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Asymptotic Results for an Inventory Model of Type (s, S) with Asymmetric Triangular Distributed Interference of Chance and Delay

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Abstract

In this study, a semi – Markovian inventory model of type (s, S) is considered and the model is expressed by a modification of a renewal – reward process $(X(t))$ with an asymmetric triangular distributed interference of chance and delay. The ergodicity of the process $X(t)$ is proved under some weak conditions. Additionally, exact expressions and three – term asymptotic expansions are found for all the moments of the ergodic distribution. Finally, obtained asymptotic results are compared with exact results for a special case.

1. INTRODUCTION

A number of very interesting problems arising in the theories of inventory, stock control, queuing theory, reliability, mathematical insurance, stochastic finance and so on can be expressed by the help of renewal processes, renewal – reward processes, random walk processes and modifications of these kind of processes.

There are important theoretical results about this object in the literature: ([1] – [6], etc.). Theoretical results on these processes (renewal processes, renewal – reward processes, random walk processes, etc.) have complex mathematical structures and are not practical for application purposes. Previous research is concentrated on two areas in order to simplify the complexity proposed by the theory. One of these areas is the simulation method which helps to find numerical results with the use of computer. The other one is asymptotic method that assists to obtain closed approximations for better use in application, which is the method employed in this study. The literature exhibits many important studies on asymptotic methods: ([4], [7], [8], [9], [10], etc.). The most efficient one of these approaches is to use the asymptotic methods and find approximated results that are close to exact expressions. In this study, three-term asymptotic expansions are used.

The first and most important specialty of this study is using a model with an asymmetric triangular distributed interference of chance. [9] and [10] investigated a similar model with a symmetric triangular distributed interference of chance. Under the case of symmetric distribution, the mode and middle point of the distribution coincide and is equal to $0.5(S + s)$ where S is the maximum stock level and s is the stock control level. However for many examples it is not a realistic assumption to keep the stock level at

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the middle point of distribution. An immediate implication of this assumption is that the stock finishes quickly and it is needed to refill the stock frequently, which leads to various types of costs such as transportation, warehouse rent, etc. and loss of customer reputation. In order to overcome this adversity, it is better to use a model with an asymmetric triangular distributed interference of chance.

As second specialty, we assumed that the delay time (lead time) has positive value. Until now in most of the studies, there was an assumption about taking lead time as zero [9] and [10]. On the other hand, it is not a valid assumption for all problems, since in real world problems it is not possible to refill a depot immediately in every time. For example, this delay time may be due to transportation, trying to provide demands from suppliers, etc. Furthermore, it must be mentioned that this lead time is not deterministic because of variability of delivery time. With respect to this variability, in this study lead time is a random variable and can have arbitrary distribution.

Now, let us express the model as follows.

The model: $X(t)$ is the stock level in a depot at time t. As an assumption, the stock level $X(t)$ at time $t = 0$ is $X(0) = z \in [s, S]$ where $0 \leq s < S < \infty$. Stock level decreases according to demand quantities η_n, $n = 1, 2, \ldots$ at random times $T_n, n = 1, 2, \ldots$ until the stock level falls below to control level s. Random variables $\theta_n, n = 1, 2, \ldots$ express the waiting time (lead time) after the stock level decreases below to s. The change of the stock level over time can be shown as: $X(T_n) = z - Y_n$. Here, $T_n = \sum_{i=1}^{n} \xi_i$, $Y_n = \sum_{i=1}^{n} \eta_i$, $n = 1,2,3, \ldots$, ξ_n and η_n are interpreted as inter – arrival time and demand quantity respectively.

When the stock level falls below the control levels at time τ_1, stock level remains stable at s for lead time θ_1 (delay time) which is random. After the lead time θ_1 the stock level is instantaneously brought to random level $\zeta_1 \in [s; S]$. Then, the first period has been completed. The second period starts from initial stock level ζ_1 at the time $\gamma_1 = \tau_1 + \theta_1$. Next demand quantities fall from new starting level ζ_1 until stock level decreases below to control levels at time τ_2 and then stock level remains same until time $\gamma_2 = \tau_2 + \theta_2$ where θ_2 is the lead time. At the time γ_2, the stock level is again immediately brought to random level ζ_2. Then, the second period has been completed. Afterwards the process continues in similar manner. Here, random variables ζ_n, $n = 1, 2, \ldots$express the discrete interference of chance and have asymmetric triangular distribution on $[s, S]$.

[9] and [10] considered a similar model with a symmetric triangular distributed interference of chance. [9] proved the weak convergence theorem for the ergodic distribution of the process. Meanwhile [10] investigated the process with same interference of chance and found three term asymptotic expansions for the ergodic moments. The reason why a triangular distributed interference of chance is chosen for these types of models is explained in [10]. According to this explanation, it is needed that ζ_n takes values close to s and S with a low probability and takes its main values from inside of the interval (s, S) with higher probability. Triangular distribution provides these qualifications. For these reasons, [9] and [10] investigated similar model with symmetric triangular distributed interference of chance. However, using an assumption such that the discrete interference of chance has symmetric triangular distribution may not be adequate for modeling some real world problems. That's why we consider a stock model with an asymmetric triangular distributed interference of chance. Our motivation for this study is to find more adequate solution for the following real world problem which is given in [10]:

The real model: A company operating in the energy sector produces, stores, fills, and distributes liquefied petroleum gas (LPG). Domestic LPG distribution is carried out pipelines and carried from the LPG production center (a city in Turkey) to the 30 dealers by tankers with the capacities of $22\ m^3$ (approximately 10-11 tons) and $35\ m^3$(approximately $17 - 18$ tons). The tankers are kept under surveillance with the GPS 24 h a day and 7 days a week. After delivering the needed amount of gas to the dealer, if more than 10% of the capacity of the tanker is left over, the tanker waits in its position until the next order of any dealer. Each dealer has a storage capacity of $S = 30\ m^3$. Random amounts of LPG (η_n) are sold from these storage tanks at random times (ξ_n). When at random moments τ_n, $n \geq 1$, the level of LPG in the tank of the dealer falls below the control level $s = S/5$, a demand signal is automatically sent online to the production center. As a response to demand, the nearest tanker to the dealer is directed to the demanding dealer. If there is no tanker near to the dealer, a full tanker is sent from the production center.

For safety concerns (in order not to allow the gas pressure to reach its maximum value), the dealers usually fill about 85% of the capacity (S) of their tanks. However, with a low probability, by taking a risk the dealers fill their tanks to the full capacity when the need arises. On the other hand, even if the amount of gas in the tanker does not meet 85% of the dealer's tank, the existing amount of gas in the tanker is loaded into the dealer's tank. Summarizing the working principle mentioned above, after each filling, 85% of the tank of each dealer is most probably filled.

The concept of filling the depot approximately 85% indicates us using an asymmetric triangular distributed interference of chance for modeling this problem. Therefore, in our opinion, the process that expresses the working principle of the depot can be considered as a stochastic process with an asymmetric triangular distributed interference of chance.

