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Research Article

Computation of Indeterminate Limit Forms for Multivariable Functions Using GUI_calcm in Calculus Education

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Abstract – Taking limit for multivariable functions is one of the essential topics of calculus courses in calculus education. When evaluating limits, one can frequently come across with the solution of indeterminate limit forms such as $\frac{0}{0}, \frac{c}{0}$ (c can be any real number). Well-known L'Hopital rule has been employed in the literature. However, this rule is lengthy and complicated process in some cases. So, finite difference methods as numerical methods have been developed and employed to solve these limits. All of the methods take too much time by hand computations. Main aim of the study is to design an indeterminate limit calculator: 'GUI_calcm' by use of Matlab GUI and APPS. The result of this study is to display the outputs obtained by each run of the calculator. These outputs are the results of each method with use of step size. Comparison of each method and performance of GUI_calcm are also presented.

Keywords: GUI_calcm, L' Hôpital rule, indeterminate limits, numerical methods.

Introduction

L' Hôpital rule is the most popular and well-known method for evaluating indeterminate limit forms: 0/0, c/0 for some functions of a single variable and series. This method is one of the important topics thought in courses of mathematics. In literature, various indeterminate forms for single variable functions were demonstrated in detail (Szabo, 1989; Huang, 1988, Young, 1910). Vyborny and Nester (1989) developed the basic rule with a counterexample. A study (Hartig, 1991) elaborated on L' Hôpital rule with use of integration. Other study (Gulmaro, 2018) suggested a proof for L' Hôpital rule. While, another study (Sheldon, 2017) explained L'Hôpital rule in different point of view. L'Hôpital rule was also described in terms of series (Takeuchi, 1995).

Similarly, it is the only method for solving indeterminate limit forms: 0/0, 1/0 for multivariable functions. For this purpose, in literature, there are some works elaborated on mulitvariable functions (Ivlev, 2013; Fine & Kass, 1966). Besides, Ivlev and Shilin (2014) and Lawlor (2020) generalized L'Hôpital rule for multivariable functions. Up to now, it is proved that L' Hôpital rule is the only basic method to overcome the complexity of cases: 0/0, 1/0 by taking the partial derivative of both numerator and denominator in literature (Lawlor, 2020; Ivlev and Shilin, 2014). This rule has also been taught as the only way to solve indeterminate limits both in engineering and also mathematics, physics education. So, neither books, digital sources nor also any other sources used in Calculus lectures propose a solution method.

Furthermore, it is such a complicated and lengthy process that takes partial derivative for multivariable functions in some situations. So applying L' Hôpital rule becomes inefficient and impractical method at that time. Moreover, L' Hôpital rule is inapplicable in case of isolated and non isolated singularities.

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For this purpose; Central Finite Difference (CFD), High Accurate Central Finite Difference (HACFD), Forward Finite Difference (FFD), High Accurate Forward Finite Difference (HAFFD), Backward Finite Difference (BFD), High Accurate Backward Finite Difference (HABFD) techniques instead of L' Hôpital rule have been recommended for multivariable functions (Bayen & Siauw, 2015; Chapra & Canale, 2010; Yogesh, 2012)

Nevertheless, each method leads to an individual computation effort and time by hand calculations. These computations take too much time. But a calculator can perform computations much easier and in a very short time. This provides users to give a decision about which method he/she should employ in a problem within a very short time. It is one of the main advantages of this study. Another advantage is to enable user compare each method in a limited time.

It has not been found in the literature such that a calculator is designed and then used in educational purpose which solves indeterminate limit problems for multivariable functions in any mathematical software. For this reason, it is impossible to make comparison with other software and any tool of artificial intelligence applications. Besides, necessity for design of such a calculator arises in the literature. One of the main targets of this study is to fulfill this requirement.

Instead there are calculators which compute different things in literature. Various calculators by Matlab GUI have been designed and also employed for various science and engineering applications in literature (Apaydın & Sevgi, 2014; Mohamed et al., 2010; Song et al., 2011). Mohamed et al. (2010) developed auto tuning PID controller using Graphical User Interface (GUI). Song et al. (2011) designed A Toolkit for resting-state functional magnetic resonance imaging data processing. Apaydın & Sevgi (2014) prepared FEM -parabolic-equation tool for path-loss calculations along multi-mixed-terrain paths. Mitchell (2014) designed a Matlab GUI for learning controller in the frequency domain. Singh et al. (2014) computed economic load dispatch. Malehmir and Schmitt (2016) designed ARTc which is an anisotropic reflectivity and transmissivity calculator. Chow (2016) performed computations in radiation therapy. Bishay (2016) designed FEApps for coding and App designing, with a deeper learning experience. Gupta and Patel (2017) prepared a teaching-learning tool for elementary psychrometric processes on psychrometric chart with use of MATLAB. Dinçkal (2018) designed a calculator which computes integrals numerically. Oke et al. (2018) determined biocoagulant dosage for water clarification using developed neuro-fuzzy network integrated with design of user interface-based calculator. John and Wara (2018) developed a smart calculator to determine the installed solar requirements for households and small businesses. Piris et al. (2021) designed a 3DHIP-Calculator which is a new tool to stochastically assess deep geothermal potential using the heat-in-place method from Voxel-based 3D geological models.

In literature, neither of the studies have purpose of designing a calculator which performs computations of indeterminate limit forms for multivariable functions with classical method: L'Hôpital rule and new improved numerical methods. Moreover, there is no calculator which also compute and display its performance in terms of CPU and elapsed time for each method. So it is not possible to compare the results of this study with other calculators designed in literature.

GUI_calcm is designed for computation of indeterminate limit forms for multivariable functions. It is the first time that a calculator was designed for this purpose. Matlab R2016a is used. Since it is the most convenient platform preferred in calculus education. (Bayen & Siauw, 2015; Yogesh, 2012). Matlab Graphical User Interface (GUI) and Matlab APPS (Bayen & Siauw, 2015; Yogesh, 2012) have been employed as tools for computation. Matlab GUI is preferred because of being a versatile, apparent and user-friendly computational platform. This calculator can be conveniently used by each computer by the contribution of Matlab APPS in computer laboratories.

The only thing for a laboratory technician to do is to install Matlab R2016a to each computer in for the usage of calculator when studying in the laboratory. Besides, GUI_calcm can be also stored in both Compact Disc and also USB by laboratory technician for usage in outside of campus.

Engineering students who registrated in Calculus course, educators, and researchers can have an opportunity to solve indeterminate limit problems in a laboratory environment and also outside of the laboratory. It is also the first time that the combination of Matlab GUI and Matlab APPS is employed for indeterminate limit computations for two variable functions with employment of finite difference methods for the educational purpose in calculus.

The embedded code behind GUI_calcm includes computation procedures of L' Hôpital rule, each finite difference approaches; CFD, HACFD, FFD, HAFFD, BFD, HABFD for two variable functions. Library users have an opportunity to select any method from GUI. User only enters the necessary inputs; point and point for L' Hôpital rule, step size for any indeterminate limit forms: 0/0, 1/0. Then calculator requires from user to select the method for solution and click the 'Calculate' button. Finally GUI_calcm gives the output in the same GUI with location: 'C'. Moreover user can also measure the performance of GUI_calcm in terms of Central Processing Unit (CPU), elapsed time of each selected method and also total CPU, elapsed time during whole computation process. CPU, elapsed time for each selected method and total CPU, elapsed time are displayed in command window after clicking 'Calculate' button.

This calculator is general and standard for any examples in mathematics education. For this purpose, after introducing calculator and its working principle, numerical results from each cases: 0/0, c/0 with also isolated and non isolated singularities for two variable functions are also presented.

Method

Indeterminate limit problem can be solved by the well-known method; L' Hôpital rule. In some cases, taking partial derivative is ineffective and impractical. It is necessary to apply L' Hôpital rule more than once to overcome indeterminate limits or L' Hôpital rule cannot be applicable in case of isolated and non isolated singularities. For this reason, new approaches including FFD, HAFFD, BFD, HABFD, CFD, HACFD methods (Dinçkal, 2025) can be preferred and also explained briefly in the following subsections as well as L' Hôpital rule.

The methods; L' Hôpital rule, CFD, HACFD, FFD, HAFFD and BFD, HABFD are thorough explained in following sections.

L'Hôpital rule

The famous method; L'Hôpital rule for two variables functions has some steps and shown as follows (Ivlev & Shilin, 2014; Lawlor, 2020)

$$\lim_{(x,y)\to(x_0,\,y_0)} \frac{f(x,y)}{g(x,y)} \tag{1}$$

Where $\mathbf{\lim\limits_{(x,y)\to(x_0,\,y_0)} f(x,y) = 0}$ or a number and $\mathbf{\lim\limits_{(x,y)\to(x_0,\,y_0)} g(x,y) = 0}$.

So one should take partial derivative with respect to x and y, at points x_0 and y_0 respectively (Ivlev & Shilin, 2014; Lawlor, 2020; Dinçkal, 2025)

$$\frac{f_x(x_0,y_0)}{g_x(x_0,y_0)} \tag{2a}$$

$$\frac{f_y(x_0,y_0)}{g_y(x_0,y_0)} \tag{2b}$$

Under the condition that eq (2a) should be equal to eq (2b) such that

$$\frac{f_x(x_0, y_0)}{g_x(x_0, y_0)} = \frac{f_y(x_0, y_0)}{g_y(x_0, y_0)} = \boldsymbol{k_1} \tag{3}$$

Presently, if indeterminate form is found again as a result of eq (3), second order partial derivatives should be taken (Ivlev & Shilin, 2014; Lawlor, 2020):

$$\frac{f_{xx}(x_0, y_0)}{g_{xx}(x_0, y_0)} = \frac{f_{xy}(x_0, y_0)}{g_{xy}(x_0, y_0)} = \frac{f_{yx}(x_0, y_0)}{g_{yx}(x_0, y_0)} = \frac{f_{yy}(x_0, y_0)}{g_{yy}(x_0, y_0)} = \boldsymbol{k_2} \tag{4}$$

L'Hôpital rule for multivariable functions confirms the rules of partial derivation. So this method can be considered as being an additional rule of partial derivation. Even so, this is still not adequate for case of isolated and non isolated singularities (Ivlev & Shilin, 2014; Lawlor, 2020).

Central Finite Difference (CFD) and High Accurate Central Finite Difference (HACFD)

The assumptions have been made by changing x variable and not changing y variable in Taylor series expansions for all finite difference methods. On the contrary, changing y variable and fixing x variable in Taylor series expansions do not make difference in the results.

The originating idea of CFD, HACFD, FFD, HAFFD, BFD and HABFD techniques is based on well-known Taylor series (Dinçkal, 2025).

The form of the Taylor series by defining a step size $h=x_{i+1}-x_i$ and expressing as

$$f(x_{i+1}, y) = f(x_i, y) + f'(x_i, y)h + \frac{f''(x_i,y)}{2!}h^2 + \frac{f^{(3)}(x_i,y)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i,y)}{n!}h^n + R_n \tag{5}$$

where the remainder term is defined as

$$R_n = \frac{f^{(n+1)}(\varepsilon_1,\varepsilon_2)}{(n+1)!}h^{n+1} \tag{6}$$

The term in eq (6) corresponds to $O((x_{i+1} - x_i)^{n+1})$ which is $O(h^{n+1})$ called as error.

For backward form, Taylor series in eq (5) can be rewritten as

$$f(x_{i-1}, y) = f(x_i, y) - f'(x_i, y)h + \frac{f''(x_i,y)}{2!}h^2 - \frac{f^{(3)}(x_i,y)}{3!}h^3 + \cdots \tag{7}$$

One of the ways to approximate the first derivative is to subtract eq (7) from the Taylor series expansion in eq (5) to obtain:

$$f(x_{i+1}, y) = f(x_{i-1}, y) + 2f'(x_i, y)h + \frac{2f^{(3)}(x_i,y)}{3!}h^3 + \cdots \tag{8}$$

which can be solved for

$$f'(x_i, y) = \frac{f(x_{i+1},y)-f(x_{i-1},y)}{2h} - \frac{f^{(3)}(x_i,y)}{3!}h^2 - \cdots \tag{9}$$

Eq (9) can be also expressed as

$$f'(x_i, y) = \frac{f(x_{i+1},y)-f(x_{i-1},y)}{2h} + O(h^2) \tag{10}$$

By use of eq (10), one of the methods including CFD for solving indeterminate limit form is formulated as

$$\lim_{(x,y)\to(x^*,\, y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{f(x^*{}_{i+1},y)-f(x^*{}_{i-1},y)}{2h}+O(h^2)}{\frac{g(x^*{}_{i+1},y)-g(x^*{}_{i-1},y)}{2h}+O(h^2)} \tag{11}$$

Both numerator and denominator in terms of CFD have errors which are $O(h^2)$. This means that errors are proportional to the square of the same step size for both $f(x, y)$ and $g(x, y)$. Error is of the order of h^2 in spite of the forward and backward approximations that are of the order of h (Chapra & Canale, 2010).

Level of accuracy depends on both decreasing the step size and also the number of terms of the Taylor series during the derivation of these formulas. Hence, it is possible to develop more accurate formulas called as HACFD by withholding more terms.

