GU J Sci 31(1): 213-220 (2018)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Pisano Periods For The K-Fibonacci And K-Lucas Sequences Mod 2ⁿ

Dursun TAŞCI [1,*], Gül ÖZKAN KIZILIRMAK [2]

[1]Gazi University, Faculty of Sciences, Departments Mathematics, 06500, Ankara, Turkey

[2]Gazi University, Faculty of Sciences, Departments Mathematics, 06500, Ankara, Turkey

Article Info

Received: 10/04/2017
Accepted: 23/12/2017

Keywords

k-Fibonacci
k-Lucas
Pisano period

Abstract

The goal of this paper is to investigate period of k-Lucas sequence with related divisibility properties and periods of k-Fibonacci and k-Lucas sequences mod 2^n.

1. INTRODUCTION

Some sequences of numbers have been studied over several years. In the literature, in mathematics and physics, there are a lot of integer sequences, which are used in almost every field modern sciences. The Fibonacci sequence is the famous integer sequence, which is defined by the following recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

With the initial conditions $F_0 = 0$ and $F_1 = 1$.

Another well-known sequence is the Lucas sequence, which satisfies the following recurrence relation

$$L_{n+1} = L_n + L_{n-1}$$

with $L_0 = 2$ and $L_1 = 1$.

There are many generalizations of the Fibonacci and Lucas sequences [1,2,4]. Two of them was given by Falcon and Plaza in [2,4] as follows:

For any integer number $k \geq 1$, the kth Fibonacci sequences $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined as for n≥ 1

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad (1)$$

with initial conditions $F_{k,0} = 0,\ F_{k,1} = 1$.

If we take $k = 1$ in (1), we get the Fibonacci sequence: $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$.

*Corresponding author, e-mail: gulozkan@gazi.edu.tr

By setting $k = 2$ in (1), we obtain the Pell sequence: $\{0, 1, 2, 5, 12, 29, 70, \dots\}$.

The k-Lucas sequence $\{L_{k,n}\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation for $n, k \geq 1$

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \tag{2}$$

with $L_{k,0} = 2,\ L_{k,1} = k$.

For $k = 1$ in (2), the classical Lucas sequence is obtained: $\{2, 1, 3, 4, 7, 11, 18, \dots\}$.
For $k = 2$ in (2), the Pell-Lucas sequence is obtained: $\{2, 2, 6, 14, 34, 82, 198, \dots\}$.

There are some properties for these numbers. Some of them are [2,4]:

- For $n \in \mathbb{N}$, $F_{k,2n+1} = (F_{k,n})^2 + (F_{k,n+1})^2$, (3)
- For $n \in \mathbb{N}$, $F_{k,n-1}\ F_{k,n+1} - (F_{k,n})^2 = (-1)^n$, (4)
- For $r > n$, $L_{k,n-r}L_{k,n+r} - (L_{k,n})^2 = (-1)^{n+r}L_{k,2r} + 2(-1)^{n+1}$, (5)
- For $n \in \mathbb{N}$, $F_{k,2n} = F_{k,n}L_{k,n}$, (6)
- For $n, m \in \mathbb{N}$, $L_{k,n}L_{k,n+m} = L_{k,2n+m} + (-1)^n L_{k,m}$, (7)
- For $m \geq 1$, $L_{k,n+1}L_{k,m} + L_{k,n}L_{k,m-1} = (k^2 + 4)\ F_{k,n+m}$. (8)

The period of the Fibonacci sequence mod m was first studied by Wall [12]. The recurrence part in the sequence creates a new sequence and gives the length of the periods of these sequences. Furthermore Kramer and Hoggatt [8] studied the periods of Fibonacci and Lucas sequences mod 2^n. Falcon and Plaza [3] studied the period length of the k-Fibonacci sequence mod m. The period of such cyclic sequences is known as Pisano period and the period-length is denoted by $\pi_k(m)$.

Motivated by the above papers, we study the Pisano period for the k-Lucas sequence and we obtain Pisano periods for the k-Fibonacci and k-Lucas sequences mod 2^n.