The aim of this study is to construct the stochastic process mathematically, which expresses the above mentioned model, and to obtain asymptotic expansions with three terms for the n^{th} ergodic moments of this process.

The rest of the paper is built as follows. In the following section, we give the mathematical construction of the process $X(t)$. The ergodicity of the process $X(t)$ is proved in Section 3. In Section4 and Section 5, exact expressions of ergodic distribution and moments are derived and also asymptotic expansions for the moments of the ergodic distribution of the process $X(t)$ are obtained respectively. In Section 6, we investigated a real world problem about semi-Markovian inventory model of type (s, S) as an example to check the closeness of the asymptotic results with exact ones. Finally, the last section is devoted to the concluding remarks and future research directions.

2. MATHEMATICAL CONSTRUCTION OF THE PROCESS $X(t)$

Let $\{\xi_n, \eta_n, \zeta_n, \theta_n\}$, $n = 1, 2, 3 \ldots$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$, such that variables in each sequences are independent and identically distributed. Suppose that ξ_n, η_n, ζ_n and θ_n can take only non-negative values. Denote the distribution functions of $\xi_n, \eta_n, \zeta_n, \theta_n$ by

$\Phi(t) = P\{\xi_1 \leq t\}, F(x) = P\{\eta_1 \leq x\}, \pi(z) = P\{\zeta_1 \leq z\}, H(t) = P\{\theta_1 \leq t\},$

respectively. Define the renewal sequences $\{T_n\}$ and $\{Y_n\}$ with the help of sequences $\{\xi_n\}$ and $\{\eta_n\}$ as follows:

$$T_0 = Y_0 = 0;\; T_n = \sum_{i=1}^{n} \xi_i;\; Y_n = \sum_{i=1}^{n} \eta_i;\; n = 1,2,3, \ldots$$

and a sequence of integer valued random variables $\{N_n\}, n = 0, 1, 2, \ldots$ as:

$N_0 = 0; N_1 = N_1(z) = \inf\{k \geq 1: z - Y_k < s\}; z \in [s, S],$

$N_{n+1} = N_{n+1}(\zeta_n) = \inf\{k \geq N_n + 1: \zeta_n - (Y_k - Y_{N_n}) < s\}; n = 1,2,3, \ldots$

Here $\inf(\emptyset) = +\infty$ is stipulated. Put

$$\tau_0 = 0; \tau_1 = \tau_1(z) = T_{N_1(z)} = \sum_{i=1}^{N_1(z)} \xi_i\,;\; \gamma_1 = \tau_1 + \theta_1;$$

$$\tau_n = \tau_n(\zeta_{n-1}) = T_{N_n} = \sum_{i=1}^{N_n} \xi_i\,; \gamma_n = \tau_n + \theta_n; n = 2,3, \ldots$$

and define $\nu(t)$ as $\nu(t) = \max\{n \geq 0: T_n \leq t\}, t > 0.$

Now let construct the stochastic process $X(t)$:

$$X(t)=\sum_{n=0}^{\infty}\max\{s,\zeta_n-(Y_{\nu(t)}-Y_{N_n})\}I_{[\gamma_n;\gamma_{n+1})}(t).$$

Here indicator function $I_A(t)$ of the set A is defined as follows:

$$I_A(t)=\begin{cases}1, t\in A\\ 0, t\notin A\end{cases}.$$

Process $X(t)$ is known as “Renewal-reward process with discrete interference of chance” in literature. Throughout this study, it is assumed that the random variable ζ_1 has asymmetric triangular distribution with parameters (s,m,S). For this reason, the process can be called “Renewal-reward process with an asymmetric triangular distributed interference of chance”. A sample trajectory of this process is shown in Figure 1.

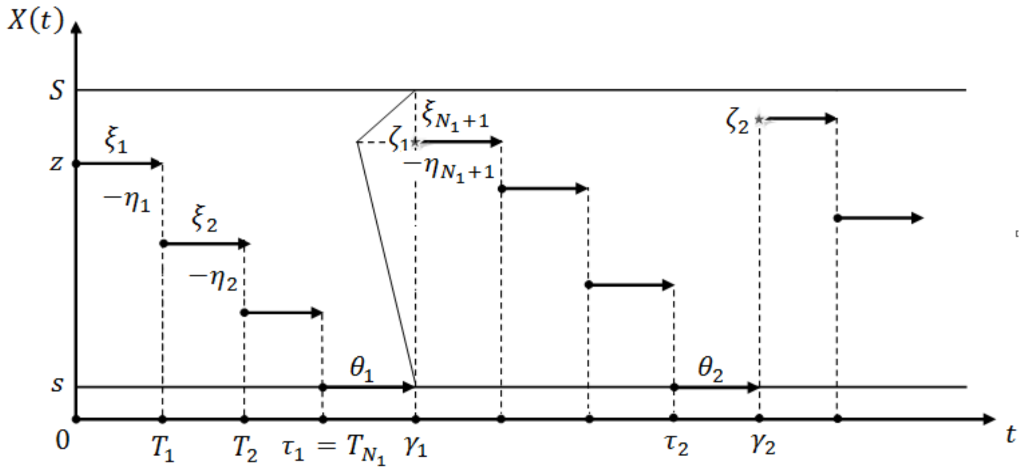

Figure 1.*A sample path of the process $X(t)$.*

3. ERGODICITY OF THE PROCESS X(t)

For investigating the stationary characteristics of the process, it is essential to prove the ergodicity of the process $X(t)$. Because of the calculation of n fold integrals, it is quite hard to do required evaluations about finite dimensional distributions. To compute stationary characteristics of the process partially removes the mentioned adversities. For this aim, first of all, we will discuss the ergodicity of the process.

Proposition 3.1: Let the initial sequence of the random variables $\{\xi_n,\eta_n,\zeta_n,\theta_n\}$ satisfy the following supplementary conditions:

1) $0<E(\xi_1)<+\infty$,
2) $0<E(\theta_1)<+\infty$,
3) $E(\eta_1)>0$,
4) η_1 is a non-arithmetic random variable,
5) Random variable ζ_1 has asymmetric triangular distribution with parameters (s,m,S).

Then the process $X(t)$ is ergodic and the following relation is true with probability 1 for every bounded and measurable function $f(x)$ $(f:[0;+\infty)\to R)$:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(u))du=\frac{1}{E(\gamma_1)}\int_s^S\int_s^S\int_0^\infty f(v)P_z\{\gamma_1>t;X(t)\in dv\}dtd\pi(z). \quad (1)$$

Proof: The process $X(t)$ belongs to a wide class of stochastic processes known as "Processes with a discrete interference of chance". This notion is firstly introduced to literature by A. N. Kolmogorov. For this class, the general ergodic theorem of type Smith's ‘key renewal theorem’ exists in the literature ([6]). According to this theorem, to prove the ergodicity of the processes with a discrete interference of chance, it is sufficient to show that the following two assumptions are hold.