By substituting first order derivative in eq (10) into eq (5), centered finite difference representation of the second order derivative based on error $O(h^2)$ can be found as

$$f''(x_i,y) = \frac{f(x_{i+1},y) - 2f(x_i,y) + f(x_{i-1},y)}{h^2} \tag{12}$$

Third order derivative based on error $O(h^2)$:

$$f^{(3)}(x_i,y) = \frac{f(x_{i+2},y) - 2f(x_{i+1},y) + 2f(x_{i-1},y) - f(x_{i-2},y)}{2h^3} \tag{13}$$

To find the high-accurate form of first derivative based on error $O(h^4)$, one should use both eq (12) and eq (13), and substitute them into eq (5):

$$f(x_{i+1},y) = f(x_i,y) + f'(x_i,y)h + \frac{\frac{f(x_{i+1},y)-2f(x_i,y)+f(x_{i-1},y)}{h^2}}{2!}h^2 + \frac{\frac{f(x_{i+2},y)-2f(x_{i+1},y)+2f(x_{i-1},y)-f(x_{i-2},y)}{2h^3})}{3!}h^3 + \cdots \tag{14}$$

So $f'(x_i)$ based on error $O(h^4)$ can be obtained from eq (14) which is

$$f'(x_i,y) = \frac{-f(x_{i+2},y) + 8f(x_{i+1},y) - 8f(x_{i-1},y) + f(x_{i-2},y)}{12h} + O(h^4) \tag{15}$$

By use of eq (15), another method including HACFD for solving indeterminate limit is written as

$$\lim_{(x,y)\to(x^*,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{-f(x^*{}_{i+2},y)+8f(x^*{}_{i+1},y)-8f(x^*{}_{i-1},y)+f(x^*{}_{i-2},y)}{12h} + O(h^4)}{\frac{-g(x^*{}_{i+2},y)+8g(x^*{}_{i+1},y)-8g(x^*{}_{i-1},y)+g(x^*{}_{i-2},y)}{12h} + O(h^4)} \tag{16}$$

Forward Finite Difference (FFD) and High Accurate Forward Finite Difference (HAFFD)

With similar fashion, first derivative by FFD based on O(h) is (Chapra & Canale, 2010; Dinçkal, 2025)

$$f'(x_i,y) = \frac{f(x_{i+1},y) - f(x_i,y)}{h} + O(h) \tag{17}$$

By use of eq (17), FFD technique for solving indeterminate limit is written as

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{f(x^*{}_{i+1},y)-f(x^*{}_{i},y)}{h} + O(h)}{\frac{g(x^*{}_{i+1},y)-g(x^*{}_{i},y)}{h} + O(h)} \tag{18}$$

More accurate form of first derivative by FFD based on O(h^2) can be formulated as . (Chapra S.C., Canale R.P. (2010))

$$f'(x_i,y) = \frac{-f(x_{i+2},y) + 4f(x_{i+1},y) - 3f(x_i,y)}{2h} + O(h^2) \tag{19}$$

By use of eq (19), HAFFD method for solving indeterminate limit become

$$\lim_{(x,y)\to(x^*,\,y^{*})} \frac{f(x,y)}{g(x,y)} = \frac{\frac{-f(x^*{}_{i+2},y)+4f(x^*{}_{i+1},y)-3f(x^*{}_{i},y)}{2h} + O(h^2)}{\frac{-g(x^*{}_{i+2},y)+4g(x^*{}_{i+1},y)-3g(x^*{}_{i},y)}{2h} + O(h^2)} \tag{20}$$

Backward Finite Difference (BFD) and High Accurate Backward Finite Difference (HABFD)

Likewise, first derivative by BFD based on O(h) is (Chapra & Canale, 2010; Dinçkal, 2025)

$$f'(x_i, y) = \frac{f(x_i,y) - f(x_{i-1},y)}{h} + O(h) \tag{21}$$

By employment of eq (19), BFD technique for solving indeterminate limit can be formulated as

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{f(x^*{}_i,y) - f(x^*{}_{i-1},y)}{h} + O(h)}{\frac{g(x^*{}_i,y) - g(x^*{}_{i-1},y)}{h} + O(h)} \tag{22}$$

More accurate form of first derivative by BFD based on O(h^2) can be written as (Chapra & Canale, 2010)

$$f'(x_i, y) = \frac{3f(x_i,y) - 4f(x_{i-1},y) + f(x_{i-2},y)}{2h} + O(h^2) \tag{23}$$

By use of eq (23), HABFD method for solving indeterminate limit become

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{3f(x^*{}_i,y) - 4f(x^*{}_{i-1},y) + f(x^*{}_{i-2},y)}{2h} + O(h^2)}{\frac{3g(x^*{}_i,y) - 4g(x^*{}_{i-1},y) + g(x^*{}_{i-2},y)}{2h} + O(h^2)} \tag{24}$$

Indeterminate Limit Calculator: GUI_calcm

Matlab R2016a (9.0.0.341360) was employed for both creation of this calculator and its implementation. The general procedure for designing an indeterminate limit calculator for two variable functions is comprised of some steps. These steps are standard and applicable for any example of the forms: $\frac{0}{0}, \frac{c}{0}$. (c can be any real number) The first step is to generate a GUI_calcm .The standard initial form of GUI_calcm for any problem is illustrated in Figure 1. In GUI_calcm, input boxes such as the point 1 ('x' in GUI) and point for L' Hôpital rule ('y' in GUI) , step size for finite difference methods ('h' in GUI), functions for numerator ('f1' in GUI) and denominator ('f2' in GUI) are created. Pop-up Menu with default method named L' Hôpital Rule includes all methods for solution of the indeterminate limits. User has opportunity to select any method from here. To start the computations, 'Calculate' button should be clicked. After clicking this button, user can see the result of the indeterminate limit forms: $\frac{0}{0}, \frac{1}{0}$ in output box; called as 'C'.

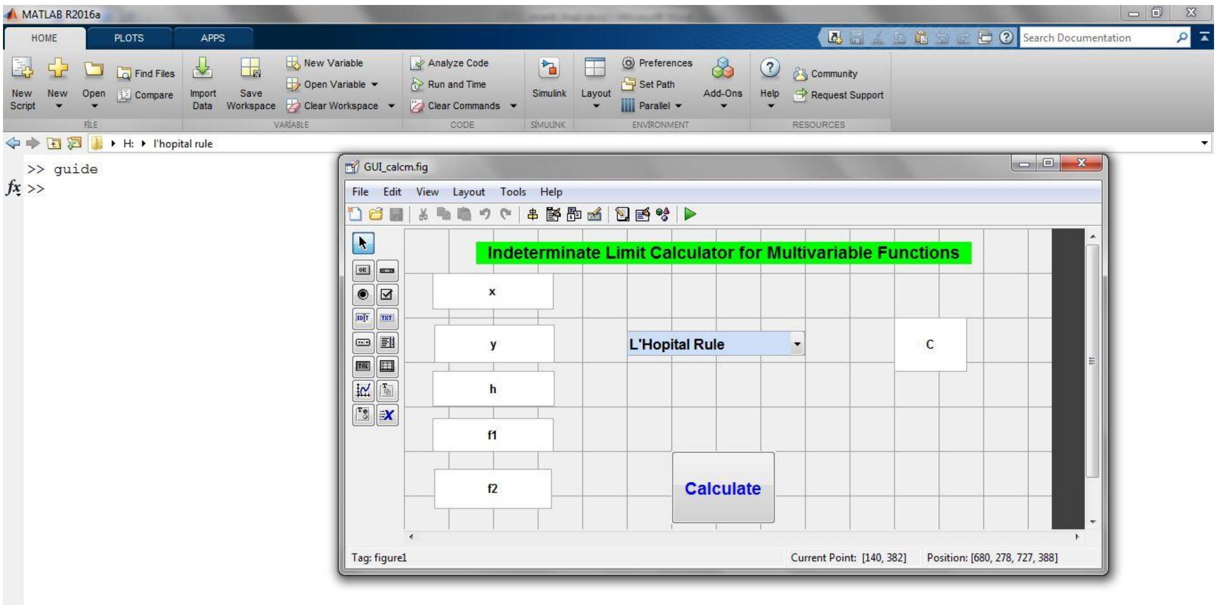

Figure 1. Initial Form of GUI_calcm

Figure 1 is the template form of the calculator. To run this calculator, user should write GUI_calcm in command window, GUI_calcm is appeared as it is displayed in Figure 2. This means that it is waiting for inputs and selection of method from Pop-up menu and then it is ready for computations.

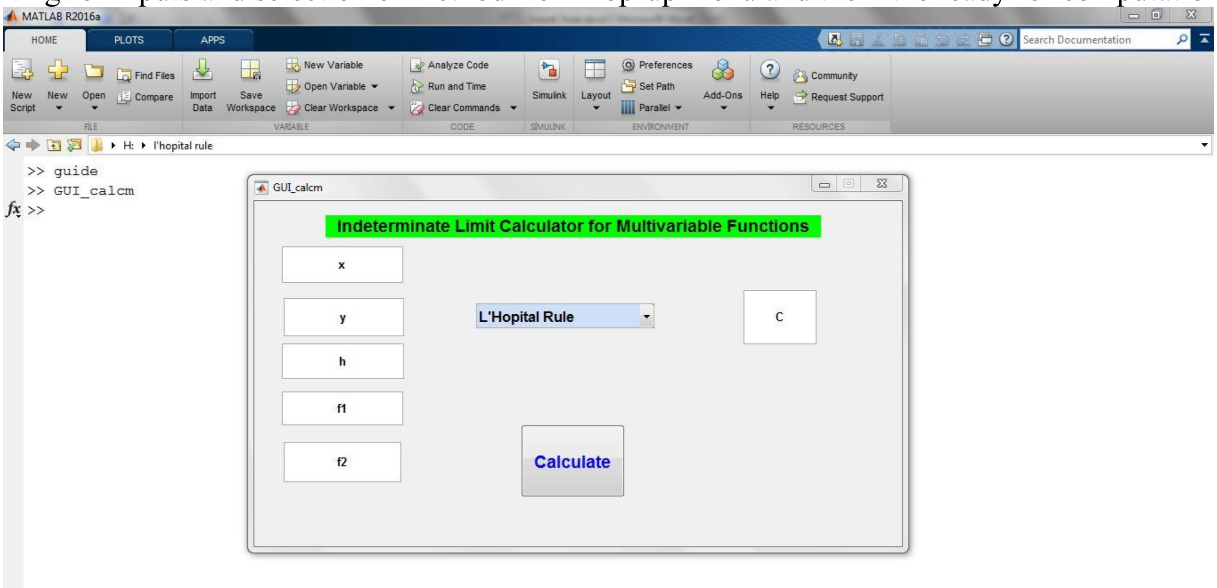

Figure 2. GUI_calcm Waiting for Inputs

Explicit Pop-up Menu selection is also given in orange frame selected in Figure 3. It is possible to see all methods, simultaneously. CFD_O(h^2) and CFD_O(h^4) represent CFD based on O(h^2) and HACFD based on O(h^4), respectively. BFD_O(h) and BFD_O(h^2) stand for BFD based on O(h) and HABFD based on O(h^2), correspondingly. FFD_O(h) and HAFFD_O(h^2) typify FFD based on O(h) and HAFFD based on O(h^2), respectively.

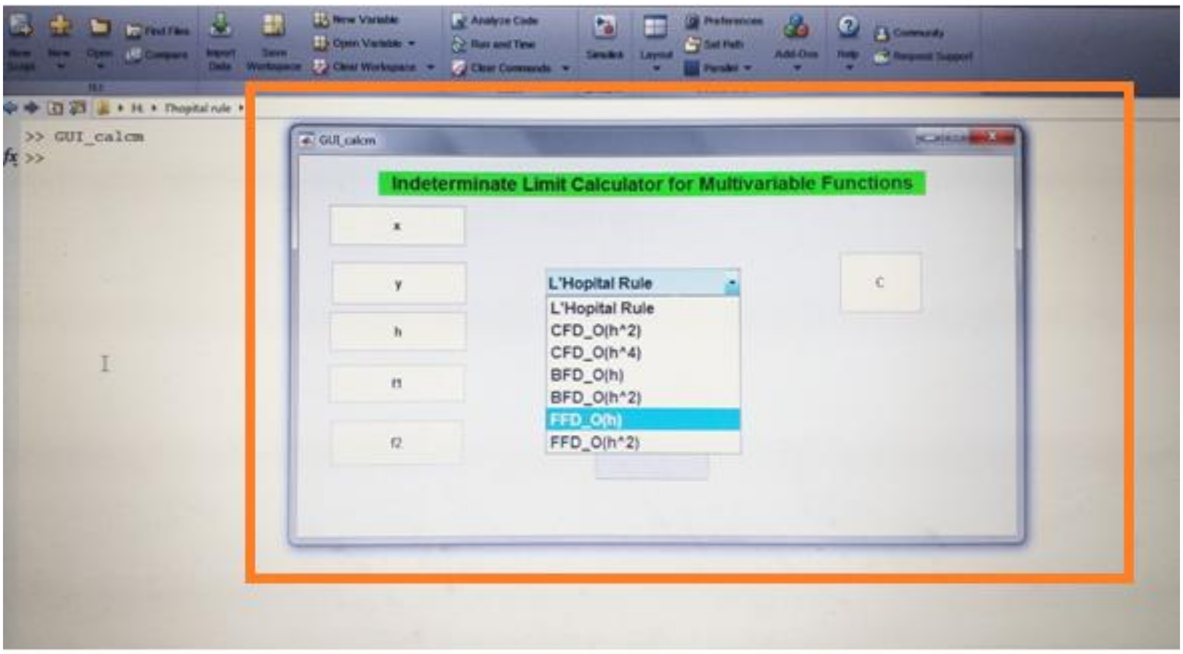

Figure 3. GUI_calcm with explicit Pop-up Menu selection

For general usage in all computers in a library, APPS version of GUI_calcm was created. It is presented in Figure 4.

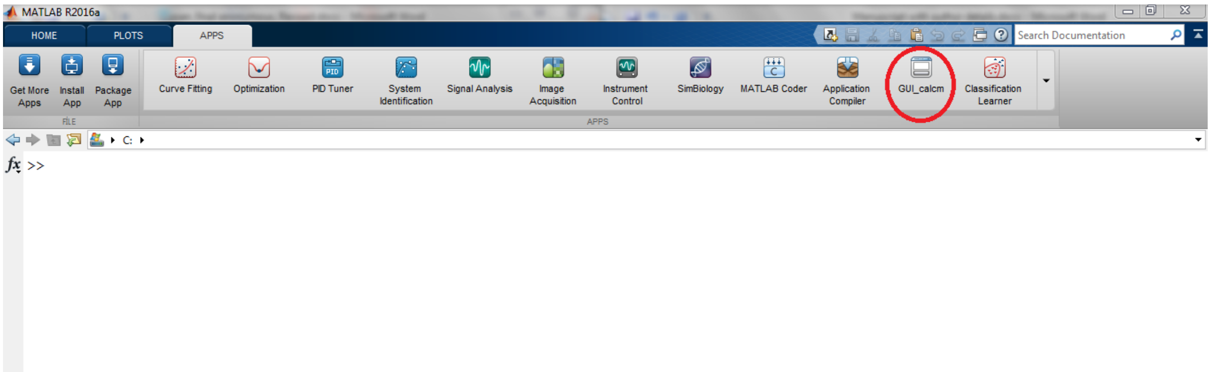

Figure 4. GUI_calcm for APPS

It is so simple to call for GUI_calcm from Matlab APPS. Only clicking GUI_calcm is sufficient. This situation is displayed in Figure 5.

By use of Matlab APPS, it is possible to call from matlab applications. So there is no need to call GUI_calcm from other devices. For this purpose, Figure 4 and 5 are presented.