2. PISANO PERIODS FOR THE K-FIBONACCI AND K-LUCAS SEQUENCES

Theorem 2.1. $\{L_{k,n} \bmod m\}_{n\in\mathbb{N}}$ is a simple periodic sequence .

Proof. From the defining relation we write,

$$L_{k,n-1} = L_{k,n+1} - kL_{k,n}\ .$$

If $L_{k,t+1} \equiv L_{k,s+1} \pmod{m}$ and $L_{k,t} \equiv L_{k,s} \pmod{m}$, then

$$L_{k,t-1} \equiv kL_{k,s-1} \pmod{m}.$$

By continiuing this way, we get $L_{k,t-s+1} \equiv L_{k,1} \pmod{m}$ and $L_{k,t-s} \equiv L_{k,0} \pmod{m}$.
So that $\{L_{k,n} \bmod m\}_{n\in\mathbb{N}}$ is a simple periodic sequence with $t - s$ period.

Corollary 2.2. For $m > 3$ every Pisano period begins with $2, 3$.

Theorem 2.3. If the prime factorization of m is $m = \prod {p_i}^{e_i}$, then

$$\pi_k\big(lcm({p_i}^{e_i})\big) = lcm\big(\pi_k({p_i}^{e_i})\big).$$

Proof. The statement $\pi_k({p_i}^{e_i})$ is the length of the period of $L_{k,n}$ (mod p) implies that the sequence $L_{k,n} \pmod{{p_i}^{e_i}}$, repeats only after blocks of length $c\pi_k({p_i}^{e_i})$ and the statement $\pi_k(m)$ is the period-length of the sequence $L_{k,n}$ (mod m), which is, $L_{k,n}$ (mod $p_i^{e_i}$) repeats after $\pi_k(m)$ terms for all values of i. Since any such number gives a period of $L_{k,n}$ (mod m), we conclude that $\pi_k(m) = lcm\left(\pi_k(p_i^{e_i})\right)$.

Corollary 2.4. If $r|m$ then $\pi_k(r)|\pi_k(m)$.

Proof. If $r \mid m$, then $m = r.p_1^{e_1}p_2^{e_2} \dots p_k^{e_k}$. From Theorem 2.3, we get
$\pi_k(m) = lcm\left(\pi_k(r), \pi_k(p_1^{e_1}), \dots, \pi_k(p_k^{e_k})\right)$ and from lcm definition $\pi_k(r)|\pi_k(m)$.

Lemma 2.5. If k is an odd integer, then for $n \in \mathbb{N}$
i. $L_{k,3n} \equiv 0 \ (mod\ 2)$ (9)
ii. $F_{k,3n} \equiv 0 \ (mod\ 2)$. (10)

Proof. i. We can give the proof by induction. For $n = 1$,

$$L_{k,3} = k^3 + 3k.$$

Since k is an odd number, $k^3 + k$ is an even integer. Thus,

$$L_{k,3} \equiv 0 \text{ (mod 2)}.$$

Suppose $L_{k,3n} \equiv 0$ (mod 2). So,

$$\begin{aligned} L_{k,3(n+1)} &= kL_{k,3n+2} + L_{k,3n+1} \\ &= k\left(kL_{k,3n+1} + L_{k,3n}\right) + L_{k,3n+1} \\ &= (k^2+1)L_{k,3n+1} + kL_{k,3n.} \end{aligned}$$

Since $(k^2 + 1)$ is an even integer and from induction hypothesis,

$$(k^2+1)L_{k,3n+1} + kL_{k,3n} \equiv 0 \text{ (mod 2)}.$$

Thus we get

$$L_{k,3(n+1)} \equiv 0 \text{ (mod 2)}.$$

ii. We can give the proof by induction. For $n = 1$, $F_{k,3} = k^2 + 1$ and thus

