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On Modal Operators over the Generalized Intuitionistic Fuzzy Set

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Abstract

In recent years, different operators (modal, topological, level, negation and other types) have been defined over intuitionistic fuzzy sets. In this note, modal types of operators over an extended generalized intuitionistic fuzzy set $GIFS_B$ are proposed. Some of the basic properties of the new operators are derived.

1. INTRODUCTION

In the real world, we frequently deal with vague, imprecise or insufficient information. Fuzzy sets (FSs) were introduced to deal with this reality. In the fuzzy sets theory, membership and non- membership degrees are complementary, i.e., the sum of both degrees of an element belonging in a fuzzy set is 1. However, there are situations where the two degrees are not complementary, mainly because of hesitation. In an intuitionistic fuzzy set (IFS), each element is assigned membership and non-membership degrees, where the sum of the two degrees does not exceed 1. Accordingly, an IFS is more applicable than a FS for representing vague information since it considers a degree of hesitation. IFSs and their generalizations can be useful in problems such as decision making problems, sales analysis, new product marketing, and financial services. The modal operators have been known to be important tools for IFSs and allow for a more detailed estimation of information.

IFSs were introduced by Atanassov [1]. Since then different types of operators have been defined over IFSs. They can be classified into several groups: modal operators, topological operators, level operators, negation operators and other types. Atanassov [2] defined several operators for the theory of IFSs. He discussed the relations between the classical negation operator and the two standard modal operators "necessity" and "possibility". Atanassov [1,3-4] defined topological operators over IFSs, and derived their basic properties. Atanassov [5] considered modal operators defined over IFSs. Cornelis et al. [6] considered extended modal operators as tools for constructing inclusion indicators over IFSs. Atanassov and Dimitrov [7] introduced different forms of negation operators. Hinde and Atanassov [8] studied some relations between intuitionistic fuzzy negations and intuitionistic fuzzy modal operators. Parvathi and Geetha [9] defined some level operators, max-min implication operators and $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$ operators on temporal IFSs. Atanassov [10] introduced two new operators that partially extend the intuitionistic fuzzy operators from modal type. Srinivasan and Palaniappan [11] defined some operators and established their properties over IFSs of root type. Sharma [12] studied the impact of the modal operator $F_{\alpha,\beta}(A)$ on

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intuitionistic fuzzy groups and proved that many properties of intuitionistic fuzzy subgroups like normality, commutativity and cyclic groups remain invariant under the modal operator. Yilmaz and Cuvalcioglu [13] introduced $\mathrm{N_B(A)}$ and $\mathrm{N_{B^*}(A)}$ operators on temporal IFSs and examined their properties. Baloui Jamkhaneh and Nadarajah [14] introduced a new generalized IFS ($\mathrm{GIFS_B}$) and some operators over $\mathrm{GIFS_B}$. Baloui Jamkhnaeh [15] and Baloui Jamkhaneh and Nadi Ghara [16] defined level operators over the $\mathrm{GIFS_B}$s. In 2017 Baloui Jamkhaneh and Garg [17] considered some new operations over the generalized intuitionistic fuzzy sets and their application to decision making process. All operations defined over IFSs were transformed to the $\mathrm{GIFS_B}$ case.

The aim of this note is to define different modal type operators over $\mathrm{GIFS_B}$ and to derive their properties. This study shows the exibility of $\mathrm{GIFS_B}$s in terms of different operators. The study is expected to encourage applicability of $\mathrm{GIFS_B}$ to the many areas of application of IFSs like sustainable energy planning, image fusion, agricultural production planning, medical diagnosis, pattern recognition, reservoir flood control operation, reliability optimization of complex systems, fault diagnosis and prediction of the best quality of two-wheelers, see Baloui Jamkhaneh and Nadarajah [14] for references and other areas of application of IFSs. The contents of this note are organized as follows: Section 2 states some definitions including the definition of $\mathrm{GIFS_B}$ given in Baloui Jamkhaneh and Nadarajah [14]; Section 3 summarizes some results on $\mathrm{GIFS_B}$ given in Baloui Jamkhaneh and Nadarajah [14]; Section 4 defines operators acting on $\mathrm{GIFS_B}$ and derives their properties; some conclusions and future work are noted in Section 5.

