}(\sigma(s)) - f_r^{-}(s)}{\mu(s)} + (1-\alpha)\frac{f_r^{-}(s) - f_r^{-}(\rho(s))}{\nu(s)} \\
&= \alpha \frac{\frac{1+r}{\sigma^2(s)} - \frac{1+r}{s^2}}{\sigma(s) - s} + (1-\alpha)\frac{\frac{1+r}{s^2} - \frac{1+r}{\rho^2(s)}}{s - \rho(s)} \\
&= \alpha \frac{\frac{1+r}{s^2+1} - \frac{1+r}{s^2}}{\sqrt{s^2+1} - s} + (1-\alpha)\frac{\frac{1+r}{s^2} - \frac{1+r}{s^2-1}}{s - \sqrt{s^2-1}}.
\end{aligned}$$

and

$$\begin{aligned}
\left(\diamond^{\alpha} f_r^{+}\right)(s) &= \alpha \frac{f_r^{+}(\sigma(s)) - f_r^{+}(s)}{\mu(s)} + (1-\alpha)\frac{f_r^{+}(s) - f_r^{+}(\rho(s))}{\nu(s)} \\
&= \alpha \frac{\frac{3-r}{s^2+1} - \frac{3-r}{s^2}}{\sqrt{s^2+1} - s} + (1-\alpha)\frac{\frac{3-r}{s^2} - \frac{3-r}{s^2-1}}{s - \sqrt{s^2-1}}.
\end{aligned}$$

4. Fuzzy Initial Value Problems on time scales with Generalized Hukuhara Diamond-Alpha Derivatives

In this section, we consider the following fuzzy initial value problem (FIVP):

$$(\diamond_{gH}^{\alpha} y)(t) = f(t, y(t)), \qquad t \in (a,b)_T \subset \mathbb{T}_{\kappa}^{\kappa} \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where $f : (a,b)_{\mathbb{T}} \times F_N(\mathbb{R}) \to F_N(\mathbb{R})$. Assume that r-level sets of y, f and $\diamond_{gH}^{\alpha} y$ are

$$\begin{aligned}
y_r(t) &= \left[y_r^{-}(t), y_r^{+}(t)\right], \\
f_r(t) &= \left[f_r^{-}(t), f_r^{+}(t)\right], \\
(\diamond_{gH}^{\alpha} y)_r(t) &= [(\diamond^{\alpha} y_r^{-})(t), (\diamond^{\alpha} y_r^{+})(t)].
\end{aligned}$$

There are two cases to be considered:

Case 1: $len(y_r)$ is increasing. By Theorem 7, we obtain

$$(\diamond_{gH}^{\alpha} y)_r(t) = [(\diamond^{\alpha} y_r^{-})(t), (\diamond^{\alpha} y_r^{+})(t)].$$

Therefore, FIVP (1)-(2) can be expressed by the system:

$$\begin{aligned}
(\diamond^{\alpha} y_r^{-})(t) &= f_r^{-}(t, y_r^{-}(t), y_r^{+}(t)), \\
(\diamond^{\alpha} y_r^{+})(t) &= f_r^{+}(t, y_r^{-}(t), y_r^{+}(t)), \\
y_r^{-}(t_0) &= y_{0r}^{-}, \\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}$$

where $r \in [0,1]$ and $t \in (a,b)_{\mathbb{T}}$.

Case 2: $len(y_r)$ is decreasing. By Theorem 7, we obtain

$$(\diamond_{gH}^{\alpha} y)_r(t) = [(\diamond^{\alpha} y_r^{+})(t), (\diamond^{\alpha} y_r^{-})(t)].$$

Therefore, FIVP (1)-(2) can be expressed by

$$\begin{aligned}
(\diamond^{\alpha} y_r^{-})(t) &= f_r^{+}(t, y_r^{-}(t), y_r^{+}(t)), \\
(\diamond^{\alpha} y_r^{+})(t) &= f_r^{-}(t, y_r^{-}(t), y_r^{+}(t)), \\
y_r^{-}(t_0) &= y_{0r}^{-}, \\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}$$

where $r \in [0,1]$ and $t \in (a,b)_{\mathbb{T}}$.