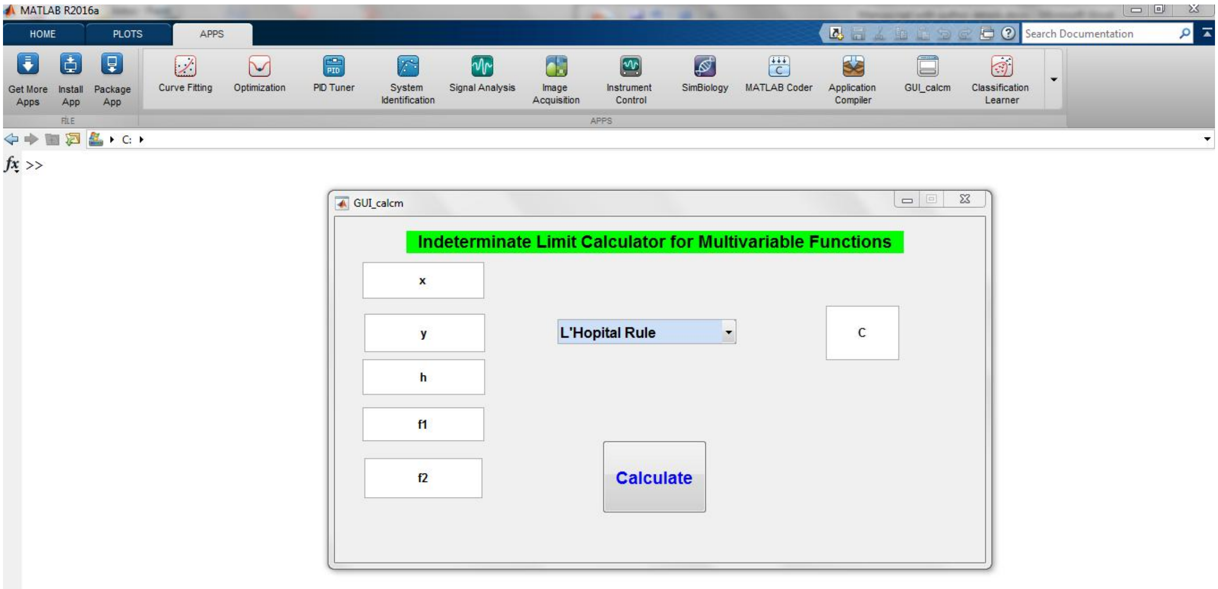

Figure5. Call for GUI_calcm from APPS

Examples from each indeterminate form are presented in following subsections.

$\frac{0}{0}$ *Case*

As an example to $\frac{0}{0}$ case; $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$ is solved by each method with step size: h=0.0001. Figure 6 displays the solution of $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$ with selection of L' Hôpital Rule. CPU, elapsed time for L' Hôpital Rule ('fintimeforLh' and 'elapsedforLh' in GUI, correspondingly) and total CPU, elapsed time ('fintime' and 'elapsed' in GUI, respectively) is appeared in command window.

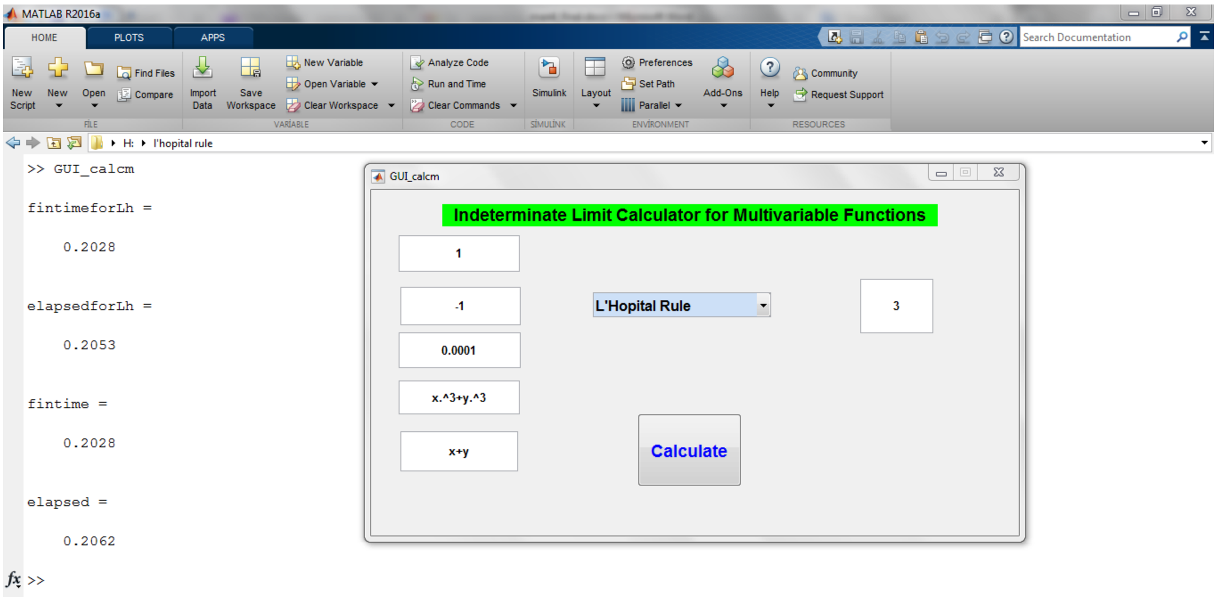

Figure 6. GUI_calcm with L' Hôpital Rule Selection for $\frac{0}{0}$ Case

Similarly, same example can be solved by CFD and HACFD methods selection with h=0.0001 which are illustrated in Figures 7 and 8, respectively. CPU for CFD and HACFD are represented by 'fintimeforCFD', 'fintimeforCFD1'. Elapsed time for CFD and HACFD stand for 'elapsedforCFD', 'elapsedforCFD1' in GUI, respectively.

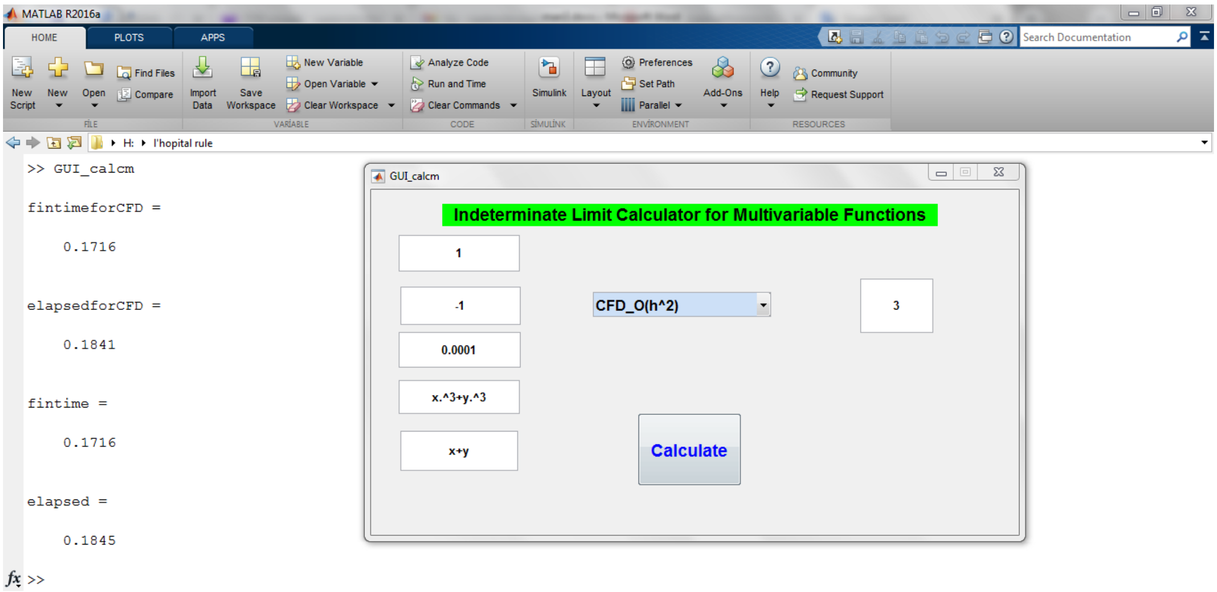

Figure 7. GUI_calcm with CFD Based on O(h^2) Rule Selection for $\frac{0}{0}$ Case

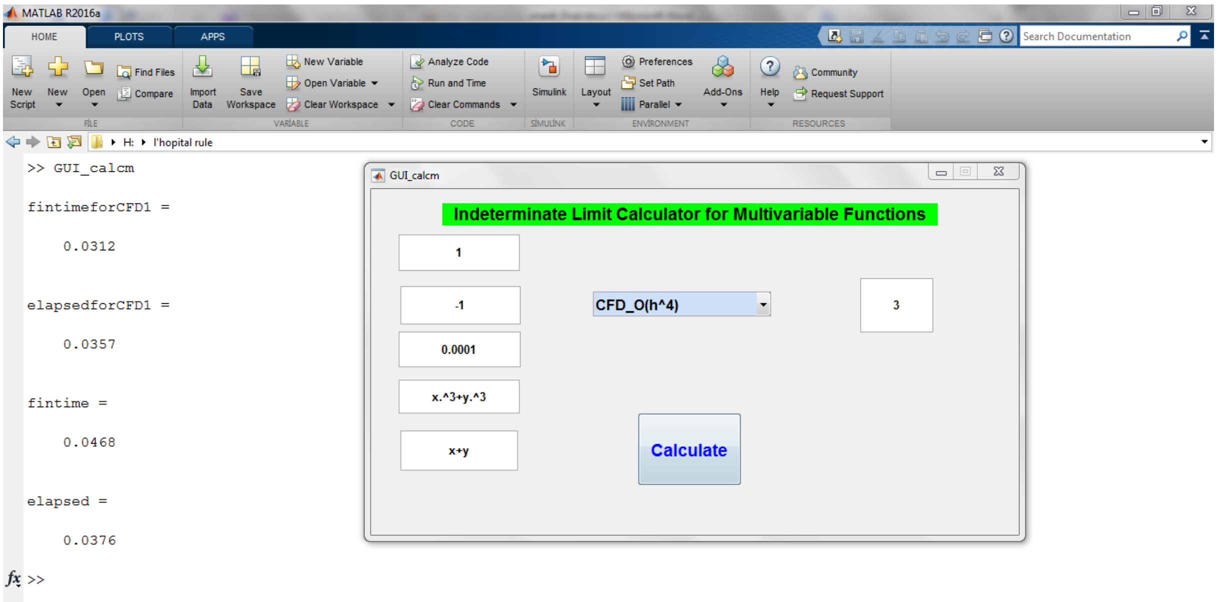

Figure 8. GUI_calcm with HACFD Based on O(h^4) Rule Selection for $\frac{0}{0}$ Case

FFD and HAFFD methods selection for example $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$ with step size: 0.0001 are displayed in Figures 9 and 10, respectively. CPU for FFD and HAFFD stand for 'fintimeforFFD', 'fintimeforFFD1'. Elapsed time for FFD and HAFFD are represented by 'elapsedforFFD', 'elapsedforFFD1' in GUI, correspondingly.

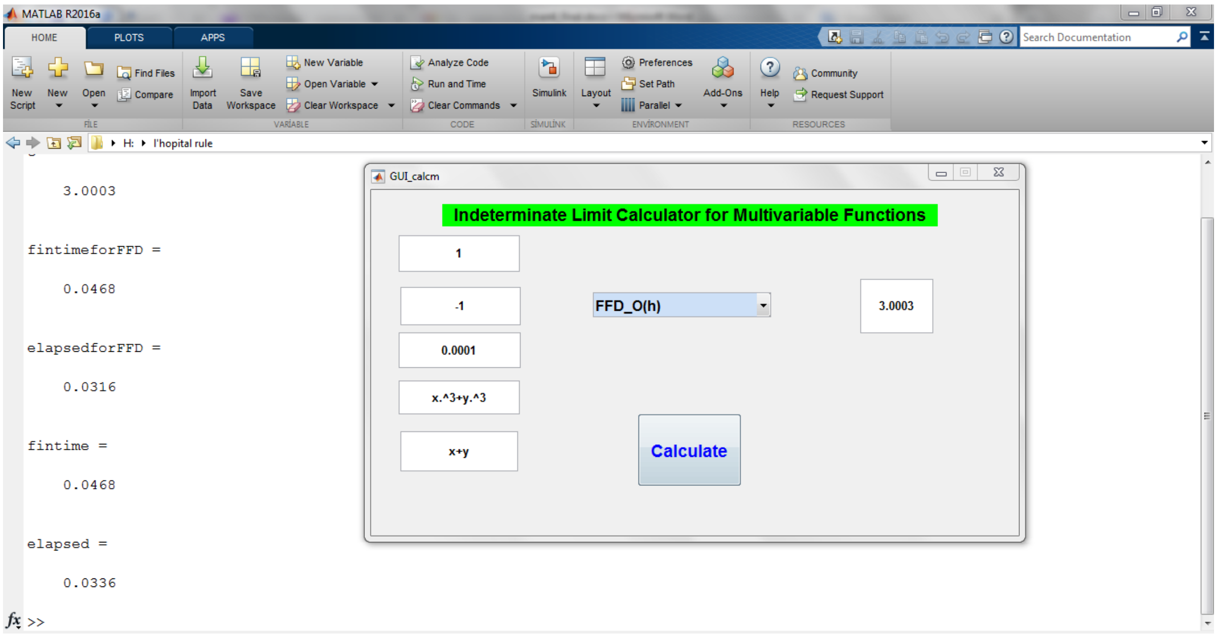

Figure 9 GUI_calcm with FFD Based on O(h) Rule Selection for $\frac{0}{0}$ Case

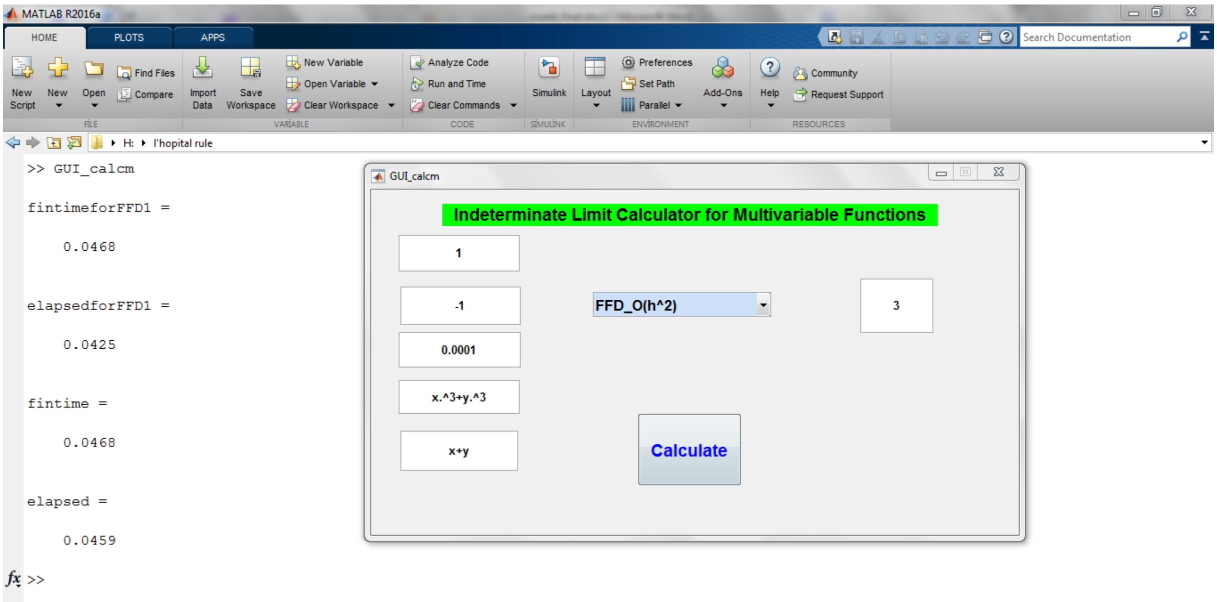

Figure 10. GUI_calcm with HAFFD Based on O(h^2) Rule selection for $\frac{0}{0}$ Case

If user prefers the methods; BFD and HABFD for example $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$ with step size: 0.0001 Figures 11 and 12 are obtained, respectively. CPU for BFD and HABFD stand for 'fintimeforBFD', 'fintimeforBFD1'. Elapsed time for BFD and HABFD are represented by 'elapsedforBFD', 'elapsedforBFD1' in GUI, respectively.