2. BASIC CONCEPTS

Definition 2.1. [1] Let X be a non-empty set. An IFS A in X is defined as an object of the form $\mathrm{A} = \{\langle \mathrm{x}, \mu_\mathrm{A}(\mathrm{x}), \nu_\mathrm{A}(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership and non-membership functions of A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.2. [14] Let X be a non empty set. Generalized intuitionistic fuzzy sets A in X, is defined as an object of the form $\mathrm{A} = \{\langle \mathrm{x}, \mu_\mathrm{A}(\mathrm{x}), \nu_\mathrm{A}(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}$ where the functions $\mu_\mathrm{A} : \mathrm{X} \to [0,1]$ and $\nu_\mathrm{A} : \mathrm{X} \to [0,1]$, denote the degree of membership and degree of non-membership functions of A respectively, and $0 \leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1$ for each $\mathrm{x} \in \mathrm{X}$, and given fixed δ. The collection of all generalized IFSs is denoted by $\mathrm{GIFS_B}(\delta, \mathrm{X})$.

Definition 2.3. [14] Let X be a non-empty set. Let A and B be two $\mathrm{GIFS_B}$s such that

$$A = \{\langle \mathrm{x}, \mu_\mathrm{A}(\mathrm{x}), \nu_\mathrm{A}(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}, \quad B = \{\langle \mathrm{x}, \mu_\mathrm{B}(\mathrm{x}), \nu_\mathrm{B}(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}.$$

Define the following operations on A and B:

i. $A \subset B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, $\forall x \in X$,
ii. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, $\forall x \in X$,
iii. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \epsilon X\}$,
iv. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \epsilon X\}$,
v. $A + B = \{\langle x, \mu_A(x)^\delta + \mu_B(x)^\delta - \mu_A(x)^\delta \mu_B(x)^\delta, \nu_A(x)^\delta \nu_B(x)^\delta\rangle : x \epsilon X\}$,
vi. $A.B = \{\langle x, \mu_A(x)^\delta \mu_B(x)^\delta, \nu_A(x)^\delta + \nu_B(x)^\delta - \nu_A(x)^\delta \nu_B(x)^\delta\rangle : x \epsilon X\}$,
vii. $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Definition 2.4. The degree of non-determinacy (uncertainty) of an element $x \in X$ to the $GIFS_B$ A is defined by $\pi_A(x) = (1 - \mu_A(x)^\delta - \nu_A(x)^\delta)^{\frac{1}{\delta}}$.
It can be easily shown that $\pi_A(x)^\delta + \mu_A(x)^\delta + \nu_A(x)^\delta = 1$.

3. REMARKS ON THE GENERALIZED INTUITIONISTIC FUZZY SETS

Let X is a non-empty finite set, $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$, as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Baloui Jamkhaneh and Nadarajah [14] introduced the following operators over $GIFS_B$ and investigated some their properties:

i. $\square A = \left\{\langle x, \mu_A(x), (1 - \mu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x \epsilon X\right\}$, (modal logic: the necessity measure),

ii. $\lozenge A = \left\{\langle x, (1 - \nu_A(x)^{\delta})^{\frac{1}{\delta}}, \nu_A(x)\rangle : x \epsilon X\right\}$, (modal logic: the possibility measure),

iii. $C(A) = \{\langle x, K, L\rangle : x \in X\}$, $K = \max_{y \in X} \mu_A(y)$, $L = \min_{y \in X} \nu_A(y)$, (topological: closure),

iv. $I(A) = \{\langle x, k, l\rangle : x \in X\}$, $k = \min_{y \in X} \mu_A(y)$, $l = \max_{y \in X} \nu_A(y)$, (topological: intersection),

v. $D_\alpha(A) = \left\{\langle x, (\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}, (\nu_A(x)^{\delta} + (1-\alpha)\pi_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x \in X\right\}$,

vi. $F_{\alpha,\beta}(A) = \left\{\langle x, \left(\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^{\delta} + \beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}$, $\alpha + \beta \leq 1$,

vii. $G_{\alpha,\beta}(A) = \left\{\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x)\rangle : x \in X\right\}$.