Assume $\mathbb{T} = \{t_0, t_1, t_2, ..., t_{N+1} : t_i < t_{i+1}, \forall i \in \overline{0,N}\}$ with $\mathbb{T}^{\kappa} = \mathbb{T}\backslash\{t_{N+1}\}$, $\mathbb{T}_{\kappa} = T\backslash\{t_0\}$ and $\mathbb{T}_{\kappa}^{\kappa} = \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$. Since $\mathbb{T}$ is an isolated time scale, according to Theorem 5 we obtain

$$(\diamond^{\alpha} y_r^{-})(t) = \alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)},$$

$$(\diamond^{\alpha} y_r^{+})(t) = \alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)}.$$

Hence, Case 1 and Case 2 become

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= f_r^{-}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= f_r^{+}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
y_r^{-}(t_0) &= y_{0r}^{-},\\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}$$

and

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= f_r^{+}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= f_r^{-}(t_i, y_r^{-}(t_i), y_r^{+}(t_i)),\\
y_r^{-}(t_0) &= y_{0r}^{-},\\
y_r^{+}(t_0) &= y_{0r}^{+},
\end{aligned}$$

respectively.

4.1. **Numerical Examples.** Now we will give some numerical examples. Triangular fuzzy numbers are widely used in fuzzy applications. They offer a straightforward and efficient means of representing and handling uncertainty and vagueness in data. Therefore, in numerical examples, fuzzy constants and initial conditions will be represented as triangular fuzzy numbers.

Example 3. *Let $\mathbb{T} = h\mathbb{Z}$ and let us consider the following FIVP:*

$$(\diamond_{gH}^{\alpha} y)(t) = -y(t), t \in (0,5)_{h\mathbb{Z}}, \quad (3)$$

$$y(0) = y(\sigma(h)) = y(h) = (-1, 0, 1). \quad (4)$$

Assume r-level sets of y and $\diamond_{gH}^{\alpha} y$ are

$$\begin{aligned}
y_r(t) &= \left[y_r^{-}(t), y_r^{+}(t)\right],\\
(\diamond_{gH}^{\alpha} y)_r(t) &= [(\diamond^{\alpha} y_r^{-})(t), (\diamond^{\alpha} y_r^{+})(t)].
\end{aligned}$$

By using the method above, we obtain the following two systems:

Case1: *Under $\diamond_{gH1}^{\alpha}$-differentiability, the FIVP yields the following system:*

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= -y_r^{+}(t_i),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= -y_r^{-}(t_i),\\
y_r^{-}(0) = y_r^{-}(h) &= -1 + r,\\
y_r^{+}(0) = y_r^{+}(h) &= 1 - r.
\end{aligned}$$

Case2: *Under $\diamond_{gH2}^{\alpha}$-differentiability, the FIVP yields the following system:*

$$\begin{aligned}
\alpha \frac{y_r^{-}(t_{i+1}) - y_r^{-}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{-}(t_i) - y_r^{-}(t_{i-1})}{\nu(t_i)} &= -y_r^{-}(t_i),\\
\alpha \frac{y_r^{+}(t_{i+1}) - y_r^{+}(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^{+}(t_i) - y_r^{+}(t_{i-1})}{\nu(t_i)} &= -y_r^{+}(t_i),\\
y_r^{-}(0) = y_r^{-}(h) &= -1 + r,\\
y_r^{+}(0) = y_r^{+}(h) &= 1 - r.
\end{aligned}$$

The approximate and true solutions to these systems for $h = \frac{1}{10}$, $\alpha = 0.8$, and $r = 0$ are illustrated in Figure 1 and Figure 2. In these figures, since the end points of the solutions do not switch, these solutions *exist on* $[0,5]$ *in both cases. We observe that switching of the end points of the solution may occur, and the error in the approximate solution may increase as we change* α*. In Case 1, when we set* $\alpha = 0.43$*, switching occurs at* $t = \frac{27}{10}$*, which implies that* $\diamond^{\alpha}_{gH1}$*-differentiability does not exist after* $t = \frac{19}{5}$*. And in Case 2, when we set* $\alpha = 0.53$*, switching occurs at* $t = \frac{19}{5}$*, which implies that* $\diamond^{\alpha}_{gH2}$*-differentiability does not exist after* $t = \frac{19}{5}$*; also see Figure 3.*