Assumption 1. Choosing a sequence of ascending random times is required such that the values of the process $X(t)$ at these times form an embedded Markov chain that is ergodic. For this purpose, it is sufficient to consider the sequence of random times $\{\gamma_n\}, n \geq 0$, defined in Section 2, which are the stopping times. The values of the process $X(t)$ at these times are equal to $\zeta_n = X(\gamma_n), n \geq 1$ which form an embedded Markov chain. In our case, the embedded Markov chain $\{\zeta_n\}, n \geq 1$ is ergodic with a stationary distribution $\pi(z) = \lim_{n\to\infty} P\{\zeta_n \leq z\} = P\{\zeta_1 \leq z\}$ because the random variables $\zeta_n, n = 1, 2, \ldots$ are independent and identically distributed random variables in the interval $[s, S]$. Therefore, the first assumption of the general ergodic theorem ([6]) is satisfied.

Assumption 2. The expected value of the times between successive stopping times $\{\gamma_n\}, n = 1,2,3, \ldots$ should be finite, that is $E(\gamma_n - \gamma_{n-1}) < \infty, n = 1,2, \ldots$ For this aim, it is sufficient to show that:

$$E(\gamma_1(z)) = E(\tau_1(z) + \theta_1) = E(\tau_1(z)) + E(\theta_1) = E(\xi_1)E(N_1(z)) + E(\theta_1) < \infty; \qquad (2)$$

$$E(\gamma_n - \gamma_{n-1}) = E(\gamma_1(\zeta_1)) = E(\tau_1(\zeta_1) + \theta_1) = E(\xi_1)E(N_1(\zeta_1)) + E(\theta_1) < \infty;\ n \geq 2, 3, \ldots \qquad (3)$$

By the conditions of Proposition 3.1, $E(\theta_1) < \infty$ and $E(\xi_1) < \infty$. Therefore, to satisfy the inequalities in Eq. (2) and Eq. (3), the inequalities $E(N_1(z)) < \infty$ and $E(N_1(\zeta_1)) < \infty$ should be hold. Note that, $E(N_1(z)) \equiv U_\eta(z - s) < \infty$ for each $z \in [s, S]$ (see, [5], p.185). Here $U_\eta(x)$ is renewal function, generated by the sequence of the random variables $\{\eta_n\}$.

At the same time, the renewal function $U_\eta(x)$ is a nondecreasing function. Therefore, for each $z \in [s, S], U_\eta(z - s) \leq U_\eta(S - s) < \infty$ is provided. Hence, we have:

$$E(N_1(\zeta_1)) \equiv \int_s^S E(N_1(z))d\pi(z) = \int_s^S U_\eta(z - s)d\pi(z) \leq U_\eta(S - s) < \infty.$$

Thereby,$E(\gamma_1(z)) < \infty$ and $E(\gamma_n - \gamma_{n-1}) < \infty, n = 2, 3, \ldots$ are provided.This shows that the Assumption 2 is also satisfied. So, the process $X(t)$ is ergodic and the relation in Eq. (1) is hold. This concludes the proof of Proposition 3.1.■

4. EXACT EXPRESSIONS FOR ERGODIC DISTRIBUTION OF $X(t)$ AND ITS MOMENTS

In section 3, the ergodicity of the process $X(t)$ is proved under the conditions of Proposition 3.1. Then, $Q_X(x)$ is the ergodic distribution function of the process $X(t)$, i.e.

$Q_X(x) = \lim_{t\to\infty} P\{X(t) \leq x\}, x \in [s, S]$.

With the help of Eq. (1), the exact formula for the ergodic distribution $Q_X(x)$ of the process $X(t)$ can be obtained.

Theorem 4.1. Let the conditions of Proposition 3.1 be satisfied. The ergodic distribution function $Q_X(x)$ of the process $X(t)$ can be written as follows for each $x \in [s, S]$:

$$Q_X(x) = 1 - \frac{E\left(U_\eta(\zeta_1 - x)\right)}{E\left(U_\eta(\zeta_1 - s)\right) + K}, x \in [s, S].$$

Here, $K = E(\theta_1)/E(\xi_1)$; $E\left(U_\eta(\zeta_1 - x)\right) \equiv \int_s^S U_\eta(z - x)d\pi(z)$; $\pi(z) = P\{\zeta_1 \leq z\}$;

$$\pi'(x) \equiv p_\zeta(x) = \begin{cases} \dfrac{2(x - s)}{(m - s)(S - s)}, & s \leq x \leq m \\ \dfrac{2(S - x)}{(S - m)(S - s)}, & m < x \leq S \end{cases}.$$

Proof. Ergodic distribution function of the process $X(t)$ is as follows:

$Q_X(x) = \lim_{t\to\infty} P\{X(t) \leq x\}$; $x \in [s, S]$.

Define indicator function:

$f(v) \equiv f_x(v) = I_{[s,x]}(v) = 1$, if $v \in [s, x]$; and $I_{[s,x]}(v) = 0$ if $v \notin [s, x], x \in [s, S]$.

Substituting the expressed indicator function instead of function $f(v)$ in Eq. (1), the following expression can be obtained for the ergodic distribution $\left(Q_X(x)\right)$of the process $X(t)$, for each $x \in [s, S]$:

$$Q_X(x) = \frac{1}{E(\gamma_1)} \int_{t=0}^{\infty} \int_{s}^{S} P_z\{\gamma_1 > t; X(t) \le x\} dt d\pi(z). \tag{4}$$

For simplicity we introduce the following notation:
$G(t, x, z) \equiv P_z\{\gamma_1 > t; X(t) \le x\}$.
The function $G(t, x, z)$ can be written as follows:

$$G(t, x, z) = G_1(t, x, z) + G_2(t, x, z). \tag{5}$$

Here $G_1(t, x, z) \equiv P_z\{t < \tau_1; X(t) \le x\}$; $G_2(t, x, z) \equiv P_z\{\tau_1 \le t < \gamma_1; X(t) \le x\}$.

Let us calculate the function $G_1(t, x, z)$, where $x \in [s, S]$:

$$G_1(t, x, z) = P_z\{t < \tau_1; X(t) \le x\} = \sum_{n=0}^{\infty} P_z\{\nu(t) = n; \tau_1 > t; X(t) \le x\}. \tag{6}$$

Here, $\nu(t) = \max\{n \ge 0;\ T_n \le t\}$. Taking into account the notations $T_n = \sum_{i=1}^{n} \xi_i$ and $Y_n = \sum_{i=1}^{n} \eta_i$ in the Eq.(6), we have:

$$G_1(t, x, z) = \sum_{n=0}^{\infty} P\{T_n \le t < T_{n+1}\} P\{z - Y_n > s;\ z - Y_n \le x\}$$

$$= \sum_{n=0}^{\infty} \left(\Phi_n(t) - \Phi_{n+1}(t)\right)\left(F_n(z - s) - F_n(z - x)\right). \tag{7}$$

Here, $\Phi_n(t) = P\{T_n \le t\}$ and $F_n(z) = P\{Y_n \le z\}$.
Now, compute the function $G_2(t, x, z)$, when $x \in [s, S]$:

$$G_2(t, x, z) = \int_0^t \sum_{n=1}^{\infty} P\{z - Y_{n-1} \ge s;\ z - Y_n < s\} P\{T_n \in du\} P\{\theta_1 > t - u\}.$$

On the other hand, $P\{z - Y_{n-1} \ge s;\ z - Y_n < s\} = F_{n-1}(z - s) - F_n(z - s)$.