Theorem 3.1. For every $GIFS_B$ A, we have

i. $\square\square A = \square A$,
ii. $\lozenge\lozenge A = \lozenge A$,
iii. $\square \lozenge A = \lozenge A$,
iv. $\lozenge \square A = \square A$,
v. $D_\alpha(\square A) = F_{\alpha,\beta}(\square A) = \square A$,
vi. $D_\alpha(\lozenge A) = F_{\alpha,\beta}(\lozenge A) = \lozenge A$,
vii. $D_\alpha(\bar{A}) = \overline{D_{1-\alpha}(A)}$,
viii. $G_{\alpha,\beta}(A \cap B) = G_{\alpha,\beta}(A) \cap G_{\alpha,\beta}(B)$,
ix. $G_{\alpha,\beta}(A \cup B) = G_{\alpha,\beta}(A) \cup G_{\alpha,\beta}(B)$,
x. $\pi_{D_\alpha(A)}(x) = 0$,
xi. $F_{\alpha,\beta}(D_\alpha(A)) = D_\alpha(A)$.

Proof. Proof is obvious.

4. MAIN RESULTS

Here, we will introduce new operators over the $GIFS_B$, which extend some operators in the research literature related to IFSs.

Definition 4.1. Letting $\alpha, \beta \in [0,1]$ and $A \in GIFS_B$, we define the operators of as follows

i. $d_\alpha(A) = \left\{\langle x, (\nu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}, (\mu_A(x)^{\delta} + (1-\alpha)\pi_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x \in X\right\}$,

ii. $f_{\alpha,\beta}(A) = \left\{\langle x, \left(\nu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\mu_A(x)^{\delta} + \beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}$, where $\alpha + \beta \leq 1$,

iii. $g_{\alpha,\beta}(A) = \left\{\langle x, \alpha^{\frac{1}{\delta}}\nu_A(x), \beta^{\frac{1}{\delta}}\mu_A(x)\rangle : x \in X\right\}$,

iv. $J_{\alpha,\beta}(A) = \left\{\langle x, \left(\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\beta\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}$,

v. $j_{\alpha,\beta}(A) = \left\{\langle x, \left(\nu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\beta\mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}$,

vi. $H_{\alpha,\beta}(A) = \left\{\langle x, \left(\alpha\mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^{\delta} + \beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}$,

vii. $h_{\alpha,\beta}(A) = \left\{\langle x, \left(\alpha\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\mu_A(x)^{\delta} + \beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}$.

Theorem 4.1. For every $A \in GIFS_B$ and for every $\alpha, \beta \in [0,1]$, we have

i. $d_\alpha(A) \in GIFS_B$,
ii. $\alpha \leq \beta \Rightarrow d_\alpha(A) \subset d_\beta(A)$,
iii. $d_0(A) = \overline{\Diamond A}$,
iv. $d_1(A) = \overline{\Box A}$,
v. $d_\alpha(\Box A) = \overline{\Box A}$,
vi. $d_\alpha(\Diamond A) = \overline{\Diamond A}$.

Proof. Proofs (i) and (ii) are obvious.

(iii) Note that

$$d_0(A) = \left\{\langle x, (\nu_A(x)^\delta + 0\pi_A(x)^\delta)^{\frac{1}{\delta}}, (\mu_A(x)^\delta + (1-0)\pi_A(x)^\delta)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$
$$= \left\{\langle x, \nu_A(x), (\mu_A(x)^\delta + \pi_A(x)^\delta)^{\frac{1}{\delta}}\rangle : x \in X\right\}.$$

Since $\pi_A(x)^\delta = 1 - \mu_A(x)^\delta - \nu_A(x)^\delta$ we have

$$d_0(A) = \left\{\langle x, \nu_A(x), (1 - \nu_A(x)^\delta)^{\frac{1}{\delta}}, \rangle : x \in X\right\} = \overline{\Diamond A}.$$

The proof is complete.

(iv) Follows by noting that

$$d_1(A) = \left\{\langle x, (\nu_A(x)^\delta + 1\pi_A(x)^\delta)^{\frac{1}{\delta}}, (\mu_A(x)^\delta + (1-1)\pi_A(x)^\delta)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

$$= \left\{\langle x, (\nu_A(x)^\delta + \pi_A(x)^\delta)^{\frac{1}{\delta}}, \mu_A(x)\rangle : x \in X\right\},$$

$$= \left\{\langle x, (1 - \mu_A(x)^\delta)^{\frac{1}{\delta}}, \mu_A(x)\rangle : x \in X\right\} = \overline{\Box A}.$$

(v) Note that

$$d_\alpha(\Box A) = \left\{\langle x, (\nu_{\Box A}(x)^\delta + \alpha\pi_{\Box A}(x)^\delta)^{\frac{1}{\delta}}, (\mu_{\Box A}(x)^\delta + (1-\alpha)\pi_{\Box A}(x)^\delta)^{\frac{1}{\delta}}\rangle : x \in X\right\}.$$

Since $\pi_{\Box A}(x) = 0$, we have

$$d_\alpha(\Box A) = \{\langle x, \nu_{\Box A}(x), \mu_{\Box A}\rangle : x \in X\} = \overline{\Box A}.$$

The proof is complete. The proof of (vi) is similar to that of (v).