The reason of developing and employing all finite difference models based on O(h^2) is to get more accurate results. In other words, main target of this paper is to obtain the exact results. It is satisfied by employing O(h^2) based rules.

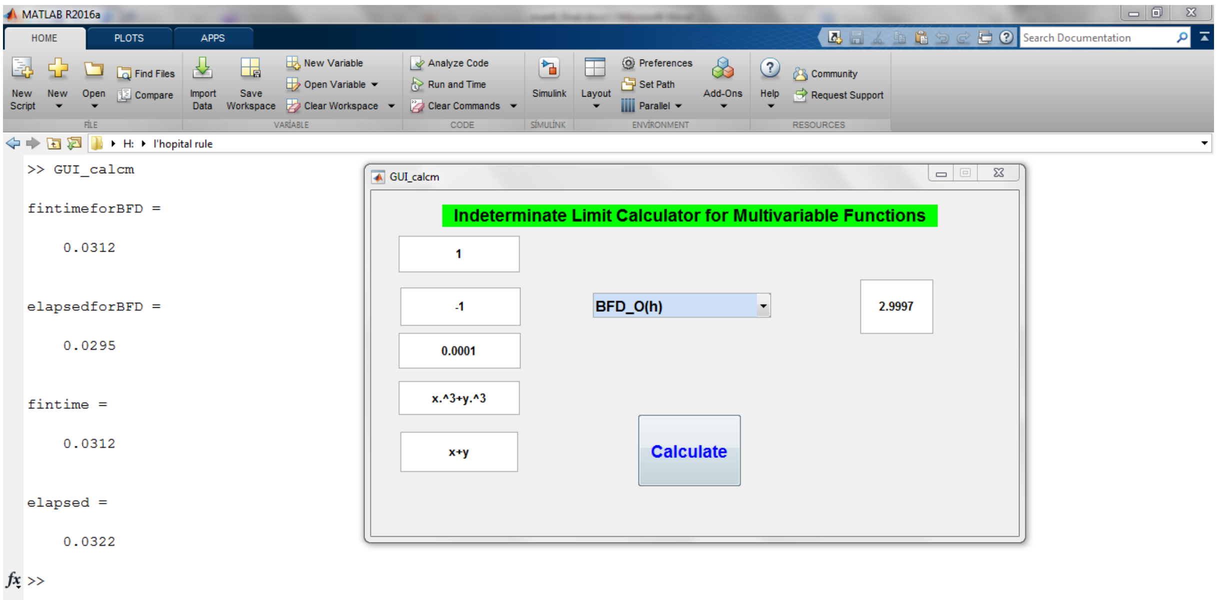

Figure 11. GUI_calcm with BFD Based on O(h) Rule Selection for $\frac{0}{0}$ Case

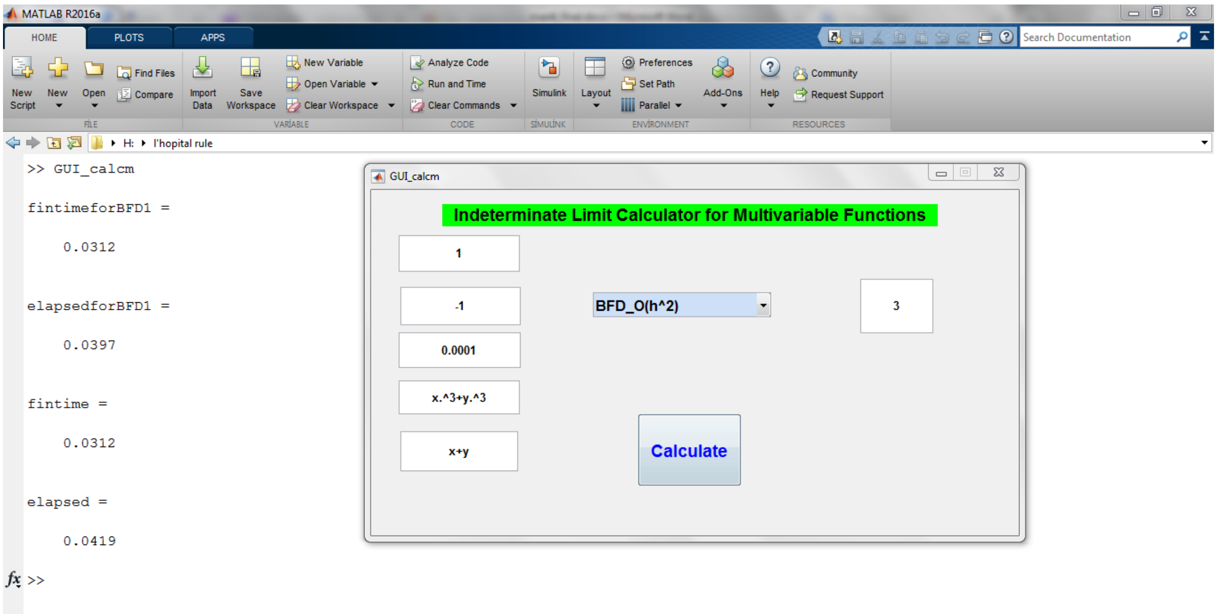

Figure 12. GUI_calcm with HABFD Based on O(h^2) Rule Selection for $\frac{0}{0}$ Case

$\frac{1}{0}$ Case

As an example to $\frac{1}{0}$ case; $\lim_{(x,y)\to(0,0)} \frac{x+\cos(y)}{\sin(x)-y}$ is solved by each method with step size: h=0.0001. Figure 13 illustrates the solution of $\lim_{(x,y)\to(0,0)} \frac{x+\cos(y)}{\sin(x)-y}$ with selection of L' Hôpital Rule. CPU, elapsed time for L' Hôpital Rule ('fintimeforLh' and 'elapsedforLh' in GUI, respectively) and total CPU, elapsed time ('fintime' and 'elapsed' in GUI, respectively) is seen in command window.

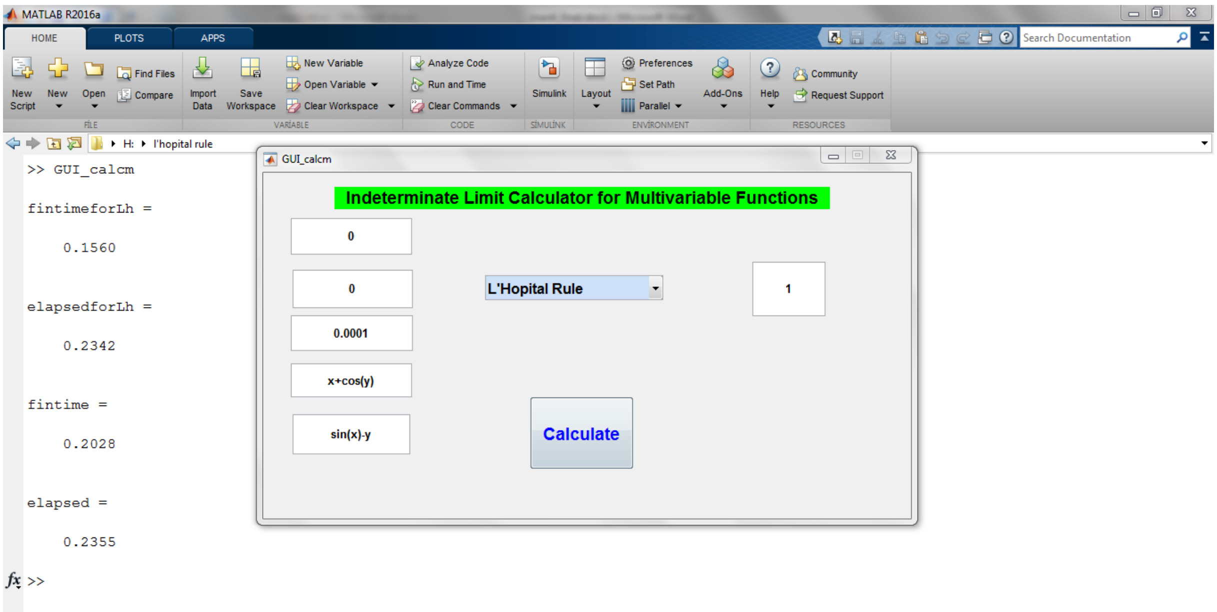

Figure 13. GUI_calcm with L' Hôpital rule Selection for $\frac{1}{0}$ Case

Similarly, same example can be solved by CFD and HACFD methods selection with h=0.0001 which are illustrated in Figures 14 and 15, respectively. CPU, elapsed time, total CPU and total elapsed time for CFD and HACFD are also calculated and seen in command window.

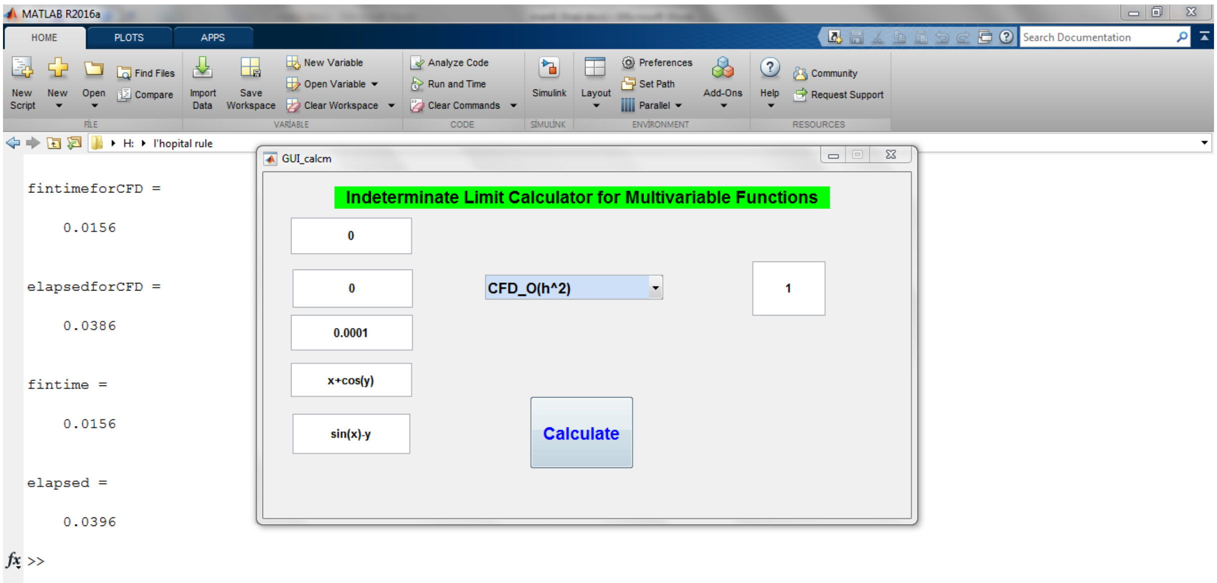

Figure 14. CFD Based on O(h^2) Rule Selection for $\frac{1}{0}$ Case in GUI_calcm

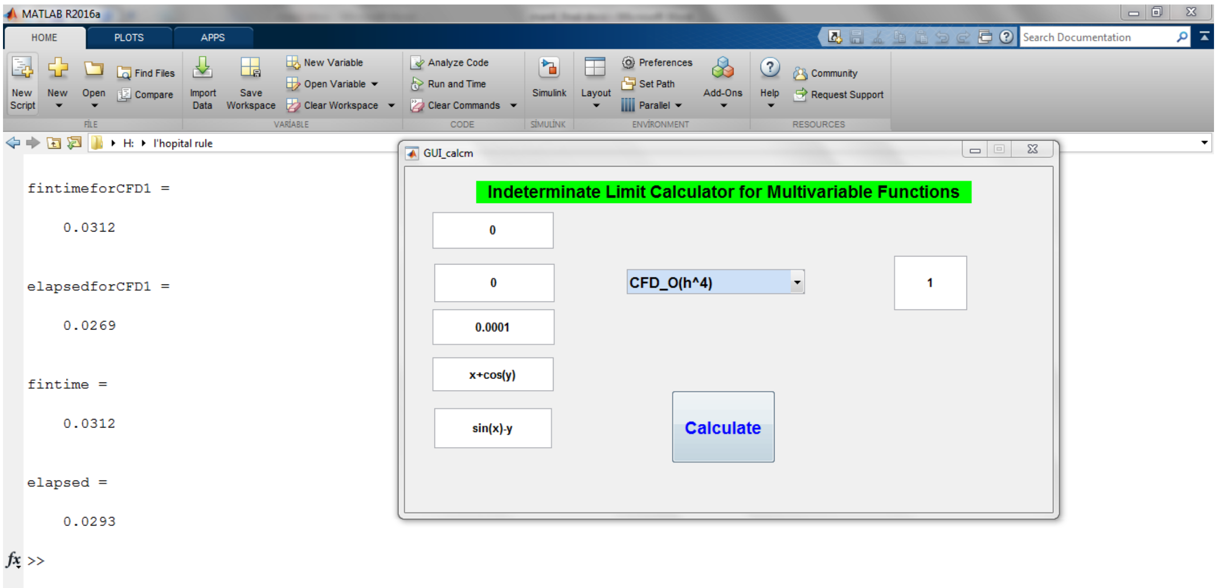

Figure 15. HACFD Based on O(h^4) Rule Selection for $\frac{1}{0}$ Case in GUI_calcm

If user prefers the methods; FFD and HAFFD for example $\lim_{(x,y)\to(0,0)} \frac{x+\cos(y)}{\sin(x)-y}$ with step size: 0.0001 Figures 16 and 17 are obtained, respectively. CPU, elapsed time, total CPU and total elapsed time for FFD and HAFFD are also measured and presented in command window.