In summary, the function $G_2(t, x, z)$ can be written as follows:

$$G_2(t, x, z) = \sum_{n=1}^{\infty} [F_{n-1}(z - s) - F_n(z - s)][\Phi_n(t) - H(t) * \Phi_n(t)], x \in [s, S]. \tag{8}$$

Here, $H(t) * \Phi_n(t) = \int_0^t H(t - u)\, d\Phi_n(u)$.

Applying Laplace transform to Eq.(7) with respect to parameter t the following expression is derived:

$$\widetilde{G}_1(\lambda, x, z) = \sum_{n=0}^{\infty} [F_n(z - s) - F_n(z - x)] \frac{\varphi^n(\lambda)\left(1 - \varphi(\lambda)\right)}{\lambda}. \tag{9}$$

Here, $\varphi(\lambda) \equiv \int_0^{\infty} e^{-\lambda t} d\Phi(t) = E\left(e^{-\lambda \xi_1}\right);\ \lambda > 0$.

Because $E(\xi_1)$ is finite, $\lim_{\lambda \to 0} \frac{1-\varphi(\lambda)}{\lambda} = E(\xi_1)$. Therefore by taking limit of Eq. (9) when λ goes to zero, we obtain:

$$\widetilde{G}_1(0, x, z) \equiv \lim_{\lambda \to 0} \widetilde{G}_1(\lambda, x, z) = \int_0^{\infty} G_1(t, x, z) dt = E(\xi_1)\left[U_\eta(z - s) - U_\eta(z - x)\right]. \tag{10}$$

In the same content, applying Laplace transform to Eq. (8) with respect to parameter t, the following expression is derived:

$$\widetilde{G}_2(\lambda, x, z) \equiv \int_0^{\infty} e^{-\lambda t}\ G_2(t, x, z) dt = \sum_{n=1}^{\infty} [F_{n-1}(z - s) - F_n(z - s)] \frac{\left(1 - H^*(\lambda)\right)\varphi^n(\lambda)}{\lambda}. \tag{11}$$

Here, $H^*(\lambda) \equiv \int_0^{\infty} e^{-\lambda t} dH(t) = E\left(e^{-\lambda \theta_1}\right)$.

Because $E(\theta_1)$ is finite, $\lim_{\lambda\to 0}\frac{(1-H^*(\lambda))}{\lambda}=E(\theta_1)$. Therefore by taking limit of Eq. (11) when λ goes to zero, we have:

$$\widetilde{G}_2(0,x,z)\equiv\lim_{\lambda\to 0}\widetilde{G}_2(\lambda,x,z)=E(\theta_1)\sum_{n=1}^{\infty}[F_{n-1}(z-s)-F_n(z-s)]=E(\theta_1),z\in[s,S]. \quad (12)$$

Therefore, by considering Eq. (10), Eq. (12) and Eq. (5), the following equation is derived for each $x\in[s,S]$ and $z\in[s,S]$:

$$\widetilde{G}(0,x,z)=\widetilde{G}_1(0,x,z)+\widetilde{G}_2(0,x,z)=E(\xi_1)\left[U_\eta(z-s)-U_\eta(z-x)\right]+E(\theta_1). \quad (13)$$

Hence, substituting Eq. (13) in Eq. (4) we have:

$$Q_X(x)=\frac{1}{E(\gamma_1)}\int_s^S\widetilde{G}(0,x,z)d\pi(z)=\frac{1}{E(\gamma_1)}\int_s^S\left\{E(\xi_1)\left[E\left(U_\eta(\zeta_1-s)\right)-E\left(U_\eta(\zeta_1-x)\right)\right]+E(\theta_1)\right\}d\pi(z) \quad (14)$$

Using Wald identity, the following equation can be obtained:

$$E(\tau_1)\equiv\int_s^S E\big(\tau_1(z-s)\big)d\pi(z)=E(\xi_1)E\left(U_\eta(\zeta_1-s)\right)$$

On the other hand,

$$E(\gamma_1)=E(\tau_1+\theta_1)=E(\xi_1)E\left(U_\eta(\zeta_1-s)\right)+E(\theta_1)=E(\xi_1)\left\{E\left(U_\eta(\zeta_1-s)\right)+K\right\}. \quad (15)$$

Here,$K=E(\theta_1)/E(\xi_1)$ is the delay coefficient.

Taking into account Eq. (15) in Eq. (14), we finally have:

$$Q_X(x)=1-\frac{E\left(U_\eta(\zeta_1-x)\right)}{E\left(U_\eta(\zeta_1-s)\right)+K},x\in[s,S].$$

This concludes the proof of Theorem 4.1.■

It is possible to obtain some important results from Theorem 4.1. Two of them are given with following corollaries. Put $\widetilde{X}(t)\equiv X(t)-s$ and $\tilde{\zeta}_1=\zeta_1-s$ and state the following results.

Corollary 4.1. Let the conditions of Proposition 3.1 be satisfied. Then the process $\widetilde{X}(t)$ is ergodic and the ergodic distribution function can be presented as follows:

$$Q_{\widetilde{X}}(y)\equiv\lim_{t\to\infty}P\{\widetilde{X}(t)\le y\}=1-\frac{E\left(U_\eta(\tilde{\zeta}_1-y)\right)}{K+E\left(U_\eta(\tilde{\zeta}_1)\right)},\ \ y\in[0,S-s]. \quad (16)$$

Let $E(\widetilde{X}^n)$ is the n^{th} order moment of the process $\widetilde{X}(t)$, i.e.

$$E(\widetilde{X}^n)\equiv\lim_{t\to\infty}E\left(\widetilde{X}^n(t)\right)=\int_s^S x^n dQ_{\widetilde{X}}(x),n=1,2,...$$

Corollary 4.2. Let the conditions of Proposition 3.1 be satisfied. The exact expression for the n^{th} order moment $E(\widetilde{X}^n),n=1,2,...$ of the ergodic distribution of the process $\widetilde{X}(t)$ can be written as follows:

$$E(\widetilde{X}^n)=\frac{n}{K+E\left(U_\eta(\tilde{\zeta}_1)\right)}\int_0^{S-s}y^{n-1}E\left(U_\eta(\tilde{\zeta}_1-y)\right)dy,n=1,2,... \quad (17)$$

Here, $\tilde{\zeta}_1$ is a random variable having triangular distribution with parameters $(0,m-s,S-s)$.

Remark 4.1. As it can be seen from Corollary 4.2, the right hand side of Eq. (17) depends on the renewal function $U_\eta(x)$. Except some special, finding the simple and obvious expression of the renewal function is quite hard. For this aim, an alternative expression for $E(\widetilde{X}^n)$ is found in the following theorem.