Theorem 4.2. For every $A \in GIFS_B$ and for every $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, we have

i. $f_{\alpha,\beta}(A) \in GIFS_B$,
ii. $0 \leq \gamma \leq \alpha \Rightarrow \mathrm{f}_{\gamma,\beta}(\mathrm{A}) \subset \mathrm{f}_{\alpha,\beta}(\mathrm{A})$,
iii. $0 \leq \gamma \leq \beta \Rightarrow \mathrm{f}_{\alpha,\beta}(\mathrm{A}) \subset \mathrm{f}_{\alpha,\gamma}(\mathrm{A})$,
iv. $d_\alpha(A) = \mathrm{f}_{\alpha,1-\alpha}(\mathrm{A})$,
v. $f_{0,1}(A) = \overline{\Diamond A}$,
vi. $f_{1,0}(A) = \overline{\Box A}$,
vii. $\mathrm{f}_{\alpha,\beta}(\mathrm{A}) = \mathrm{F}_{\alpha,\beta}(\bar{A})$,
viii. $\overline{f_{\alpha,\beta}(\overline{A})} = f_{\beta,\alpha}(A)$,
ix. $f_{0,0}(A) = \overline{A}$,

x. $f_{\alpha,\beta}(\Box A) = \overline{\Box A}$,

xi. $f_{\alpha,\beta}(\lozenge A) = \overline{\lozenge A}$.

Proof. (i) Follows by noting that

$$\mu_{f_{\alpha,\beta}(A)}(x)^{\delta} + \nu_{f_{\alpha,\beta}(A)}(x)^{\delta} = [(\nu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}]^{\delta} + [(\mu_A(x)^{\delta} + \beta\pi_A(x)^{\delta})^{\frac{1}{\delta}}]^{\delta},$$

$$= \nu_A(x)^{\delta} + \mu_A(x)^{\delta} + \pi_A(x)^{\delta}(\alpha+\beta),$$

$$\le \mu_A(x)^{\delta} + \nu_A(x)^{\delta} + \pi_A(x)^{\delta} = 1.$$

Proofs of (ii) and (iii) are obvious.

(iv) Follows by noting that

$$f_{\alpha,1-\alpha}(A) = \left\{\langle x, (\nu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}, (\mu_A(x)^{\delta} + (1-\alpha)\pi_A(x)^{\delta})^{\frac{1}{\delta}} \rangle : x \in X\right\} = d_{\alpha}(A).$$

(v) By (iv) we have $f_{0,1}(A) = d_0(A)$ and $f_{1,0}(A) = d_1(A)$. It follows by Theorem4.1-(iii) that $f_{0,1}(A) = \overline{\lozenge A}$.

(vi) It follows by Theorem4.1-(iv) that $f_{1,0}(A) = \overline{\Box A}$.

(viii) Since

$$f_{\beta,\alpha}(A) = \left\{\langle x, (\nu_A(x)^{\delta} + \beta\pi_A(x)^{\delta})^{\frac{1}{\delta}}, (\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}} \rangle : x \in X\right\},$$

and

$$f_{\alpha,\beta}(\bar{A}) = \left\{\langle x, (\mu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}, (\nu_A(x)^{\delta} + \beta\pi_A(x)^{\delta})^{\frac{1}{\delta}} \rangle : x \in X\right\},$$

we have

$$\overline{f_{\alpha,\beta}(\bar{A})} = \left\{\langle x, (\nu_A(x)^{\delta} + \alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}, (\mu_A(x)^{\delta} + \beta\pi_A(x)^{\delta})^{\frac{1}{\delta}} \rangle : x \in X\right\},$$

and $\overline{f_{\alpha,\beta}(\bar{A})} = f_{\beta,\alpha}(A)$.