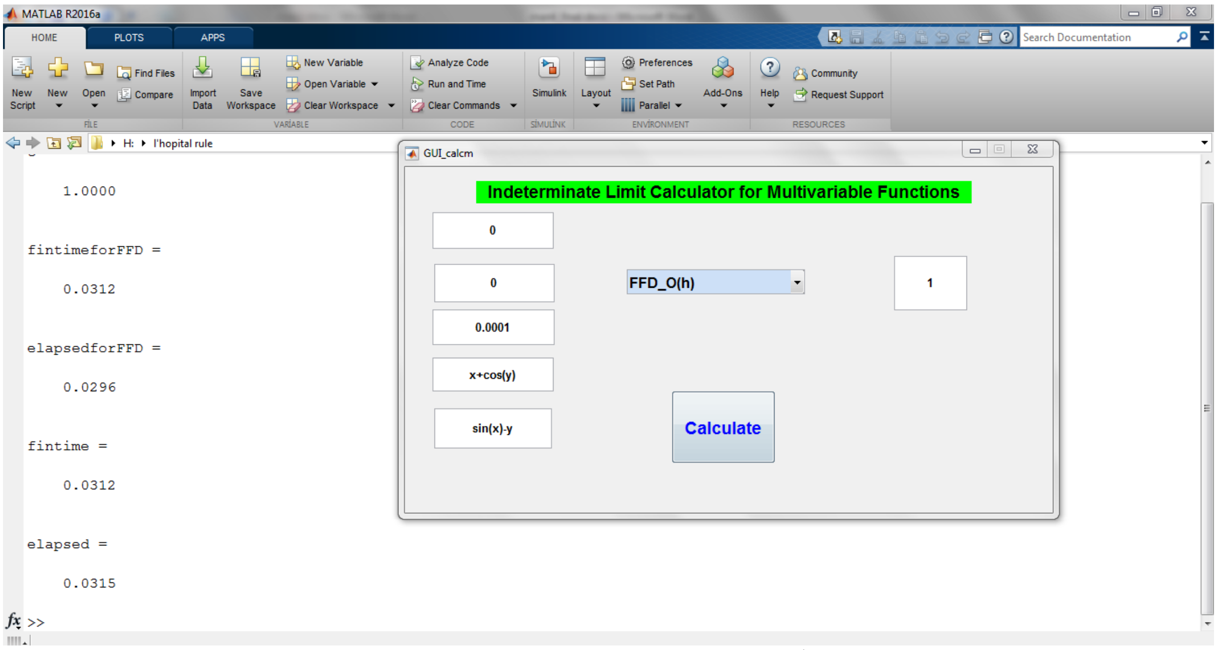

Figure 16. FFD Based on O(h) Rule Selection for $\frac{1}{0}$ Case in GUI_calcm

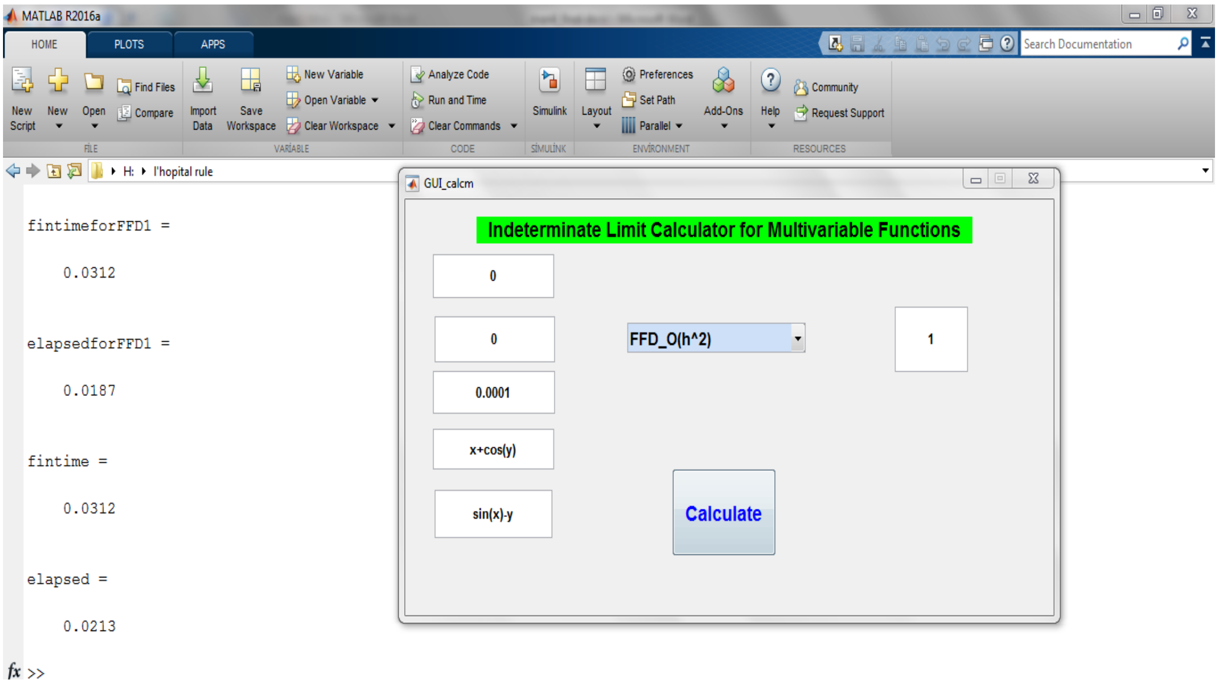

Figure 17. HAFFD Based on O(h^2) Rule Selection for $\frac{1}{0}$ Case in GUI_calcm

BFD and HABFD methods selection for example $\lim_{(x,y)\to(0,0)} \frac{x+\cos(y)}{\sin(x)-y}$ with step size: 0.0001 are displayed in Figures 18 and 19, respectively. CPU, elapsed time , total CPU and total elapsed time for BFD and HABFD with all process are also presented in Figure 18 and 19, respectively.

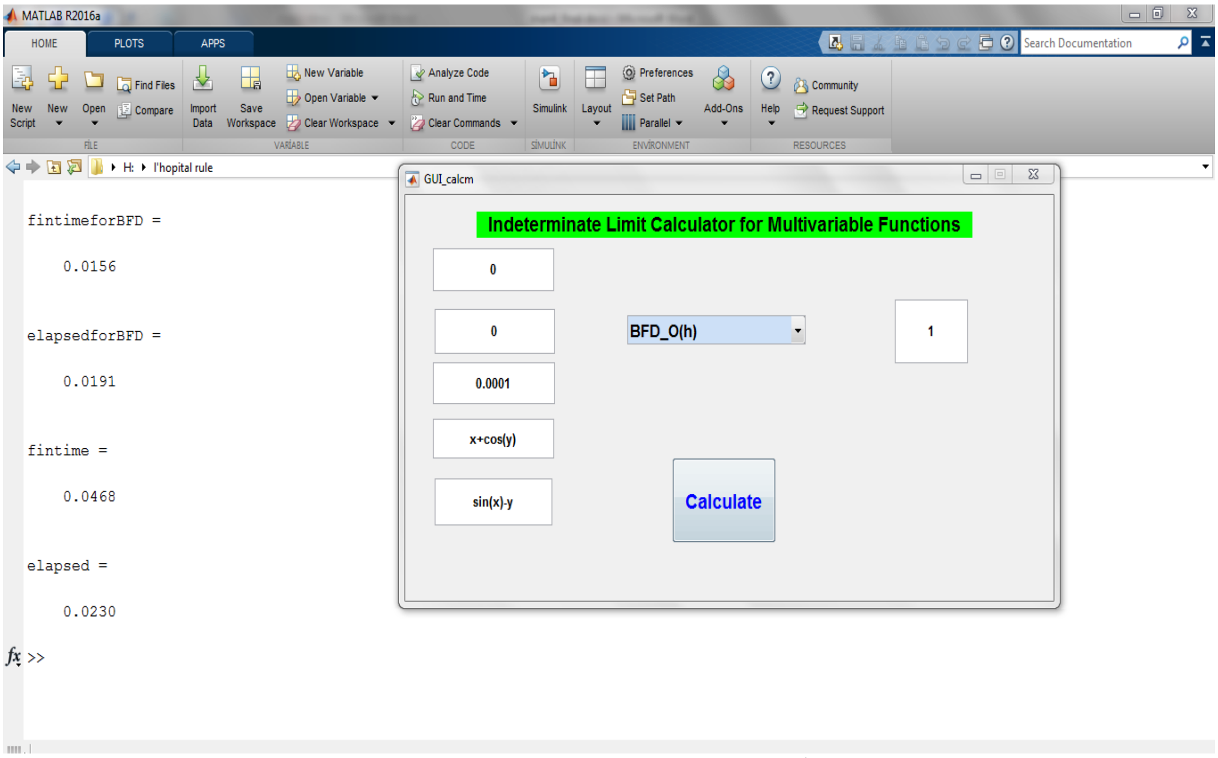

Figure 18. BFD Based on O(h) Rule Selection for $\frac{1}{0}$ Case in GUI_calcm

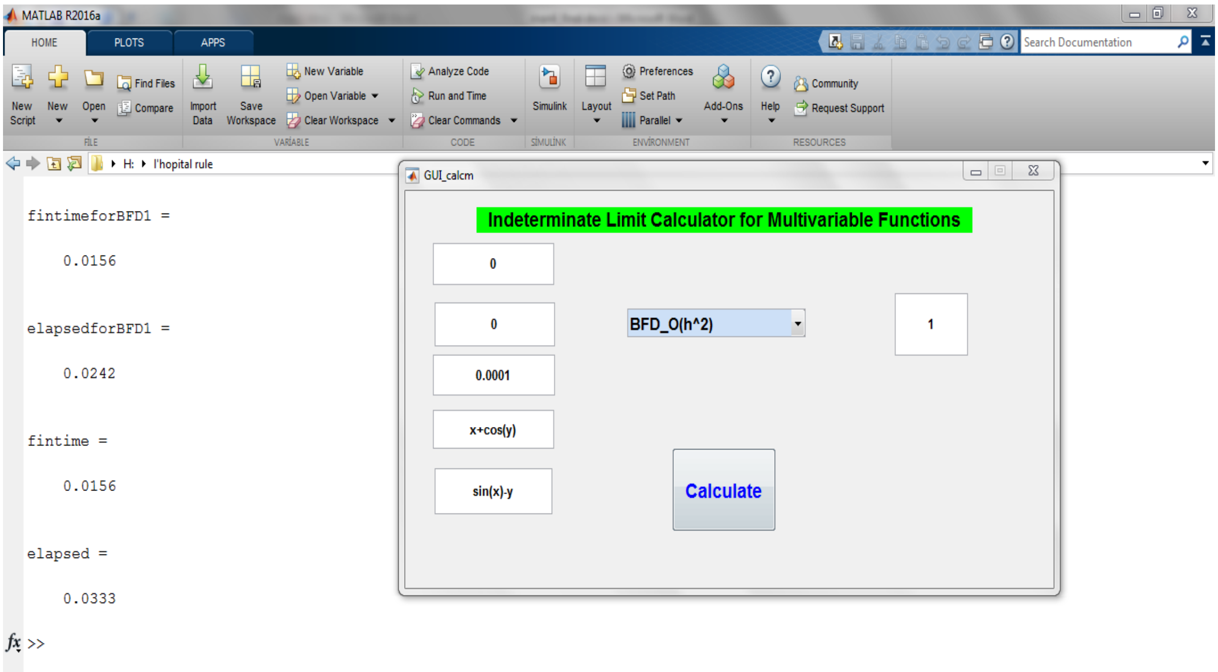

Figure 19. HABFD Based on O(h^2) Rule Selection for $\frac{1}{0}$ Case in GUI_calcm

Multivariable Functions with Isolated and Non Isolated Singularities

In some situations such as two variable functions with isolated or non isolated singularities, L' Hôpital rule is inapplicable. It is proved in Figure 20. The example is: $\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^4-2(\sin(x))^2(\sin(y))^2+y^4}$. The answer is 'NaN'. This means that result is not a number and an indeterminate form is again obtained. In command window, GUI_calcm warns the user with the sentence: 'due to 0/0 result, L' Hôpital rule cannot be employed'. Nevertheless, CPU, elapsed time for L' Hôpital rule and also for all process can be calculated. These situations are displayed in Figure 20. For these reasons, finite difference methods and their developed versions are required to overcome the problem 'NAN'. This problem is overcomed by FFD, HAFFD, BFD and HABFD. This situation is proved by Figures 23-26. Calculator warns the user such as 'due to 0/0 result, L Hopital rule cannot be employed'. So calculator recommends other method or methods indirectly. This situation is an opportunity for users to not waste their time with inapplicable methods.