Theorem 4.2. Let the conditions of Proposition 3.1 be satisfied. The exact expression of the n^{th} order ergodic moment of the process $\widetilde{X}(t)$ can be presented as follows:

$$E(\widetilde{X}^n) = \frac{nE\left(U_n(\tilde{\zeta}_1)\right)}{K + E\left(U_\eta(\tilde{\zeta}_1)\right)}, n = 1,2,3, \dots \tag{18}$$

Here, $E\left(U_n(\tilde{\zeta}_1)\right) = \int_0^{S-s} U_n(y)d\tilde{\pi}(y); U_n(y) \equiv y^{n-1} * U_\eta(y) = \int_0^y t^{n-1}U_\eta(y-t)dt$ and $\tilde{\pi}(y) = P(\tilde{\zeta}_1 \leq y) = \pi(s+y), y \in [0, S-s]$.

Proof: Calculate the integral in the Eq.(17):

$$I_n \equiv \int_0^{S-s} y^{n-1}E\left(U_\eta(\tilde{\zeta}_1 - y)\right)dy = \int_0^{S-S} y^{n-1}\left\{\int_y^{S-s} U_\eta(z-y)\tilde{p}(z)dz)\right\}dy$$

$$= \int_0^{S-s} \tilde{p}(z)\left\{\int_0^z y^{n-1}U_\eta(z-y)dy)\right\}dz = \int_0^{S-s} U_n(z)d\tilde{\pi}(z) = E\left(U_n(\tilde{\zeta}_1)\right). \tag{19}$$

Here, $\tilde{p}(z) = \tilde{\pi}'(z) = \begin{cases} \frac{z}{\beta^2(1+\alpha)}, & 0 < z \leq m-s \\ \frac{(2\beta - z)}{\beta^2(1-\alpha)}, & m-s < z \leq S-s \end{cases}$,

$\beta = \frac{S-s}{2}, \alpha = \frac{m-a}{\beta} \in [-1,1]$, and $a = \frac{S+s}{2}$ is the middle point. Note that, we call α as the asymmetry coefficient and when $\alpha = 0$ the asymmetry returns to symmetry. Substituting Eq. (19) in Eq. (17), Eq. (18) is obtained. This concludes the proof of Theorem 4.2. ■

5. ASYMPTOTIC EXPANSIONS FOR THE ERGODIC MOMENTS OF THE PROCESS $\widetilde{X}(t)$

The main aim of this section is to obtain asymptotic expansions for n^{th} order moment of the ergodic distribution of the process $\widetilde{X}(t)$. For this purpose, some required preliminary information is given by following propositions.

Proposition 5.1. Let $A(z) = \int_0^z U_\eta(y)dy$ and $V(z) = \int_0^z yU_\eta(y)dy$. Then, the Laplace transforms $\widetilde{A}(\lambda)$ and $\widetilde{V}(\lambda)$ of the function $A(z)$ and $V(z)$ can be written as follows:

$$\widetilde{A}(\lambda) = \frac{1}{\lambda^2(1-\varphi(\lambda))}; \widetilde{V}(\lambda) = \frac{1-\varphi(\lambda) - \lambda\varphi'(\lambda)}{\lambda^3(1-\varphi(\lambda))^2}. \tag{20}$$

Here $\varphi(\lambda) = E(\exp(-\lambda\eta_1)),\ \lambda > 0$. ■

Using Tauber – Abel theorem for the Laplace transforms in Eq.(20), we can give the following proposition.

Proposition 5.2. If $m_3 < \infty$, then the following asymptotic expansions for $A(z)$ and $V(z)$ can be written, when $z \to \infty$:

$$A(z) = \frac{1}{2m_1}z^2 + \frac{m_2}{2m_1^2}z + \frac{A}{m_1} + o(1);\ V(z) = \frac{1}{3m_1}z^3 + \frac{m_2}{4m_1^2}z^2 + o(z). \tag{21}$$

Here $A = m_{21}^2 - \frac{1}{2}m_{31}; m_{k1} = \frac{m_k}{km_1}, m_k = E(\eta_1^k), k = 1,2,3.$ ■

Lemma 5.1. Let the conditions of Proposition 3.1 be satisfied and additionally $m_3 = E(\eta_1^3) < +\infty$. Then three term asymptotic expansion for $E\left(U_\eta(\tilde{\zeta}_1)\right)$ can be written as follows, when $\beta \to \infty$:

$$E\left(U_\eta(\tilde{\zeta}_1)\right) = \frac{3+\alpha}{3m_1}\beta + \frac{m_2}{2m_1^2} + o\left(\frac{1}{\beta}\right). \tag{22}$$

Proof. By definition of $\tilde{\zeta}_1$, the following equation can be written:

$$E\left(U_\eta(\tilde{\zeta}_1)\right)=\int_0^{S-s}U_\eta(z)d\tilde{\pi}(z)=\int_0^{m-s}U_\eta(z)d\tilde{\pi}(z)+\int_{m-s}^{S-s}U_\eta(z)d\tilde{\pi}(z). \tag{23}$$

Now, let us calculate the first integral in Eq.(23):

$$I_1\equiv\int_0^{m-s}U_\eta(z)d\tilde{\pi}(z)=\frac{1}{\beta^2(1+\alpha)}\int_0^{\beta(1+\alpha)}z\,U_\eta(z)dz=\frac{1}{\beta^2(1+\alpha)}V\big(\beta(1+\alpha)\big). \tag{24}$$

Taking account the expansion for $V(z)$ from Eq.(23) into Eq.(24), the following expansion is derived:

$$I_1=\frac{1}{\beta^2(1+\alpha)}\left\{\frac{1}{3m_1}\big(\beta(1+\alpha)\big)^3+\frac{m_2}{4m_1^2}\big(\beta(1+\alpha)\big)^2+o\big(\beta(1+\alpha)\big)\right\}$$
$$=\frac{(1+\alpha)^2}{3m_1}\beta+\frac{m_2(1+\alpha)}{4m_1^2}+o\left(\frac{1}{\beta}\right). \tag{25}$$

Now, calculate the second integral in Eq. (23):

$$I_2\equiv\int_{m-s}^{S-s}U_\eta(z)d\tilde{\pi}(z)=\frac{1}{\beta^2(1-\alpha)}\big[D(2\beta)-D\big(\beta(1+\alpha)\big)\big]. \tag{26}$$

Here, $D(x)=\int_0^x U_\eta(z)(2\beta-z)dz$. By applying Proposition 5.2, we can obtain the following expansion for the integral $D(2\beta)$, as $\beta\to\infty$:

$$D(2\beta)=2\beta\,A(2\beta)-V(2\beta)=\frac{(2\beta)^3}{6m_1}+\frac{m_2}{4m_1^2}(2\beta)^2+\frac{A}{m_1}(2\beta)+o(\beta). \tag{27}$$