Since $\pi_{\Box A}(x) = 0$ and $\pi_{\lozenge A}(x) = 0$, the proof (vii), (x) and (xi) are obvious.

Theorem 4.3. For every $A \in GIFS_B$, and real numbers $\alpha, \beta, \gamma \in [0,1]$

i. $g_{\alpha,\beta}(A) \in GIFS_B$,

ii. $\alpha \le \gamma \Rightarrow g_{\alpha,\beta}(A) \subset g_{\gamma,\beta}(A)$,

iii. $\beta \le \gamma \Rightarrow g_{\alpha,\beta}(A) \supset g_{\alpha,\gamma}(A)$,

iv. $\tau \in [0,1] \Rightarrow g_{\alpha,\beta}\left(g_{\gamma,\tau}(A)\right) = \overline{g_{\gamma,\tau}(g_{\alpha,\beta}(\bar{A}))}$,

v. $g_{\alpha,\beta}(C(A)) = I(g_{\alpha,\beta}(A)$,

vi. $g_{\alpha,\beta}(I(A)) = C(g_{\alpha,\beta}(A))$,

vii. $\overline{g_{\alpha,\beta}(\bar{A})} = g_{\beta,\alpha}(A)$,

viii. $g_{1,1}(A) = \bar{A}$,

ix. $g_{\alpha,\beta}(A \cap B) = g_{\alpha,\beta}(A) \cap g_{\alpha,\beta}(B)$,

x. $g_{\alpha,\beta}(A \cup B) = g_{\alpha,\beta}(A) \cup g_{\alpha,\beta}(B)$.

Proof. (i) Since

$$g_{\alpha,\beta}(A) = \left\{ \langle x, \alpha^{\frac{1}{\delta}}\nu_A(x), \beta^{\frac{1}{\delta}}\mu_A(x) \rangle : x \in X \right\},$$

$$\begin{aligned}\mu_{g_{\alpha,\beta}(A)}(x)^\delta + \nu_{g_{\alpha,\beta}(A)}(x)^\delta &= (\alpha^{\frac{1}{\delta}}\nu_A(x))^\delta + \left(\beta^{\frac{1}{\delta}}\mu_A(x)\right)^\delta,\\ &= \alpha\nu_A(x)^\delta + \beta\mu_A(x)^\delta,\\ &\leq \nu_A(x)^\delta + \mu_A(x)^\delta \leq 1.\end{aligned}$$

We have $g_{\alpha,\beta}(A) \in GIFS_B$.

(ii) Note that

$$g_{\alpha,\beta}(A) = \left\{ \langle x, \alpha^{\frac{1}{\delta}}\nu_A(x), \beta^{\frac{1}{\delta}}\mu_A(x) \rangle : x \in X \right\},$$

and

$$g_{\gamma,\beta}(A) = \left\{ \langle x, \gamma^{\frac{1}{\delta}}\nu_A(x), \beta^{\frac{1}{\delta}}\mu_A(x) \rangle : x \in X \right\}.$$

Since $\alpha \leq \gamma$ we have $\alpha^{\frac{1}{\delta}} \leq \gamma^{\frac{1}{\delta}}$, $\alpha^{\frac{1}{\delta}}\nu_A(x) \leq \gamma^{\frac{1}{\delta}}\nu_A(x)$ and so $g_{\alpha,\beta}(A) \subset g_{\gamma,\beta}(A)$.

The proof of (iii) is similar to that of (ii).

(iv) Since

$$g_{\gamma,\tau}(A) = \left\{ \langle x, \gamma^{\frac{1}{\delta}}\nu_A(x), \tau^{\frac{1}{\delta}}\mu_A(x) \rangle : x \in X \right\},$$

$$\begin{aligned}g_{\alpha,\beta}\left(g_{\gamma,\tau}(A)\right) &= \left\{ \langle x, \alpha^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\gamma^{\frac{1}{\delta}}\nu_A(x) \rangle : x \in X \right\},\\ &= \left\{ \langle x, (\alpha\tau)^{\frac{1}{\delta}}\mu_A(x), (\beta\gamma)^{\frac{1}{\delta}}\nu_A(x) \rangle : x \in X \right\},\\ &= \overline{g_{\beta\gamma,\alpha\tau}(A)},\end{aligned}$$

and

$$g_{\alpha,\beta}(\bar{A}) = \left\{ \langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x) \rangle : x \in X \right\},$$