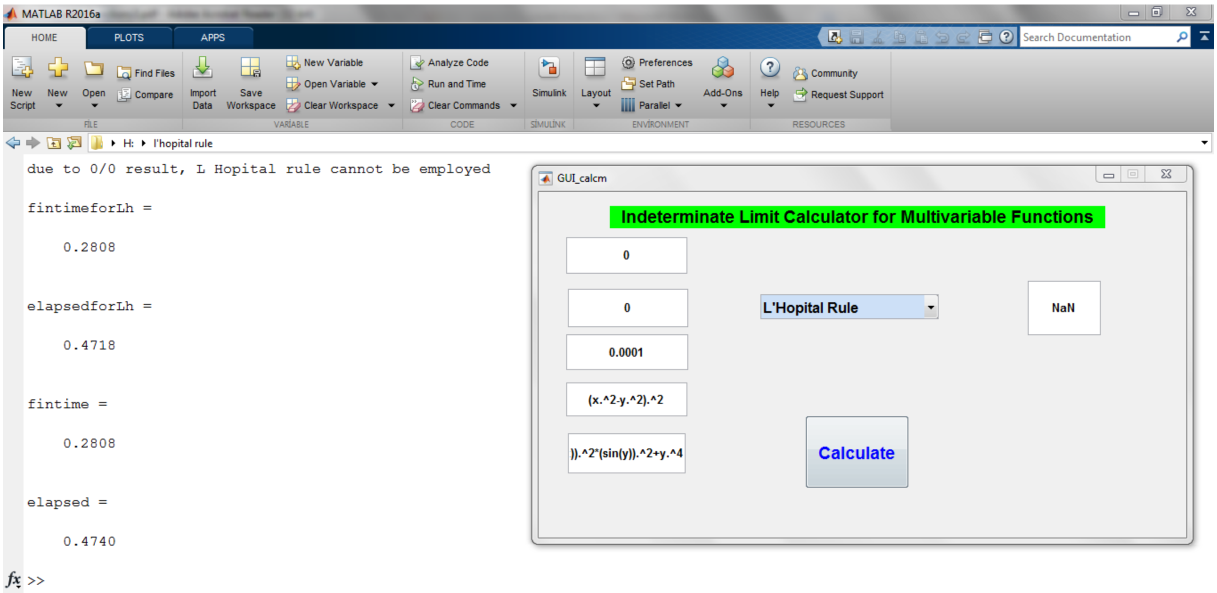

Figure 20. Example for Isolated Singularity Case that L' Hôpital Rule does not work with GUI_calcm

CFD and HACFD methods selection with step size: 0.0001 for example: $\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^4-2(\sin(x))^2(\sin(y))^2+y^4}$. The answer is again 'NaN'. This means that result is not a number and an indeterminate form is again obtained. Even so, CPU, elapsed time for CFD and HACFD and total CPU, elapsed time can be calculated and displayed in Figures 21 and 22, respectively.

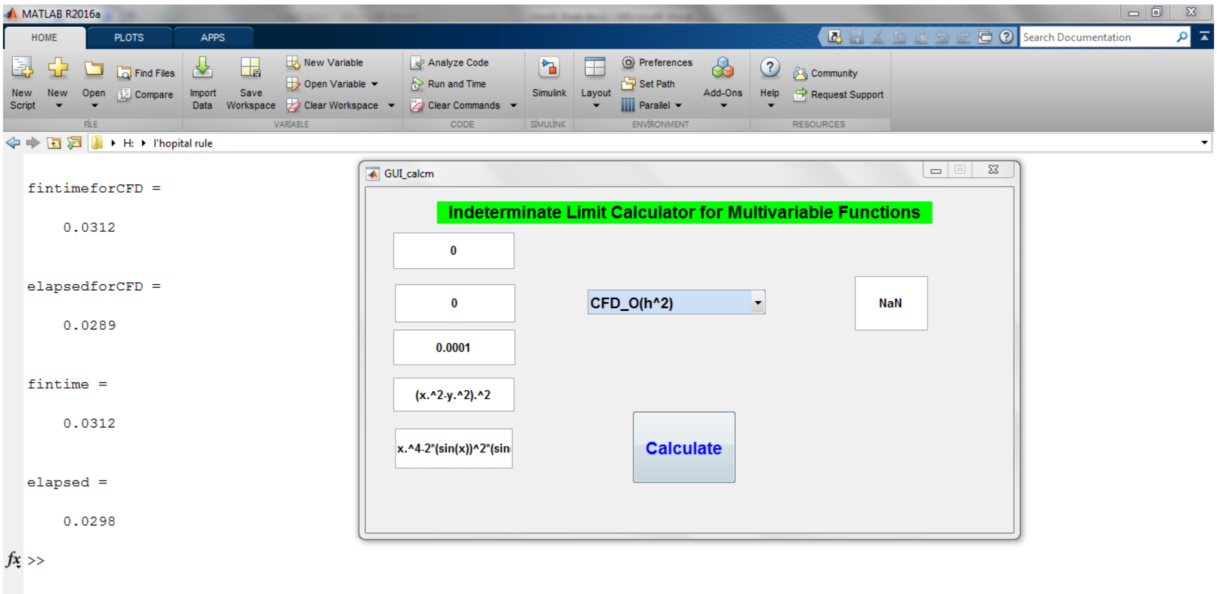

Figure 21. GUI_calcm with CFD based on O(h^2) rule selection for isolated singularity

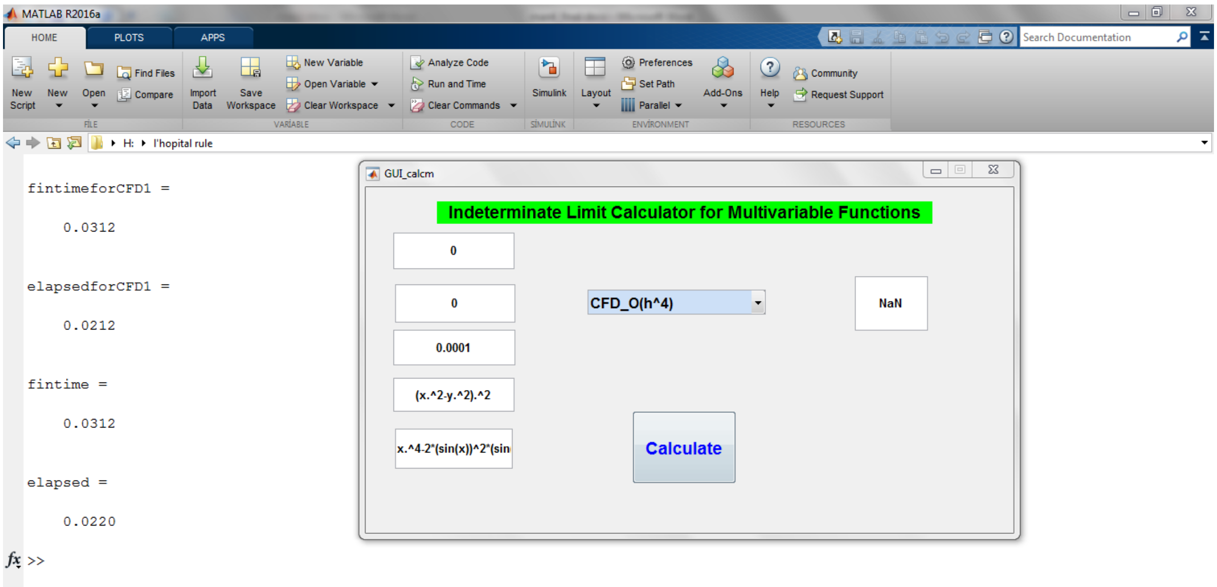

Figure 22. GUI_calcm with HACFD based on O(h^4) rule selection for isolated singularity

If user selects the methods; FFD and HAFFD for example: $\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^4-2(\sin(x))^2(\sin(y))^2+y^4}$ with step size: 0.0001 Figures 23 and 24 are obtained, respectively. CPU and elapsed time for FFD and HAFFD and total CPU and elapsed time are also computed and presented in command window.

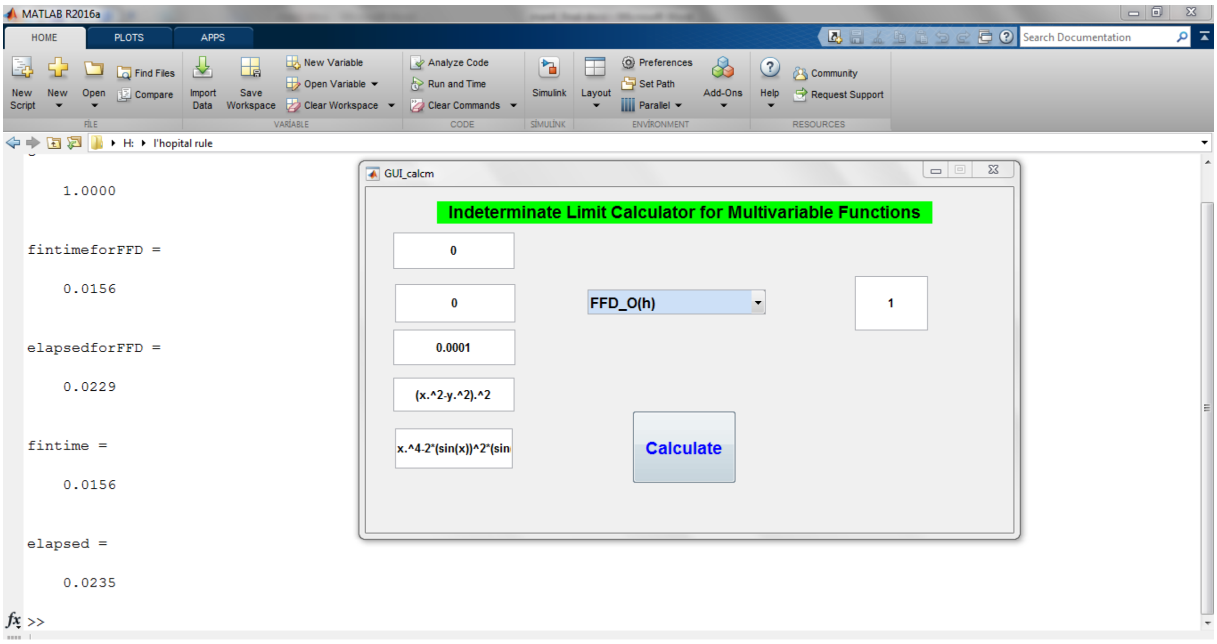

Figure 23. GUI_calcm with FFD Based on O(h) Rule Selection for Isolated Singularity

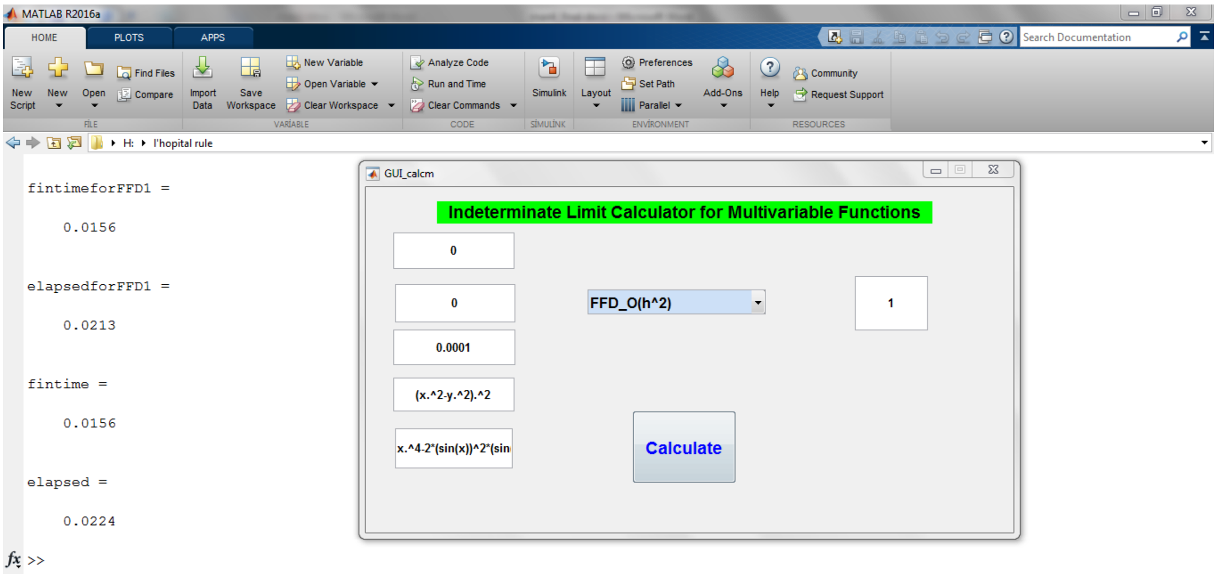

Figure 24. GUI_calcm with HAFFD Based on O(h^2) Rule Selection for Isolated Singularity

Similarly, same example can be solved by BFD and HABFD methods selection with h=0.0001 which are illustrated in Figures 25 and 26, respectively. CPU and elapsed time for BFD and HABFD and also total CPU and elapsed time are also calculated and displayed in command window.

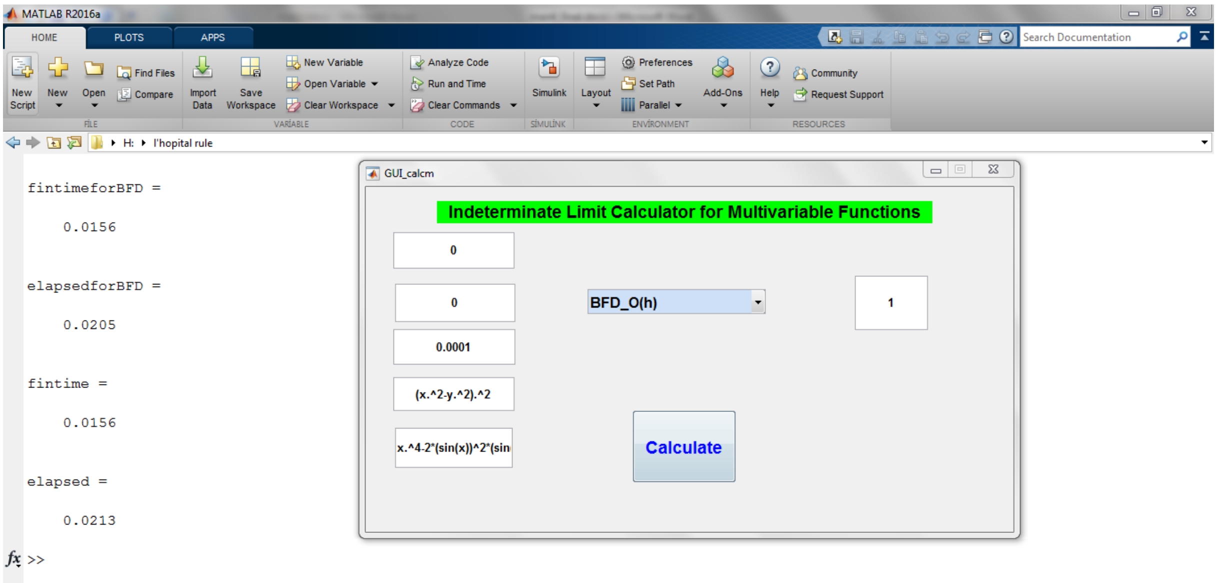

Figure 25. GUI_calcm with BFD based on O(h) rule selection for isolated singularity