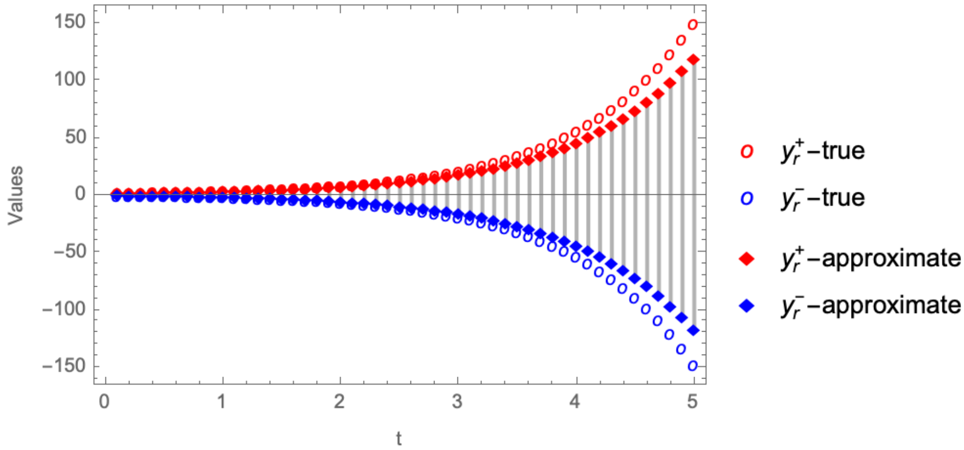

FIGURE 1. 0-level solutions to Case 1.

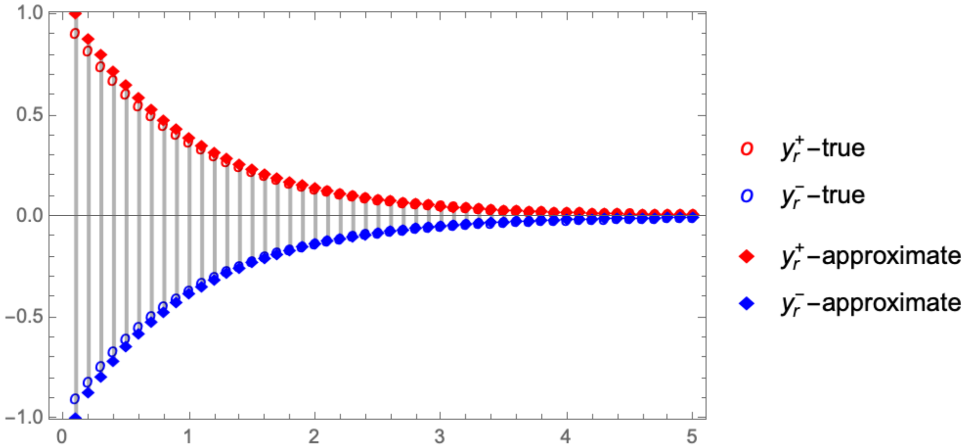

FIGURE 2. 0-level solutions to Case 2.