$$\begin{aligned}g_{\gamma,\tau}\left(g_{\alpha,\beta}(\bar{A})\right) &= \left\{ \langle x, \gamma^{\frac{1}{\delta}}\beta^{\frac{1}{\delta}}\nu_A(x), \tau^{\frac{1}{\delta}}\alpha^{\frac{1}{\delta}}\mu_A(x) \rangle : x \in X \right\},\\ &= \left\{ \langle x, (\gamma\beta)^{\frac{1}{\delta}}\nu_A(x), (\alpha\tau)^{\frac{1}{\delta}}\mu_A(x) \rangle : x \in X \right\},\\ &= g_{\gamma\beta,\alpha\tau}(A),\end{aligned}$$

we have $g_{\alpha,\beta}\left(g_{\gamma,\tau}(A)\right) = \overline{g_{\gamma,\tau}\left(g_{\alpha,\beta}(\bar{A})\right)}$.

(v) Follows by noting that

$$C(A) = \left\{\langle x, max_{y\in X}\mu_A(y), min_{y\in X}\nu_A(y)\rangle : x \in X\right\},$$

and

$$g_{\alpha,\beta}\big(C(A)\big) = \left\{\langle x, \alpha^{\frac{1}{\delta}} min_{y\in X}\nu_A(y), \beta^{\frac{1}{\delta}} max_{y\in X}\mu_A(y)\rangle : x \in X \right\},$$

$$= \left\{\langle x, min_{y\in X}\alpha^{\frac{1}{\delta}}\nu_A(y), max_{y\in X}\beta^{\frac{1}{\delta}}\mu_A(y)\rangle : x \in X \right\},$$

$$= I(g_{\alpha,\beta}(A)).$$

(vi) Follows by noting that

$$I(A) = \left\{\langle x, \underset{y\in X}{min}\mu_A(y), \underset{y\in X}{max}\nu_A(y)\rangle : x \in X \right\},$$

and

$$g_{\alpha,\beta}\big(I(A)\big) = \left\{\langle x, \alpha^{\frac{1}{\delta}} \underset{y\in X}{max}\nu_A(y), \beta^{\frac{1}{\delta}} \underset{y\in X}{min}\mu_A(y)\rangle : x \in X \right\},$$

$$= \left\{\langle x, \underset{y\in X}{max}\alpha^{\frac{1}{\delta}}\nu_A(y), \underset{y\in X}{min}\beta^{\frac{1}{\delta}}\mu_A(y)\rangle : x \in X \right\},$$

$$= C(g_{\alpha,\beta}(A)) \quad , \quad \alpha,\beta \in [0,1] .$$

(vii) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be a $GIFS_B$. Then

$$\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\},$$

$$g_{\beta,\alpha}(A) = \left\{\langle x, \beta^{\frac{1}{\delta}}\nu_A(x), \alpha^{\frac{1}{\delta}}\mu_A(y)\rangle : x \in X\right\},$$

$$g_{\alpha,\beta}(\bar{A}) = \left\{\langle x, \alpha^{\frac{1}{\delta}}\mu_A(x), \beta^{\frac{1}{\delta}}\nu_A(x)\rangle : x \in X\right\},$$

and

$$\overline{g_{\alpha,\beta}(\bar{A})} = \left\{\langle x, \beta^{\frac{1}{\delta}}\nu_A(x), \alpha^{\frac{1}{\delta}}\mu_A(x)\rangle : x \in X\right\},$$

so $\overline{g_{\alpha,\beta}(\bar{A})} = g_{\beta,\alpha}(A)$. Proof of (viii), (x) and (xi) are obvious.

Theorem 4.4. For every $A \in GIFS_B$, and $\alpha, \beta, \gamma \in [0,1]$

i. $J_{\alpha,\beta}(A) \in GIFS_B$,
ii. $\alpha \le \gamma \Rightarrow J_{\alpha,\beta}(A) \subset J_{\gamma,\beta}(A)$,
iii. $\beta \le \gamma \Rightarrow J_{\alpha,\beta}(A) \supset J_{\alpha,\gamma}(A)$,
iv. $\lozenge A = J_{1,1}(A)$,
v. $A = J_{0,1}(A)$.