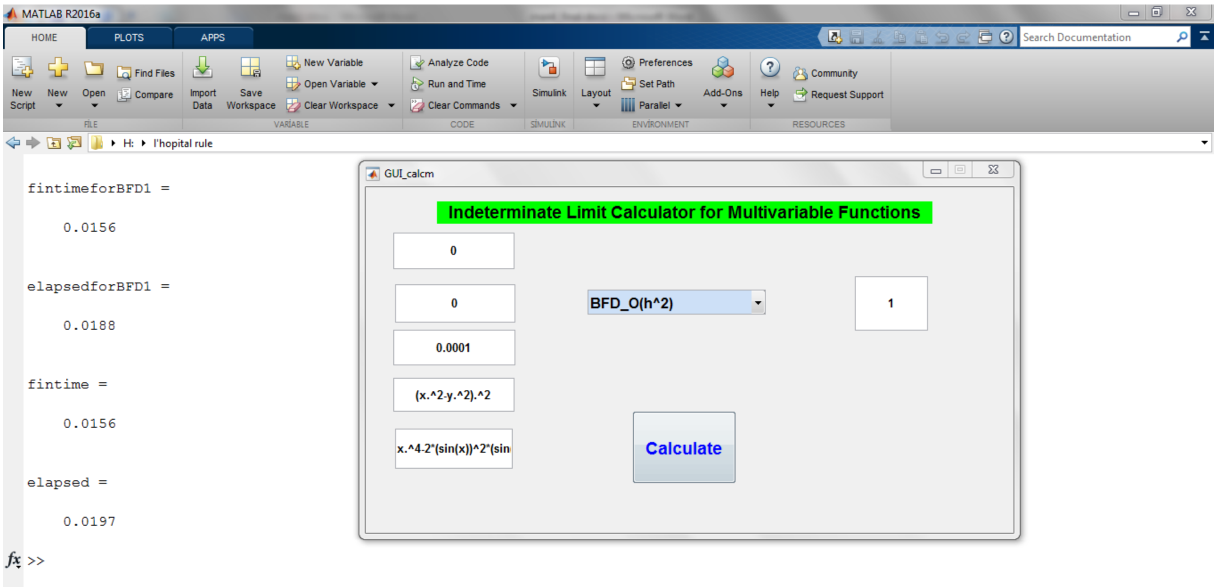

Figure 26. GUI_calcm with HABFD based on O(h^2) rule selection for isolated singularity

Findings

After several runs of calculator, some results in terms of various step sizes from cases:0/0, 1/0 can be obtained, separately. Those are presented in the following subsections.

$\frac{0}{0}$ Case

With use of GUI_calcm, effect of step size on results is displayed seen in Figure 27 . Exact results are always obtained by L' Hôpital Rule whatever the step size is. However a decrease in step size leads to an increase in accuracy for converge the exact results by finite difference techniques. This situation is proved in Figure 27.

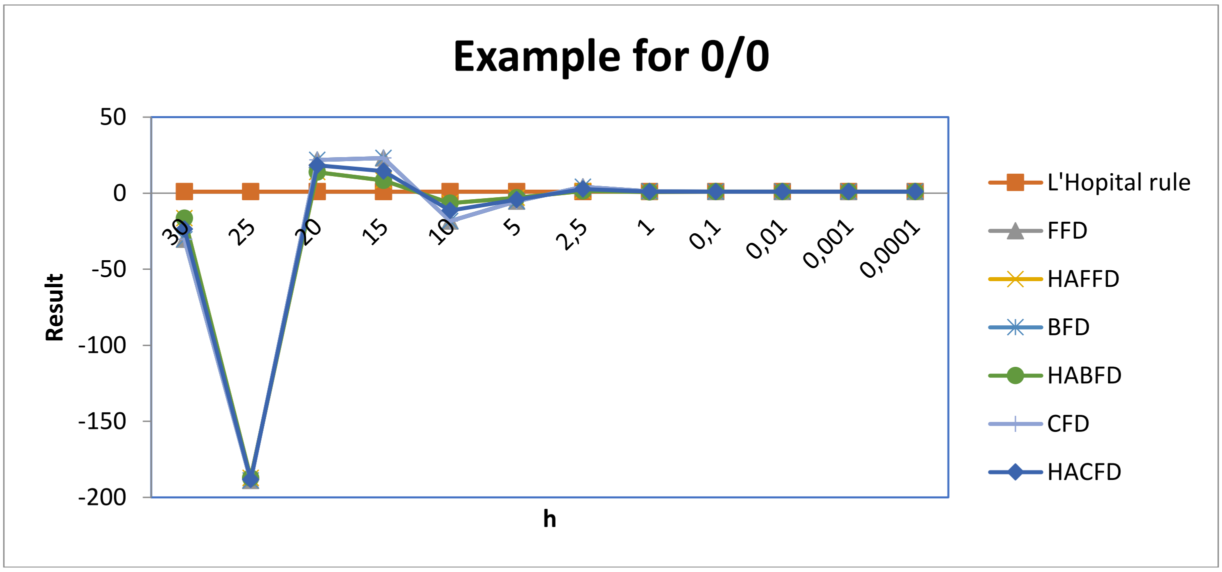

Figure 27. Results of Each Method with Various Step Sizes for Example: $\lim_{(x,y)\to(0,0)} \frac{x-y}{\sin(x)-\sin(y)}$

$\frac{2}{0}$ Case

Effect of step size on results is given for example: $\lim_{(x,y)\to(1,-1)} \frac{x-y}{x^3+y^3}$ in Figure 28. Like $\frac{0}{0}$ case, exact results are always found by L' Hôpital Rule whatever the step size is. Besides, a decrease in step size leads to an increase in accuracy for converge the exact results by all finite difference techniques.

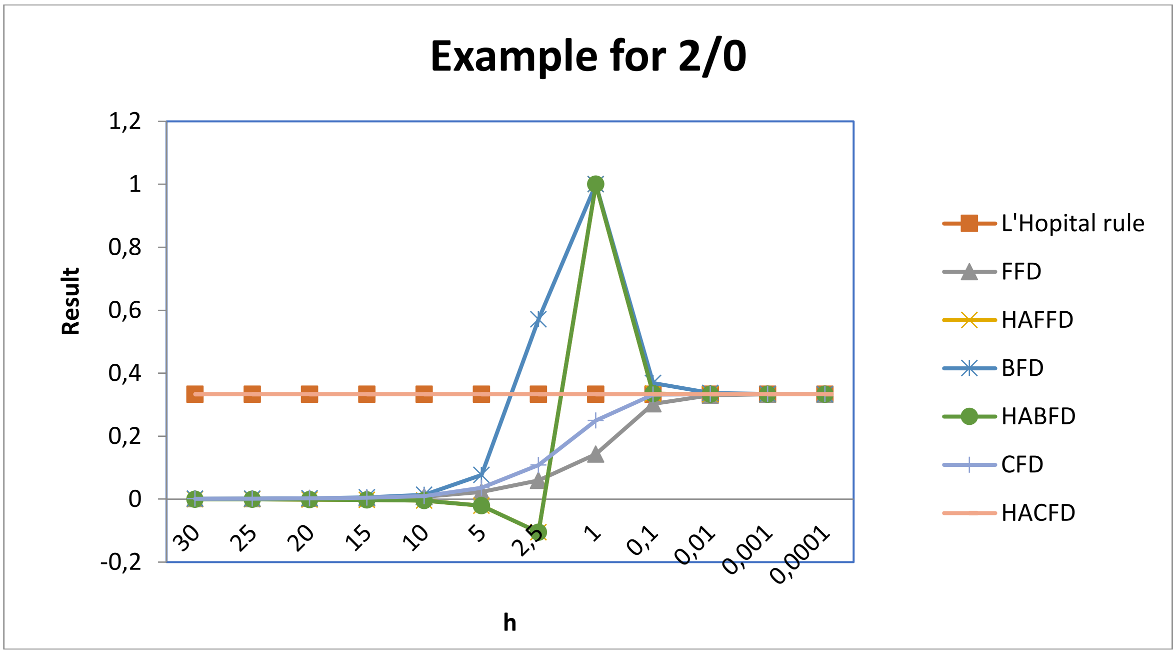

Figure 28. Results of Each Method with Various Step Sizes for Example: $\lim_{(x,y)\to(1,-1)} \frac{x-y}{x^3+y^3}$

Multivariable Functions with Isolated and Non Isolated Singularities

After several run of GUI_calcm, indeterminate forms for two variable functions with isolated and non isolated singularities can be obtained. Numerical results are presented in Table 1 (Dinçkal, 2025).

Table 1. Numerical Results for Other Indeterminate Forms for Functions with Isolated and Non Isolated Singularities

Example	L' Hôpital Rule	CFD	HACFD	FFD	HAFFD	BFD	HABFD
$\lim_{(x,y)\to(0,0)} \frac{x^\alpha y}{x^6+x^2y^2+y^6}$ $0 \le \alpha \le 7$	-	-	-	0	0	0	0
$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^4+2(\sin(x))^2(\sin(y))^2+y^4}$	-	-	-	1	1	1	1
$\lim_{(x,y)\to(0,0)} \frac{x^\alpha y}{x^4+y\sin(y)}$ $0 \le \alpha \le 4$	-	-	-	0	0	0	0

L' Hôpital rule gives the exact result for all step sizes. This case is proved by Figures 27 and 28.

Performance of GUI_calcm

Since there is no calculator designed for the purpose of indeterminate limit calculation in any software platform. Performance of the vehicle has been evaluated in terms of each method presented in this paper. For this reason, the examples given in Table 2 were employed.

Table 2. Several Examples for GUI_calcm Performance

Example no	Example
Example 1	$\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$
Example 2	$\lim_{(x,y)\to(0,0)} \frac{x+\cos(y)}{y\sin(x)}$
Example 3	$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^4+2(\sin(x))^2(\sin(y))^2+y^4}$
Example 4	$\lim_{(x,y)\to(0,0)} \frac{xy}{x^6+x^2y^2+y^6}$
Example 5	$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^6+x^2y^2+y^6}$
Example 6	$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+x^2y^2+y^6}$
Example 7	$\lim_{(x,y)\to(0,0)} \frac{x^6y}{x^6+x^2y^2+y^6}$

GUI_calcm performance was measured in terms of CPU and elapsed time. These processes are achieved for L'Hôpital rule, CFD, HACFD, FFD, HAFFD, BFD and HABFD correspondingly. They are presented in Figures 29-35 respectively.

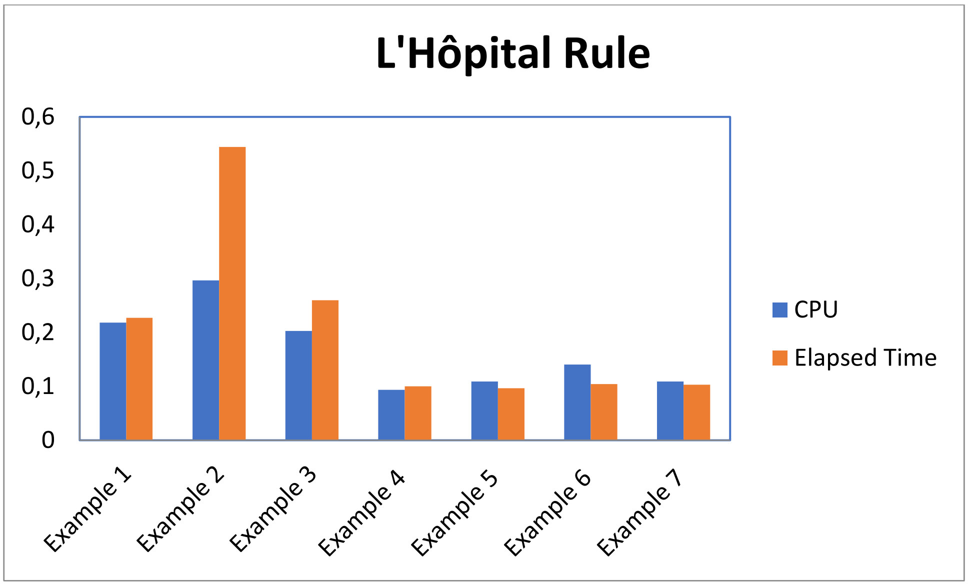

Figure 29. Performance of Calculator for L'Hôpital Rule.

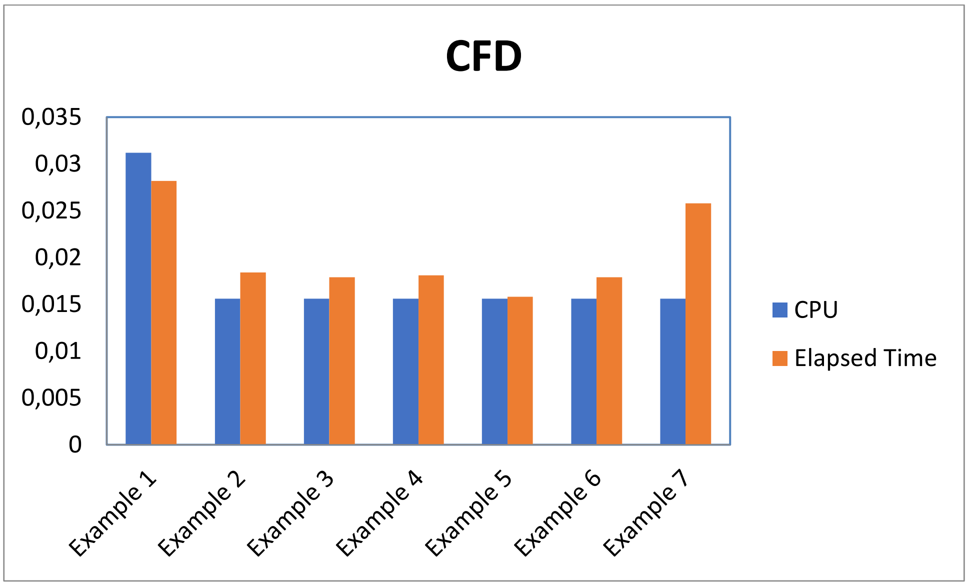

Figure 30. Performance of Calculator for CFD.