Example 4. *Let* $\mathbb{T} = h\mathbb{Z}$ *and let us consider the following FIVP:*

$$(\diamond^{\alpha}_{gH} y)(t) = -y(t) + (1,2,3)e^{-t}, t \in (0,5)_{h\mathbb{Z}}, \tag{5}$$

$$y(0) = y(\sigma(h)) = y(h) = (-2,0,2). \tag{6}$$

Assume r*-level sets of* y *and* $\diamond^{\alpha}_{gH} y$ *are*

$$y_r(t) = \left[y_r^-(t), y_r^+(t)\right],$$
$$(\diamond^{\alpha}_{gH} y)_r(t) = [(\diamond^{\alpha} y_r^-)(t), (\diamond^{\alpha} y_r^+)(t)].$$

By using the method above, we obtain the following two systems:

Case1: *Under* $\diamond^{\alpha}_{gH1}$*-differentiability, the FIVP yields the following system:*

$$\alpha \frac{y_r^-(t_{i+1}) - y_r^-(t_i)}{\mu(t_i)} + (1-\alpha)\frac{y_r^-(t_i) - y_r^-(t_{i-1})}{\nu(t_i)} = -y_r^+(t_i) + (1+r)e^{-t},$$

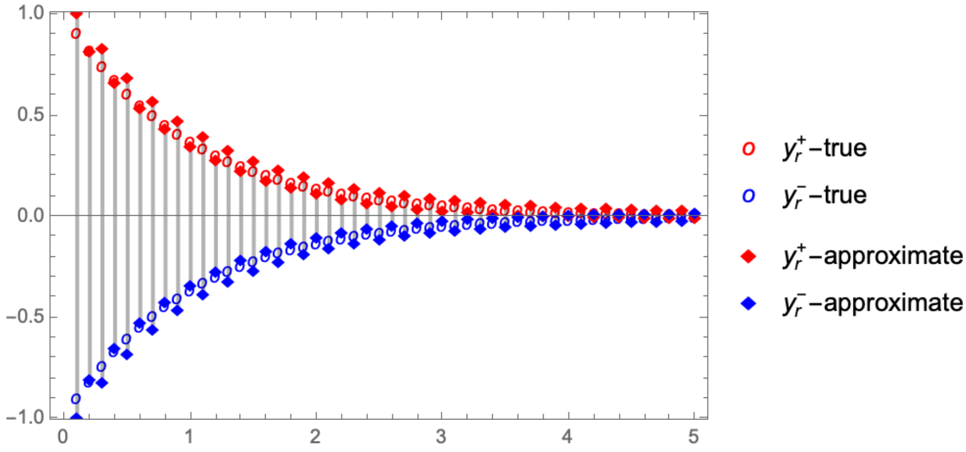

FIGURE 3. 0-level solutions to Case 2.

$$\begin{aligned}
\alpha\frac{y_r^+(t_{i+1})-y_r^+(t_i)}{\mu(t_i)}+(1-\alpha)\frac{y_r^+(t_i)-y_r^+(t_{i-1})}{\nu(t_i)} &= -y_r^-(t_i)+(3-r)e^{-t},\\
y_r^-(0)=y_r^-(h) &= -2+2r,\\
y_r^+(0)=y_r^+(h) &= 2-2r.
\end{aligned}$$

Case2: *Under $\diamond^{\alpha}_{gH2}$-differentiability, the FIVP yields the following system:*

$$\begin{aligned}
\alpha\frac{y_r^-(t_{i+1})-y_r^-(t_i)}{\mu(t_i)}+(1-\alpha)\frac{y_r^-(t_i)-y_r^-(t_{i-1})}{\nu(t_i)} &= -y_r^+(t_i)+(1+r)e^{-t},\\
\alpha\frac{y_r^+(t_{i+1})-y_r^+(t_i)}{\mu(t_i)}+(1-\alpha)\frac{y_r^+(t_i)-y_r^+(t_{i-1})}{\nu(t_i)} &= -y_r^-(t_i)+(3-r)e^{-t},\\
y_r^-(0)=y_r^-(h) &= -2+2r,\\
y_r^+(0)=y_r^+(h) &= 2-2r.
\end{aligned}$$

Figure 4 and Figure 5 illustrate the approximate and true solutions of these systems for $h=\frac{1}{15}$, $\alpha=0.6$, and $r=0$. In both figures, we have fuzzy solutions within the interval $[0,5]$ as there is no switching at the endpoints. In Case 1, setting $\alpha=0.45$ causes switching at $t=\frac{49}{15}$, indicating that $\diamond^{\alpha}_{gH1}$-differentiability in a neighborhood of $t=\frac{49}{15}$. In Case 2, setting $\alpha=0.53$ causes switching at $t=\frac{19}{5}$, indicating that $\diamond^{\alpha}_{gH2}$-differentiability in a neighborhood of $t=\frac{19}{5}$.