$$F_{k,3} \equiv 0 \text{ (mod 2)}.$$

Suppose $F_{k,3n} \equiv 0$ (mod 2). So,

$$\begin{aligned} F_{k,3(n+1)} &= kF_{k,3n+2} + F_{k,3n+1} \\ &= k\left(kF_{k,3n+1} + F_{k,3n}\right) + F_{k,3n+1} \\ &= (k^2+1)F_{k,3n+1} + kF_{k,3n} \end{aligned}$$

and thus we have

$$F_{k,3(n+1)} \equiv 0 \text{ (mod 2)}.$$

Lemma 2.6. If k is an even integer, then for $n \in \mathbb{N}$
i. $L_{k,2^n} \equiv 0$ (mod 2) (11)
ii. $F_{k,2^n} \equiv 0$ (mod 2). (12)

Proof. i. We can give the proof by induction. For $n = 1$, $L_{k,2} = k^2 + 2$ and thus

$L_{k,2} \equiv 0 \pmod 2$.

Suppose $L_{k,2^n} \equiv 0 \pmod 2$.

For $m = 0$ and n is replaced by 2^n, we have the Eq. (7)

$L_{k,2^{n+1}} = (L_{k,2^n})^2 + 2(-1)^{2^n+1}$

and thus

$L_{k,2^{n+1}} \equiv 0 \pmod 2$.

ii. We can give the proof by induction. For $n = 1$, $F_{k,2} = k$ and thus

$F_{k,2} \equiv 0 \pmod 2$.

Suppose $F_{k,2^n} \equiv 0 \pmod 2$.

For n is replaced by 2^n, we get the Eq. (6)

$F_{k,2^{n+1}} = F_{k,2^n} L_{k,2^n}$.

From the Eq. (11) and induction hypothesis can be formulated as

$F_{k,2^{n+1}} \equiv 0 \pmod 2$.

Lemma 2.7. If k is odd integer,
i. $F_{k,3.2^{n-1}} \equiv 0 \pmod{2^n}$ (13)
ii. $F_{k,3.2^{n-1}+1} \equiv 1 \pmod{2^n}$. (14)

Proof. i. We can give the proof by induction. For $n = 1$, $F_{k,3} = k^2 + 1$ and

$F_{k,3} \equiv 0 \pmod 2$.

Suppose $F_{k,3.2^{n-1}} \equiv 0 \pmod{2^n}$.

For n is replaced by 3.2^{n-1}, we have the Eq. (6)

$F_{k,3.2^n} = F_{k,3.2^{n-1}} L_{k,3.2^{n-1}}$.

From the Eq. (9) and induction hypothesis, $F_{k,3.2^n} \equiv 0 \pmod{2^{n+1}}$ is satisfies.

ii. We can give the proof by induction. For $n = 1$, $F_{k,4} = k^3 + 2k$ and thus
$F_{k,4} \equiv 1 \pmod 2$.

Suppose $F_{k,3.2^{n-1}+1} \equiv 1 \pmod{2^n}$.

For n is replaced by 3.2^{n-1}, we get the Eq. (3)

$F_{k,3.2^n+1} = (F_{k,3.2^{n-1}})^2 + (F_{k,3.2^{n-1}+1})^2$ (15)

From the Eq. (10) and Eq. (13),

$(F_{k,3.2^{n-1}})^2 \equiv 0 \pmod{2^{n+1}}$

is satisfies. For n is replaced by 3.2^{n-1}, we have the Eq. (4)

$\left(F_{k,3.2^{n-1}+1}\right)\left(F_{k,3.2^{n-1}-1}\right) - (F_{k,3.2^{n-1}})^2 = (-1)^{3.2^{n-1}} = 1.$

Since $F_{k,3.2^{n-1}-1} = F_{k,3.2^{n-1}+1} - k\, F_{k,3.2^{n-1}}$ and $F_{k,3.2^{n-1}+1} \equiv 1 \pmod{2^n}$, then