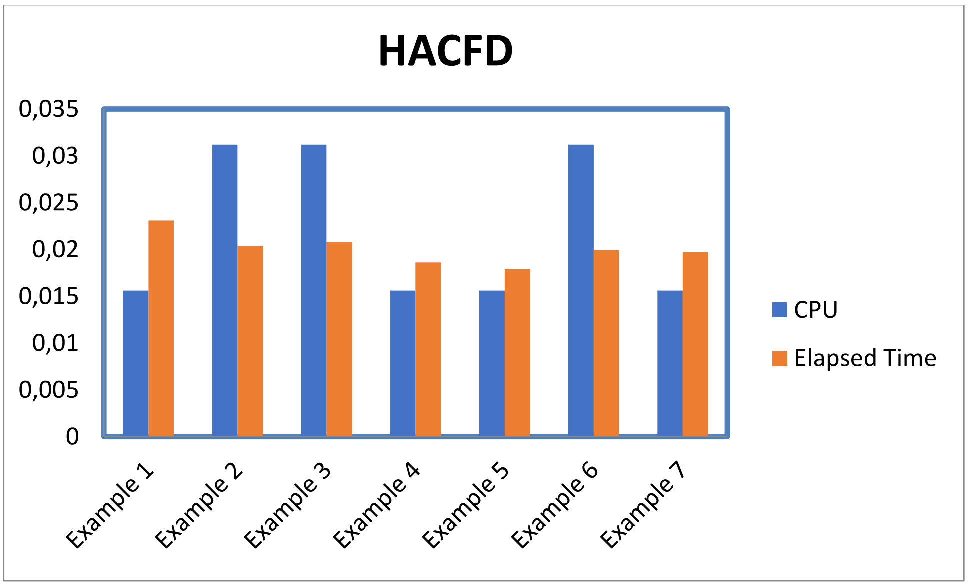

Figure 31. Performance of Calculator for HACFD.

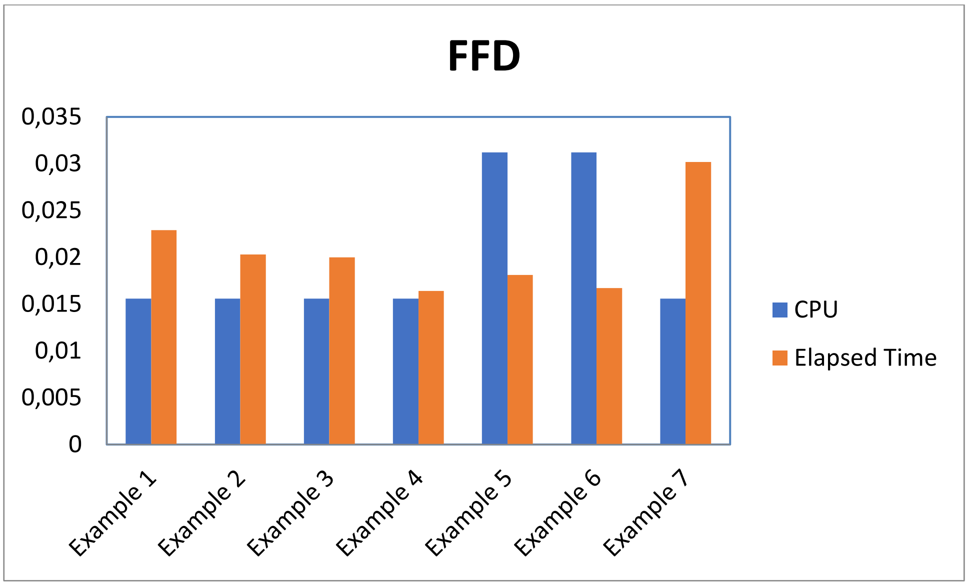

Figure 32. Performance of Calculator for FFD.

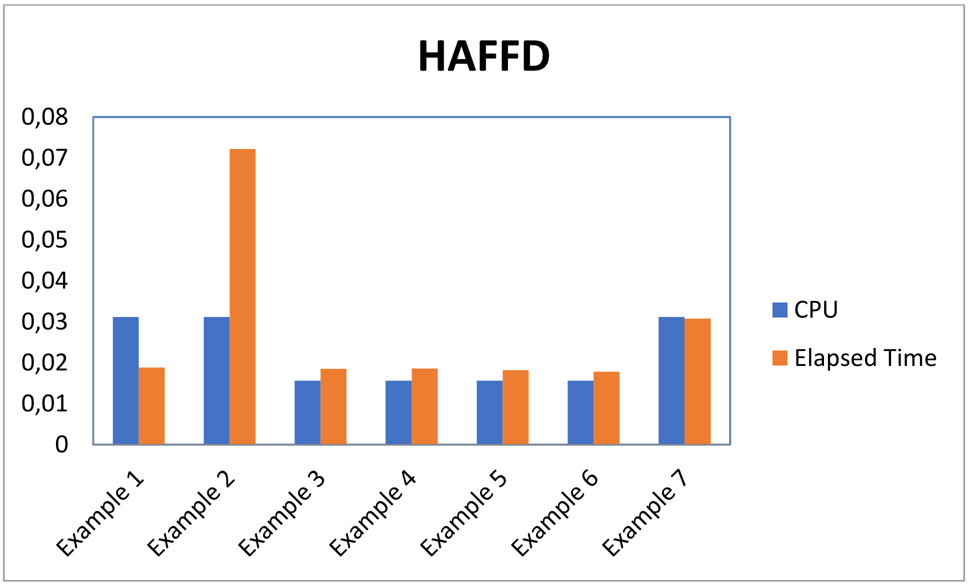

Figure 33. Performance of Calculator for HAFFD.

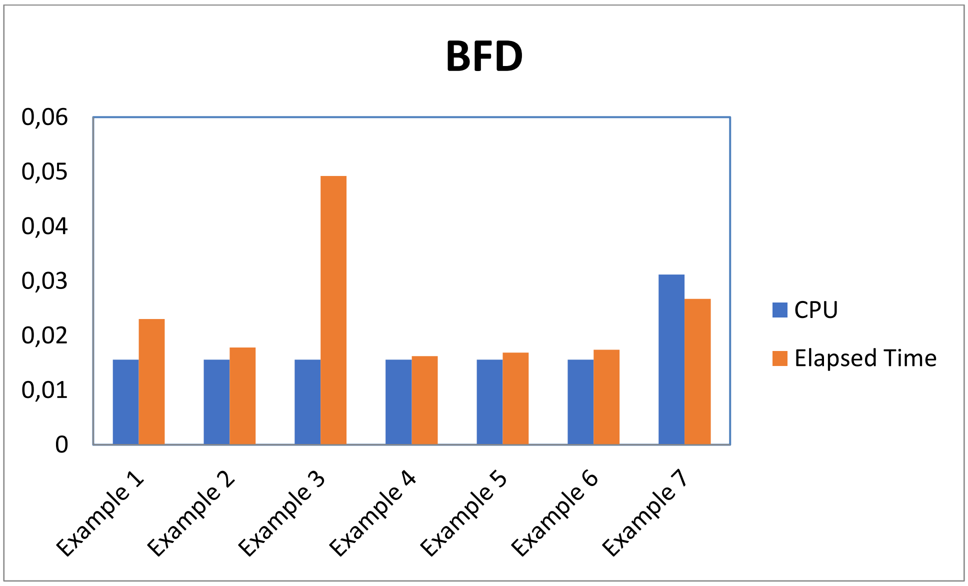

Figure 34. Performance of Calculator for BFD.

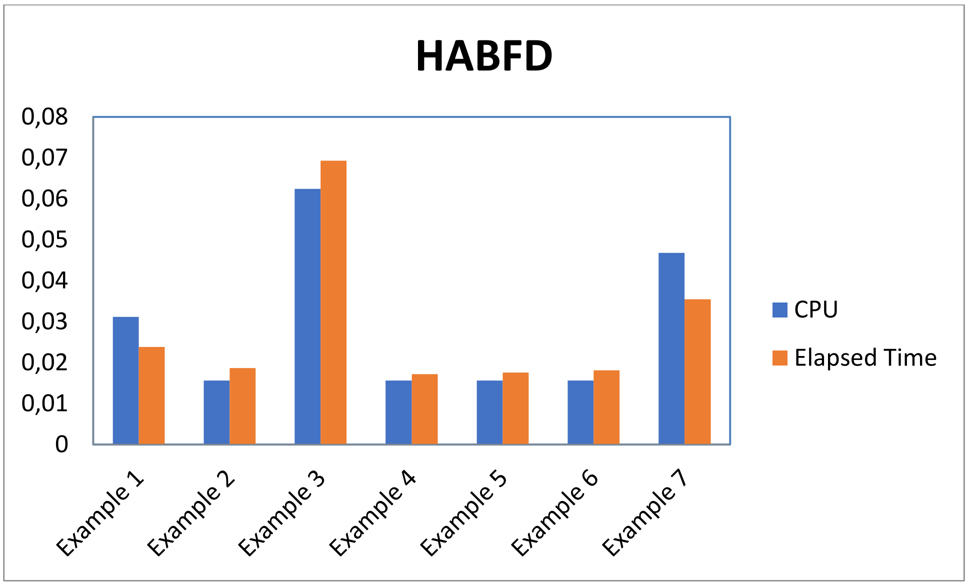

Figure 35. Performance of Calculator for HABFD.

The performance analyses were done for h=0.0001. Since it is the smallest step size that GUI_calcm performs calculations for smallest as welll as closest points to obtain accurate results. Nevertheless, GUI_calcm realized computations in a very short time for each method. This is proved for each method in Figures 29-35.

Average CPU time for L'Hôpital rule is 0.17 sec. approximately and average elapsed time for same method is 0.2 sec. Whereas for all numerical methods, average CPU and elapsed time are approximately equal to 0.02 sec. This also demonstrates that the performance of GUI_calcm for all numerical methods is more efficient than for L'Hôpital rule.

Furthermore the effect of decreasing step size significantly causes a great accuracy for all finite difference methods. So step size should be small as possible to obtain higher accuracy for all finite difference methods. For this reason h is selected as '0.0001'.

Conclusion, Discussion and Suggestions

As it is mentioned in introduction, there is no work related with design of calculator for solution of indeterminate limit computations in literature. It is the first time that this paper emphasizes on both methods (L'Hôpital rule and new developed methods; finite difference rules) and calculator designed by use of embedded algorithms of these methods.

Since, there are many works related with calculators for different purposes in science and engineering applications. Matlab GUI was used as computation platform to develop auto tuning PID controller (Mohamed et al., 2010). Matlab GUI was also employed for Various calculators by Matlab GUI have been designed and also employed for preparation of FEM -parabolic-equation tool for path-loss calculations along multi-mixed-terrain paths (Apaydın & Sevgi, 2014). A Toolkit for resting-state functional magnetic resonance imaging data processing was designed by use of Matlab (Song et al., 2011). Learning controller in the frequency domain was designed by Matlab GUI (Mitchell, 2014). Economic load dispatch was computed by also Matlab (Singh et al., 2014). Gupta and Patel (2017) prepared a teaching-learning tool for elementary psychrometric processes on psychrometric chart with use of MATLAB.

Calculators were also designed by use of different computational platforms in literature (Dinçkal, 2018; Chow, 2016; Bishay, 2016; Malehmir & Schmitt, 2016; John & Wara, 2018; Oke et al., 2018; Piris et al., 2021) Those works have been mentioned in introduction.

Neither of these studies elaborated on designing of a calculator for computation of indeterminate limits form for multivariate functions. Such that this calculator make all calculations automatically, once user select the method from pop-up menu.

GUI_calcm can be used as a marking scheme for any users to solve the problems including indeterminate limits for two variable functions. One can have seven options for computation of limits. If L' Hôpital rule cannot be applicable to calculate the limit, GUI_calcm provides user to select another method such as any one of the finite difference methods to obtain the answer. Furthermore, emphasis was not only on usage of L' Hôpital rule but also other techniques such as FFD, HAFFD, BFD, HABFD, CFD and HACFD separately to compute indeterminate limits for any problems including indeterminate limits for multivariable functions. Step size has no effect on L' Hôpital rule. But it has a significant effect on all finite difference methods. Smaller the step size in each finite difference methods leads to high accuracy in results. It is demonstrated in both Figure 27 and Figure 28 for each cases. Even if there exists singularity and nonsingularity in problems, FFD, HAFFD, BFD and HABFD give exact results by running GUI_calcm. Whereas CFD and HACFD do not work in case of isolated and non isolated singularities. It is the first time that singularity and nonsingularity problems are solved by the methods: FFD, HAFFD, BFD and HABFD with GUI_calcm.

It is proved in Table 1.

GUI_calcm also provides all users to measure performance and speed of the calculator in terms of CPU and elapsed time for each method while obtaining the answer of indeterminate limit forms: $\frac{0}{0}$ and $\frac{c}{0}$.(c can be any real number) GUI_calcm is practical in terms of less computational elapsed and CPU time even if the problem includes singularity and nonsingularity.

Moreover GUI_calcm has been created by a novel algorithm in Matlab GUI with Matlab APPS for the first time in the literature. GUI_calcm is also designed as one of the APPS of Matlab. Thus, there is no need to procure other storage devices for working file transfers. GUI_calcm is ready to use in all computers in a laboratory. This would be a great benefit for any kind of users such that only Matlab R2016a installation is sufficient.

This calculator is so user-friendly that users only require entering relevant inputs and clicking 'Calculate' button. The purpose of GUI_calcm is to be an electronic source for the computation of indeterminate limits of two dimensional functions in courses of calculus in calculus education and also research studies in calculus.

Once, the existence of GUI_calcm is revealed, it will be preferred by researchers, mathematicians, scientists, physicist and also students.

Since there is no book and digital source offered a calculator for the problems of indeterminate limit forms and restricted only for use of L' Hôpital rule to overcome the problems without calculator, GUI_calcm including many method options can be used conveniently to solve the problems in any calculus books.

Main suggestions for this study are listed as follows:

1) GUI_calcm with algorithms embedded in the calculator can be involved into exercise problems of Calculus textbooks. This provides students as well as instructors an environment to understand new numerical methods, not spend too much time for solution of these problems by hand-calculations and finally verify their calculation results by various methods in very short time.

2) After installing Matlab into computer laboratories of any University, this calculator can be uploaded with use of Matlab APPS. Both instructors and students can use it any time.

3) The developed methods as well as L'Hôpital rule can be included in Calculus course curriculum. Instructor can teach not only L'Hôpital rule, but also developed new methods (Dinçkal, 2025).

Thus, students can be aware of other methods and their applicability to indeterminate limit forms for multivariate functions.

4) To make the calculations rapidly and conveniently, GUI_calcm can be used in computer laboratories at recitation hours.

After these suggestions are put into practice, the usability and convenience of this study can be proved, clearly.

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