Proof. (i) Since

$$J_{\alpha,\beta}(A) = \left\{\langle x, \left(\mu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\beta\nu_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}.$$

and

$$\mu_{J_{\alpha,\beta}(A)}(x)^{\delta}+\nu_{J_{\alpha,\beta}(A)}(x)^{\delta}=((\mu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}})^{\delta}+\left(\beta^{\frac{1}{\delta}}\nu_A(x)\right)^{\delta},$$

$$=(\mu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})+\beta\nu_A(x)^{\delta},$$

$$\leq \mu_A(x)^{\delta}+\pi_A(x)^{\delta}+\nu_A(x)^{\delta}=1,$$

we have $J_{\alpha,\beta}(A)\in GIFS_B$.

(ii) Note that

$$J_{\alpha,\beta}(A)=\left\{\langle x,(\mu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}},(\beta\nu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x\in X\right\}.$$

and

$$J_{\gamma,\beta}(A)=\left\{\langle x,(\mu_A(x)^{\delta}+\gamma\pi_A(x)^{\delta})^{\frac{1}{\delta}},(\beta\nu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x\in X\right\}.$$

Since $\alpha\leq\gamma$ we have $(\mu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}}\leq(\mu_A(x)^{\delta}+\gamma\pi_A(x)^{\delta})^{\frac{1}{\delta}}$ and so $J_{\alpha,\beta}(A)\subset J_{\gamma,\beta}(A)$.

The proof of (iii) is similar to that (ii). Proofs of (iv) and (v) are obvious.

Theorem 4.5. For every $A\in GIFS_B$, and $\alpha,\beta,\gamma\in[0,1]$

i. $j_{\alpha,\beta}(A)\in GIFS_B$,
ii. $\alpha\leq\gamma\Rightarrow j_{\alpha,\beta}(A)\subset j_{\gamma,\beta}(A)$,
iii. $\beta\leq\gamma\Rightarrow j_{\alpha,\beta}(A)\supset j_{\alpha,\gamma}(A)$,
iv. $j_{\alpha,\beta}(\overline{A})=J_{\alpha,\beta}(A)$,
v. $\overline{\Box A}=j_{1,1}(A)$,
vi. $\bar{A}=j_{0,1}(A)$.

Proof. (i) Since

$$j_{\alpha,\beta}(A)=\left\{\langle x,(\nu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}},(\beta\mu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x\in X\right\}.$$

and

$$\mu_{j_{\alpha,\beta}(A)}(x)^{\delta}+\nu_{j_{\alpha,\beta}(A)}(x)^{\delta}=((\nu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}})^{\delta}+\left(\beta^{\frac{1}{\delta}}\mu_A(x)\right)^{\delta},$$

$$=(\nu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})+\beta\mu_A(x)^{\delta},$$

$$\leq \mu_A(x)^{\delta}+\pi_A(x)^{\delta}+\nu_A(x)^{\delta}=1.$$

Finally, it can be concluded that $j_{\alpha,\beta}(A)\in GIFS_B$.

(ii) Note that

$$j_{\alpha,\beta}(A)=\left\{\langle x,(\nu_A(x)^{\delta}+\alpha\pi_A(x)^{\delta})^{\frac{1}{\delta}},(\beta\mu_A(x)^{\delta})^{\frac{1}{\delta}}\rangle : x\in X\right\},$$

and

$$\mathrm{j}_{\gamma,\beta}(\mathrm{A}) = \left\{\langle x, \left(\nu_A(x)^\delta + \gamma\pi_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\beta\mu_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}.$$

Since $\alpha \le \gamma$ we have $\left(\nu_A(x)^\delta + \alpha\pi_A(x)^\delta\right)^{\frac{1}{\delta}} \le \left(\nu_A(x)^\delta + \gamma\pi_A(x)^\delta\right)^{\frac{1}{\delta}}$ and so $\mathrm{j}_{\alpha,\beta}(\mathrm{A}) \subset \mathrm{j}_{\gamma,\beta}(\mathrm{A})$. The proof of (iii) is similar to that (ii). Proofs of (iv), (v) and (vi) are obvious.