$F_{k,3.2^{n-1}+1} F_{k,3.2^{n-1}} \equiv 0 \pmod{2^{n+1}}$

is satisfies. Since,

$$\left(F_{k,3.2^{n-1}+1}\right)\left(F_{k,3.2^{n-1}+1} - k\, F_{k,3.2^{n-1}}\right) - \left(F_{k,3.2^{n-1}}\right)^2 = \left(F_{k,3.2^{n-1}+1}\right)^2 - k\, F_{k,3.2^{n-1}+1} F_{k,3.2^{n-1}} - (F_{k,3.2^{n-1}})^2$$

and $(F_{k,3.2^{n-1}})^2 \equiv 0 \pmod{2^{n+1}}$, then we get

$$\begin{aligned}\left(F_{k,3.2^{n-1}+1}\right)\left(F_{k,3.2^{n-1}+1} - k\, F_{k,3.2^{n-1}}\right) - \left(F_{k,3.2^{n-1}}\right)^2 &\equiv \left(F_{k,3.2^{n-1}+1}\right)^2 \pmod{2^{n+1}} \\ &\equiv 1 \pmod{2^{n+1}} .\end{aligned}$$

From the Eq. (15) we have $F_{k,3.2^n+1} \equiv \left(F_{k,3.2^{n-1}+1}\right)^2 \pmod{2^{n+1}}$ and thus we have

$F_{k,3.2^n+1} \equiv 1 \pmod{2^{n+1}}$.

Lemma 2.8. If k is an even integer,
i. $F_{k,2^n} \equiv 0 \pmod{2^n}$ (16)
ii. $F_{k,2^n+1} \equiv 1 \pmod{2^n}$. (17)

Proof. i. We can give the proof by induction. For $n = 1$, $F_{k,2} = k$ and since k is an even integer,

$F_{k,2} \equiv 0 \pmod{2}$.

Suppose $F_{k,2^n} \equiv 0 \pmod{2^n}$.

For n is replaced by 2^n, we have the Eq. (6)

$F_{k,2^{n+1}} = F_{k,2^n}\, L_{k,2^n}$.

From the Eq. (11) and induction hypothesis we get

$F_{k,2^{n+1}} \equiv 0 \pmod{2^{n+1}}$.

ii. We can give the proof by induction. For $n = 1$, $F_{k,3} = k^2 + 1$ and $F_{k,3} \equiv 1 \pmod{2}$.

Suppose $F_{k,2^n+1} \equiv 1 \pmod{2^n}$.

For n is replaced by 2^n, we have the Eq. (3)

$$F_{k,2^{n+1}+1} = (F_{k,2^n})^2 + (F_{k,2^n+1})^2 \tag{18}$$

From the Eq. (12) and the Eq. (16),

$(F_{k,2^n})^2 \equiv 0 \pmod{2^{n+1}}$

is satisfies. For n is replaced by 2^n, we have the Eq. (4)

$\left(F_{k,2^n+1}\right)\left(F_{k,2^n-1}\right) - (F_{k,2^n})^2 = (-1)^{2^n} = 1.$

From the induction hypothesis and the Eq. (16)

$F_{k,2^n+1}\, F_{k,2^n} \equiv 2^n \pmod{2^{n+1}}$

is satisfies. Since k is an even integer, we get

$k\, F_{k,2^n+1}\, F_{k,2^n} \equiv 0 \pmod{2^{n+1}}.$

Thus we have

$\left(F_{k,2^n+1}\right)\left(F_{k,2^n+1} - k\, F_{k,2^n}\right) - (F_{k,2^n})^2 = (F_{k,2^n+1})^2 - k\, F_{k,2^n+1}\, F_{k,2^n} - (F_{k,2^n})^2$

and since $(F_{k,2^n})^2 \equiv 0 \pmod{2^{n+1}}$, then we get

$\left(F_{k,2^n+1}\right)\left(F_{k,2^n+1} - k\, F_{k,2^n}\right) - (F_{k,2^n})^2 \equiv (F_{k,2^n+1})^2 \pmod{2^{n+1}} \equiv 1 \pmod{2^{n+1}}.$

From the Eq. (18) we have $F_{k,2^{n+1}+1} \equiv (F_{k,2^n+1})^2 \pmod{2^{n+1}}$ and thus we get

$F_{k,2^{n+1}+1} \equiv 1 \pmod{2^{n+1}}.$

Theorem 2.9. The period of the k- Fibonacci sequences mod 2^n is

$$\pi_k(2^n) = \begin{cases} if\ k\ odd, & 3.2^{n-1} \\ if\ k\ even, & 2^n \end{cases}$$

Proof. The proof is obtain from Lemma 2.7 and Lemma 2.8.