In a similar way, the following asymptotic expansion can be also obtained for $D\big(\beta(1+\alpha)\big)$, as $\beta\to\infty$:

$$D\big(\beta(1+\alpha)\big)=2\beta\,A\big(\beta(1+\alpha)\big)-V\big(\beta(1+\alpha)\big)$$
$$=\frac{2(1+\alpha)(2+\alpha-\alpha^2)}{6m_1}\beta^3+\frac{m_2(3+2\alpha-\alpha^2)}{4m_1^2}\beta^2+\frac{2A}{m_1}\beta+o(\beta) \tag{28}$$

where, $A=m_{21}^2-\frac{1}{2}m_{31};\ m_{k1}=\frac{m_k}{km_1};\ m_k=E\big(\eta_1^k\big),k=1,2,\ldots$

Substituting expansions Eq. (27) and Eq. (28) into Eq.(26), the following expansion is derived, as $\beta\to\infty$:

$$I_2=\frac{1}{\beta^2(1-\alpha)}\big[D(2\beta)-D\big(\beta(1+\alpha)\big)\big]=\frac{(2-\alpha-\alpha^2)}{3m_1}\beta+\frac{m_2(1-\alpha)}{4m_1^2}+o\left(\frac{1}{\beta}\right). \tag{29}$$

Considering the expansions Eq. (25) and Eq. (29) in Eq. (23), the Eq. (22) is obtained.

Therefore, the Lemma 5.1 is proved.■

Now let us investigate the asymptotic behavior of the function $U_n(y)=y^{n-1}*U_\eta(y)$.

Proposition 5.3 ([10]). Let the condition $E(\eta_1^3)<\infty$ be satisfied. Then, the following asymptotic expansion can be written, as $z\to\infty$

$$U_n(y)\equiv\frac{y^{n+1}}{n(n+1)m_1}+\frac{m_2y^n}{2nm_1^2}+\frac{A}{nm_1}+o(y^{n-1}),n=1,2\ldots \tag{30}$$

where $A=m_{21}^2-\frac{1}{2}m_{31}$. ■

Put $A_n(y)=\int_0^y U_n(z)dz$.

Proposition 5.4. Let the condition $E(\eta_1^3)<\infty$ be satisfied. Then, for each $n=1,2,3\ldots$ the following asymtotic expansion can be written, as $z\to\infty$:

$$A_n(z)=\frac{1}{n(n+1)(n+2)m_1}z^{n+2}+\frac{m_2}{2n(n+1)m_1^2}z^{n+1}+\frac{A}{nm_1}z^n+o(z^n). \tag{31}$$

Proof. The Laplace transform $\widetilde{A}_n(\lambda)$ of $A_n(z)$ has the following form:

$$\tilde{A}_n(\lambda) = \frac{1}{\lambda}\left(\tilde{U}_n(\lambda)\right) = \frac{(n-1)!}{\lambda^{n+2}(1-\varphi(\lambda))}, \tag{32}$$

where $\varphi(\lambda) = E\left(e^{-\lambda\eta_1}\right)$, $\lambda > 0$. Using Taylor expansion of functions $\varphi(\lambda)$ and $\varphi'(\lambda)$ as $\lambda \to 0$ and applying Tauber-Abel theorem to Eq. (32), the expansion in Eq.(31) can be obtained.

This concludes the proof of Proposition 5.4. ■

Put $V_n(z) = \int_0^z yU_n(y)dy$.

Proposition 5.5. Let the condition $E(\eta_1^3) < \infty$ be satisfied. Then, for each $n = 1,2,3\ldots$ the following asymptotic expansion can be written, as $z \to \infty$:

$$V_n(z) = \frac{z^{n+3}}{n(n+1)(n+3)m_1} + \frac{m_2 z^{n+2}}{2n(n+2)m_1^2} + \frac{Az^{n+1}}{(n+1)m_1} + o(z^{n+1}). \tag{33}$$

Proof. The Laplace transform $\tilde{V}_n(\lambda)$ of $V_n(z)$ has the following form:

$$\tilde{V}_n(\lambda) = \frac{(n-1)!\left\{(n+1)\left(1-\varphi(\lambda)\right) - \lambda\varphi'(\lambda)\right\}}{\lambda^{n+3}\left(1-\varphi(\lambda)\right)^2}. \tag{34}$$

Using Taylor expansion of function $\varphi(\lambda)$ as $\lambda \to 0$ and applying Tauber-Abel theorem to Eq. (34), the expansion in Eq.(33) can be obtained.

This concludes the proof of Proposition 5.5. ■

Lemma 5.2. Let the conditions of Proposition 3.1 be satisfied and additionally $E(\eta_1^3) < +\infty$. Then the three – term asymptotic expansion for $E\left(U_n(\tilde{\zeta}_1)\right)$ can be written as follows, when $\beta \to \infty$:

$$E\left(U_n(\tilde{\zeta}_1)\right) = A_{1n}\beta^{n+1} + A_{2n}\beta^n + A_{3n}\beta^{n-1} + o(\beta^{n-1}). \tag{35}$$

Here,

$$A_{1n} = \frac{(1+\alpha)^{n+2}}{n(n+1)(n+3)m_1} + \frac{2(n+3)C_{n+2}(\alpha) - (n+2)C_{n+3}(\alpha)}{n(n+1)(n+2)(n+3)m_1};$$

$$A_{2n} = \frac{m_2(1+\alpha)^{n+1}}{2n(n+2)m_1^2} + \frac{m_2\{2(n+2)C_{n+1}(\alpha) - (n+1)C_{n+2}(\alpha)\}}{2n(n+1)(n+2)m_1^2};$$

$$A_{3n} = \frac{A(1+\alpha)^n}{(n+1)m_1} + \frac{A\{2(n+1)C_n(\alpha) - nC_{n+1}(\alpha)\}}{n(n+1)m_1};$$

$C_n(\alpha) = \frac{2^n - (1+\alpha)^n}{1-\alpha}$, $A = m_{21}^2 - \frac{1}{2}m_{31}, m_{k1} = \frac{m_k}{km_1}$, and $m_k = E\left(\eta_1^k\right), k = 1,2,3,\ldots$

Proof. Write $E\left(U_n(\tilde{\zeta}_1)\right)$ in the following form:

$$E\left(U_n(\tilde{\zeta}_1)\right) = \int_0^{m-s} U_n(z)d\,\tilde{\pi}(z) + \int_{m-s}^{S-s} U_n(z)d\tilde{\pi}(z). \tag{36}$$

Using expansion in Eq. (33), the first integral in Eq.(36) can be presented as follows:

$$I_{n1}(\beta) \equiv \int_0^{m-s} U_n(z)d\tilde{\pi}(z) = \frac{1}{\beta^2(1+\alpha)}\int_0^{\beta(1+\alpha)} zU_n(z)dz = \frac{1}{\beta^2(1+\alpha)}V_n\left(\beta(1+\alpha)\right)$$

$$= \frac{(1+\alpha)^{n+2}}{n(n+1)(n+3)m_1}\beta^{n+1} + \frac{m_2(1+\alpha)^{n+1}}{2n(n+2)m_1^2}\beta^n + \frac{A(1+\alpha)^n}{(n+1)m_1}\beta^{n-1} + o(\beta^{n-1}). \tag{37}$$