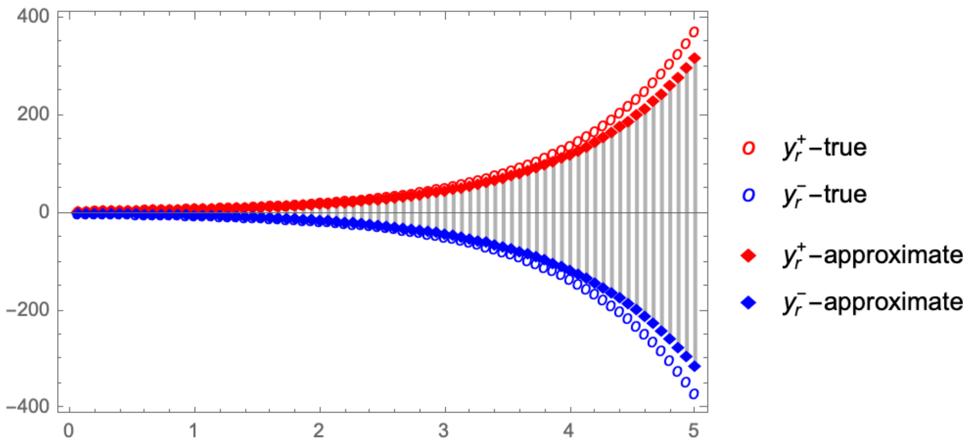

FIGURE 4. 0-level solutions to Case 1.

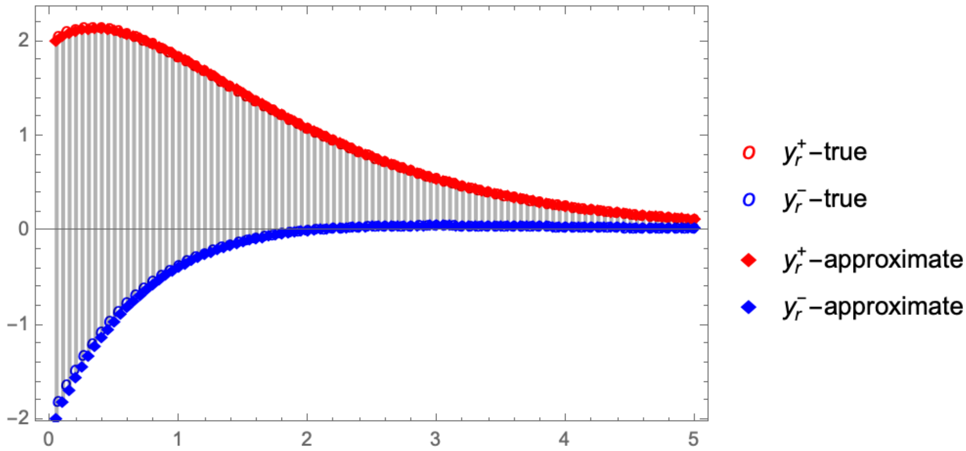

FIGURE 5. 0-level solutions to Case 2.

5. Conclusions

We have introduced the diamond-alpha derivative for fuzzy number valued functions on time scales by employing the generalized Hukuhara difference. Additionally, we have established some fundamental properties of this derivative and applied it to a specific class of fuzzy initial value problems on time scales. Numerical examples demonstrate the existence of approximate solutions under certain parameter settings with potential switching in the end points of the level sets of the solutions as the parameter α varies. Such switching can affect the accuracy of approximate solutions and the existence of $\diamond^{\alpha}_{gH1}$-differentiability or $\diamond^{\alpha}_{gH2}$-differentiability. These results enhance the understanding of the behavior of $\diamond^{\alpha}_{gH}$-differentiability in fuzzy functions and its applications within fuzzy differential equations on time scales.

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