Lemma 2.10. If k is odd integer, then $L_{k,3.2^{n-1}} \equiv 2 \pmod{2^n}$.

Proof. We can give the proof by induction. When $n = 1$, $L_{k,3} = k^3 + 3k$ and

$L_{k,3} \equiv 0 \equiv 2 \pmod 2$.

Suppose $L_{k,3.2^{n-1}} \equiv 2 \pmod{2^n}$.

For $m = 0$ and n is replaced by 3.2^{n-1}, we have the Eq. (7)

$$\begin{aligned} L_{k,3.2^n} &= (L_{k,3.2^{n-1}})^2 + 2(-1)^{3.2^{n-1}+1} \\ &= (L_{k,3.2^{n-1}})^2 - 2. \end{aligned}$$

Using the induction hypothesis we get $(L_{k,3.2^{n-1}})^2 \equiv 4 \pmod{2^{n+1}}$. Thus we have

$L_{k,3.2^n} \equiv 2 \pmod{2^{n+1}}$.

Lemma 2.11. If k is odd integer, then $L_{k,3.2^{n-1}+1} \equiv k \pmod{2^n}$.

Proof. For $m = 1$ and n is replaced by 3.2^{n-1}, we get the Eq. (8)

$$kL_{k,3.2^{n-1}+1} + 2L_{k,3.2^{n-1}} = (k^2+4)\,F_{k,3.2^{n-1}+1}$$

From Lemma 2.10 and the Eq. (14), we have

$$L_{k,3.2^{n-1}+1} \equiv k \pmod{2^n}.$$

Theorem 2.12. If k is odd integer, then the Pisano period of the k- Lucas sequences mod 2^n is 3.2^{n-1}.

Proof. The proof is obtain from Lemma 2.10 and Lemma 2.11.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Bolat, C., Köse, H., "On the properties of k-fibonacci numbers", International Journal of Contemporary Mathematical Sciences, 5(22): 1097-1105, (2010).

[2] Falcon, Plaza, A., "On the fibonacci k-numbers", Chaos, Solutions and Fractals, , 32: 1615-1624, (2007).

[3] Falcon, S., Plaza, A., "K-fibonacci sequences modulo m", Chaos, Solutions and Fractals, 41: 497-504, (2009).

[4] Falcon, S., "On the k-lucas numbers", International Journal of Contemporary Mathematical Sciences, 21: 1039-1050, (2011).

[5] Falcon, S., Plaza, A., "The k-fibonacci sequences and the pascal 2-triangle", Chaos, Solutions and Fractals, 33: 38-49, (2007).

[6] Falcon, S., Plaza, A., "On k-fibonacci sequences and polynomials and their derivatives", Chaos, Solutions and Fractals, 39: 1005-1019, (2007).

[7] Kocer, E.G., Tunces, S., "Bivariate fibonacci and lucas like polynomials", GU J Sci, 29(1): 109-113, (2016).

[8] Kramer, J., Hoggat, V. E., "Special cases of fibonacci periodicity", The Fibonacci Quarterly, 10(5): 519-522, (1972).

[9] Niven, I., Zuckerman, H.S., Montegomery, H.L., "An ıntroduction to the theory of numbers", 5 nd ed., John Wiley Sons, Inc., (1991).

[10] Rogers, N., Campbell, C. W., "The period of the fibonacci sequence modulo j", Phd, The University of Arizona, Tucson, USA, (2007).

[11] Vajda, S., “Fibonacci and lucas numbers, and the golden section, theory and applications”, Ellis Horwood Limited, (1989).

[12] Wall, D. D., “Fibonacci series modulo m”, The American Mathematical Monthly, 67: 525-532, (1960).