Using expansion in Eq. (31) and Eq. (33), the second integral in Eq.(36) can be written as follows:

$$I_{n2}(\beta) \equiv \frac{1}{\beta^2(1-\alpha)}\int_{\beta(1+\alpha)}^{2\beta} U_n(z)(2\beta - z)dz$$

$$=\frac{1}{\beta^2(1-\alpha)}\{2\beta[A_n(2\beta)-A_n(\beta(1+\alpha))]-[V_n(2\beta)-V_n(\beta(1+\alpha))]\}$$
$$=\frac{1}{(1-\alpha)}\left\{\frac{[2^{n+2}-(1+\alpha)^{n+2}]}{n(n+1)(n+2)m_1}\beta^{n+1}+\frac{m_2[2^{n+1}-(1+\alpha)^{n+1}]}{2n(n+1)m_1^2}\beta^n\right.$$
$$\left.+\frac{A[2^n-(1+\alpha)^n]}{nm_1}\beta^{n-1}\right\}-\frac{1}{(1-\alpha)}\left\{\frac{[2^{n+3}-(1+\alpha)^{n+3}]}{n(n+1)(n+3)m_1}\beta^{n+1}\right.$$
$$\left.+\frac{m_2[2^{n+2}-(1+\alpha)^{n+2}]}{2n(n+2)m_1^2}\beta^n+\frac{A[2^{n+1}-(1+\alpha)^{n+1}]}{(n+1)m_1}\beta^{n-1}\right\}+o(\beta^{n-1}). \quad (38)$$

For simplicity, introduce the notation $C_n(\alpha)=\frac{2^n-(1+\alpha)^n}{1-\alpha}$ and substitute it in Eq. (38):

$$I_{n2}(\beta)=\frac{2(n+3)C_{n+2}(\alpha)-(n+2)C_{n+3}(\alpha)}{n(n+1)(n+2)(n+3)m_1}\beta^{n+1}$$
$$+\frac{2(n+2)m_2C_{n+1}(\alpha)-(n+1)m_2C_{n+2}(\alpha)}{2n(n+1)(n+2)m_1^2}\beta^n$$
$$+\frac{2(n+1)AC_n(\alpha)-nAC_{n+1}(\alpha)}{n(n+1)m_1}\beta^{n-1}+o(\beta^{n-1}). \quad (39)$$

Here, $A=m_{21}^2-\frac{1}{2}m_{31},\ m_{k1}=\frac{m_k}{km_1}$ and $m_k=E(\eta_1^k), k=1,2,3,\dots$

By substituting expansions Eq. (37) and Eq. (39) into Eq. (36), we get the asymptotic expansion in Eq. (35). This concludes the proof of Lemma 5.2. ■

Finally the following theorem establishes the main result of our study.

Theorem 5.1. Let the conditions of Proposition 3.1 be satisfied and additionally $E(\eta_1^3)<+\infty$. Then, the following asymptotic expansions for n^{th} order ergodic moments of the process $\widetilde{X}(t)$ can be written, when $\beta\to\infty$:

$$E(\widetilde{X}^n)=D_{1n}(\alpha)\beta^n+D_{2n}(\alpha)\beta^{n-1}+D_{3n}(\alpha)\beta^{n-2}+o(\beta^{n-2}). \quad (40)$$

Here,

$$D_{1n}(\alpha)=\frac{6C_{n+2}(\alpha)}{(3+\alpha)(n+1)(n+2)(n+3)},$$
$$D_{2n}(\alpha)=\frac{3m_2C_{n+1}(\alpha)}{m_1(3+\alpha)(n+1)(n+2)}-\frac{9(2Km_1^2+m_2)C_{n+2}(\alpha)}{m_1(3+\alpha)^2(n+1)(n+2)(n+3)},$$
$$D_{3n}(\alpha)=\frac{6AC_n(\alpha)}{(3+\alpha)(n+1)}-\frac{9m_2(2Km_1^2+m_2)C_{n+1}(\alpha)}{2m_1(3+\alpha)^2(n+1)(n+2)}$$
$$+\frac{27(2Km_1^2+m_2)^2C_{n+2}(\alpha)}{2m_1^2(3+\alpha)^3(n+1)(n+2)(n+3)},$$
$$\beta=\frac{S-s}{2},\alpha=\frac{m-a}{\beta},a=\frac{S+s}{2},C_n(\alpha)=\frac{2^n-(1+\alpha)^n}{1-\alpha},n=1,2,\dots,K=E(\theta_1)/E(\xi_1)$$

and m is the mode of the distribution of random variable ζ_1.

Proof. We get the exact form of $E(\widetilde{X}^n)$ in the Theorem 4.2 as follows:

$$E(\widetilde{X}^n)=\frac{nE(U_n(\tilde{\zeta}_1))}{K+E(U_\eta(\tilde{\zeta}_1))}. \quad (41)$$

We obtained the asymptotic expansions for $E\left(U_\eta(\tilde{\zeta}_1)\right)$ and $E\left(U_n(\tilde{\zeta}_1)\right)$ in the Lemma 5.1 and Lemma 5.2, respectively. Using Lemma 5.1 we can write the following asymptotic expansion, as $\beta\to\infty$:

$$\frac{1}{K+E\left(U_\eta(\tilde{\zeta}_1)\right)}=\left(K+\frac{3+\alpha}{3m_1}\beta+\frac{m_2}{2m_1^2}+o\left(\frac{1}{\beta}\right)\right)^{-1}$$
$$=\frac{3m_1}{(3+\alpha)\beta}\left[1-\frac{3m_1}{(3+\alpha)\beta}\frac{(2Km_1^2+m_2)}{2m_1^2}+\frac{9m_1^2}{(3+\alpha)^2\beta^2}\frac{(2Km_1^2+m_2)^2}{4m_1^4}+o\left(\frac{1}{\beta^2}\right)\right]. \quad (42)$$

Substituting Eq. (35) and Eq. (42) into Eq. (41), we have:

$$E(\widetilde{X}^n) = \frac{6C_{n+2}(\alpha)}{(3+\alpha)(n+1)(n+2)(n+3)}\beta^n$$
$$+\left[\frac{3m_2C_{n+1}(\alpha)}{m_1(3+\alpha)(n+1)(n+2)} - \frac{9(2Km_1^2+m_2)C_{n+2}(\alpha)}{m_1(3+\alpha)^2(n+1)(n+2)(n+3)}\right]\beta^{n-1}$$
$$+\left[\begin{array}{c}\dfrac{6AC_n(\alpha)}{(3+\alpha)(n+1)} - \dfrac{9m_2(2Km_1^2+m_2)C_{n+1}(\alpha)}{2m_1(3+\alpha)^2(n+1)(n+2)} \\ +\dfrac{27(2Km_1^2+m_2)^2C_{n+2}(\alpha)}{2m_1^2(3+\alpha)^3(n+1)(n+2)(n+3)}\end{array}\right]\beta^{n-2} + o(\beta^{n-2})$$

This concludes the proof of Theorem 5.1. ■

6. COMPARISON OF THE EXACT AND THE ASYMPTOTIC RESULTS

The obvious drawback of exact formulas is their complex mathematical structures which poses difficulty in dealing with real world problems. In particular we are faced with n-fold convolutions of series, which can be approximated by closed form formulas that have better use in application. We apply asymptotic methods to get three – term asymptotic expansions. In order to evaluate the precision of the asymptotic results, it is important to compare the obtained formulas with exact ones. A real world example from Section 1 is used to check the efficiency of the formulas. Our study has specialties such that having asymmetric triangular interference of chance and delay. Therefore an appropriate problem is designed to fit the assumptions of the model as follows:

Suppose that the random variable ζ_1 follows asymmetric triangular distribution with parameters (s, m, S) where s is the control level, S is the maximum stock level and m is the mode of the distribution. Moreover the random variable η_1 has exponential distribution with parameter $\lambda = 1$. Besides, assume that the stock control level is $s = 0$ and $m = 0{,}85S$. Here the mode m is computed with the contribution of $\alpha = 0{,}7$ which is the degree of asymmetry of triangular distribution. So, we can reach the real world problem's needs because, dealers who are mentioned in problem usually want to fill about 85% of the capacity (S) of their tanks. When we change the value of α, we can obtain different m values for other problems. By computing m with contribution of α, our study becomes a generalization of previous studies about triangular distributed interference of chance. As another difference from other studies, there is a delay time θ_1. We assume that delay time θ_1 has the same distribution with inter-arrival times between demands $\xi_1$in this problem. So, the delay coefficient $(K = E(\theta_1)/E(\xi_1))$ is equal to 1. When η_1 has exponential distribution with a parameter $\lambda = 1$, the renewal function is $U_\eta(x) = x + 1$ which provides convenience in calculation. By the aid of this information, the exact formulas for the first two initial ergodic moments of the process $\widetilde{X}(t)$ that are found by Eq. (18) are as follows:

$$E(\widetilde{X}) = \frac{0.8575\beta^2 + 1.23332\beta}{1.23332\beta + 2}, \quad E(\widetilde{X}^2) = \frac{0.84976\beta^3 + 1.715\beta^2}{1.23332\beta + 2}.$$

In addition, the approximations for the first two initial moments of the process $\widetilde{X}(t)$ that are obtained from Eq. (40) by omitting the reminder term are as follows:

$$E\left(\widehat{\widetilde{X}}\right) = 0.69527027\beta - 0.127465303 + \frac{0.2067004915}{\beta},$$
$$E\left(\widehat{\widetilde{X}}^2\right) = 0.689\beta^2 + 0.273243242\beta - 0.4409715.$$

In order to control the realization of our study, we denote exact expression values as $E(\widetilde{X})$ and $E(\widetilde{X}^2)$ and the approximated values as $E\left(\widehat{\widetilde{X}}\right)$ and $E\left(\widehat{\widetilde{X}}^2\right)$. And then, we computed the error term Δ_k by taking difference of exact and approximated values that is $\Delta_k = \left|E(\widetilde{X}^k) - E\left(\widehat{\widetilde{X}}^k\right)\right|, k = 1, 2$. By using Δ_k, we calculated relative error term percentage $\delta_k(\%)$ that is $\delta_k = \frac{\Delta_k}{E(\widetilde{X}^k)}100\%, k = 1, 2$ and calculated accuracy percentage $(AP_k = 100 - \delta_k, k = 1, 2)$ that gives the closeness of formulas. Δ_k, $\delta_k(\%)$ and $AP_k(\%)$ values are computed for different $\beta = (S - s)/2$ values given in Table 1 and Table 2 as follows:

Table 1.*Comparison of the exact and the approximated values for the 1^{st} ergodic moment.*

β	$E(\tilde{X})$	$E(\hat{\tilde{X}})$	Δ_1	$\delta_1(\%)$	$AP_1(\%)$
20	13.7876	13.7883	0.0006	0.0047	99.9953
15	10.3141	10.3154	0.0013	0.0122	99.9878
10	6.8431	6.8459	0.0028	0.0414	99.9586
5	3.3801	3.3902	0.0101	0.2989	99.7011
4	2.6904	2.7053	0.0149	0.5536	99.4464
3	2.0031	2.0272	0.0242	1.2066	98.7934
2	1.3202	1.3664	0.0463	3.5052	96.4948

Table 2.*Comparison of the exact and the approximated values for the 2^{nd} ergodic moment.*

β	$E(\tilde{X}^2)$	$E(\hat{\tilde{X}}^2)$	Δ_2	$\delta_2(\%)$	$AP_2(\%)$
20	280,-.6578	280.6218	0.0360	0.0128	99.9872
15	158.7253	158.6806	0.0448	0.0282	99.9718
10	71.2518	71.1893	0.0625	0.0877	99.9123
5	18.2568	18.1481	0.1087	0.5952	99.4048
4	11.8018	11.6739	0.1279	1.0838	98.9162
3	6.7332	6.5776	0.1555	2.3098	97.6902
2	3.0578	2.8594	0.1984	6.4890	93.5110

Remark 6.1: Observing Table 1 and Table 2, one can easily realize the efficiency of the asymptotic results. For instance, for the first initial moment, even if the β value is equal to 3 the accuracy percentage is near to %98.8. For the second initial moment, even if the β value is equal to 3 the accuracy percentage is near to %97.7. Moreover, when β value is greater than 5, the accuracy percentages are close to %99 for both of the initial moments. Note that, when we are applying asymptotic method, we assume that $\beta \to \infty$. However, considering calculated accuracy percentages, we notice that it is not a strict requirement to take β as infinity. Valuable results can be obtained for even small values for β too.

7. CONCLUSION

In the study [10], a renewal-reward process with symmetric triangular distributed interference of chance is investigated and obtained asymptotic expansions for the moments of the process. Its results are interesting not only for theoretical aspect but also by the means of application usage. Nevertheless it does not address all of the features of a more realistic model. To model most of real world problems, processes with an asymmetric triangular distributed interference of chance are more applicable and useful (see, Section 1). That's why we considered such a process in this study with an asymmetric triangular distributed interference of chance on one hand and the case with positive lead time on other hand. Under these two assumptions, the ergodicity of the process is proved. The exact formulas and the asymptotic expansions for the ergodic moments of the process are found. The closeness of the approximated formulas and the exact formulas is tested by an example. It is observed that these formulas are sufficiently near to each other with high accuracy percentages even the parameter β takes small values. The studies in the literature ([9], [10], etc.) became a special case of our study, when the degree of asymmetry $\alpha = 0$ and delay coefficient $K = 0$. As future studies, the proof of weak convergence theorem for our process which is important for application and the investigation of similar results for random walk processes can be carried out. Also, as an improvement to our study, backordering and shortage in the stock as well as imperfect stock quality can be allowed in the future research.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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