Theorem 4.6. For every $\mathrm{A} \in \mathrm{GIFS_B}$, and $\alpha, \beta, \gamma \in [0,1]$

i. $\mathrm{H}_{\alpha,\beta}(\mathrm{A}) \in \mathrm{GIFS_B}$,
ii. $\alpha \le \gamma \Rightarrow \mathrm{H}_{\alpha,\beta}(\mathrm{A}) \subset \mathrm{H}_{\gamma,\beta}(\mathrm{A})$,
iii. $\beta \le \gamma \Rightarrow \mathrm{H}_{\alpha,\beta}(\mathrm{A}) \supset \mathrm{H}_{\alpha,\gamma}(\mathrm{A})$,
iv. $\mathrm{H}_{1,0}(\mathrm{A}) = \mathrm{A}$,
v. $\mathrm{H}_{1,1}(\mathrm{A}) = A$.

Proof. (i) Since

$$\mathrm{H}_{\alpha,\beta}(\mathrm{A}) = \left\{\langle x, \left(\alpha\mu_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

and

$$\mu_{\mathrm{H}_{\alpha,\beta}(A)}(x)^\delta + \nu_{\mathrm{H}_{\alpha,\beta}(A)}(x)^\delta = \left(\left(\alpha\mu_A(x)^\delta\right)^{\frac{1}{\delta}}\right)^\delta + \left(\left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\right)^\delta,$$

$$= \alpha\mu_A(x)^\delta + \nu_A(x)^\delta + \beta\pi_A(x)^\delta,$$

$$\le \mu_A(x)^\delta + \pi_A(x)^\delta + \nu_A(x)^\delta = 1,$$

we have $\mathrm{H}_{\alpha,\beta}(\mathrm{A}) \in \mathrm{GIFS_B}$.

(ii) Note that

$$\mathrm{H}_{\alpha,\beta}(\mathrm{A}) = \left\{\langle x, \left(\alpha\mu_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

and

$$\mathrm{H}_{\gamma,\beta}(\mathrm{A}) = \left\{\langle x, \left(\gamma\mu_A(x)^\delta\right)^{\frac{1}{\delta}}, \left(\nu_A(x)^\delta + \beta\pi_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}.$$

Since $\alpha \le \gamma$ we have $\left(\alpha\mu_A(x)^\delta\right)^{\frac{1}{\delta}} \le \left(\gamma\mu_A(x)^\delta\right)^{\frac{1}{\delta}}$ and so $\mathrm{H}_{\alpha,\beta}(\mathrm{A}) \subset \mathrm{H}_{\gamma,\beta}(\mathrm{A})$. The proof of (iii) is similar to that (ii). Proofs of (iv) and (v) are obvious.

Theorem 4.7. For every $\mathrm{A} \in \mathrm{GIFS_B}$, and $\alpha, \beta, \gamma \in [0,1]$

i. $\mathrm{h}_{\alpha,\beta}(\mathrm{A}) \in \mathrm{GIFS_B}$,
ii. $\alpha \le \gamma \Rightarrow \mathrm{h}_{\alpha,\beta}(\mathrm{A}) \subset \mathrm{h}_{\gamma,\beta}(\mathrm{A})$,
iii. $\beta \le \gamma \Rightarrow \mathrm{h}_{\alpha,\beta}(\mathrm{A}) \supset \mathrm{h}_{\alpha,\gamma}(\mathrm{A})$,
iv. $\mathrm{h}_{\alpha,\beta}(\overline{\mathrm{A}}) = \mathrm{H}_{\alpha,\beta}(\mathrm{A})$,
v. $\mathrm{h}_{1,0}(\mathrm{A}) = \overline{A}$,
vi. $\mathrm{h}_{1,1}(\mathrm{A}) = \overline{\Diamond A}$.

Proof. (i) Since

$$\mu_{\mathrm{h}_{\alpha,\beta}(A)}(x)^{\delta}+\nu_{\mathrm{h}_{\alpha,\beta}(A)}(x)^{\delta}=\left(\alpha\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}})^{\delta}+\left(\left(\mu_A(x)^{\delta}+\beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\right)^{\delta},$$

$$=\alpha\nu_A(x)^{\delta}+\mu_A(x)^{\delta}+\beta\pi_A(x)^{\delta},$$

$$\leq \nu_A(x)^{\delta}+\mu_A(x)^{\delta}+\pi_A(x)^{\delta}=1,$$

we have $\mathrm{h}_{\alpha,\beta}(\mathrm{A})\in \mathrm{GIFS_B}$.

(ii) Note that

$$\mathrm{h}_{\alpha,\beta}(\mathrm{A})=\left\{\langle x,\left(\alpha\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}},\left(\mu_A(x)^{\delta}+\beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}.$$

and

$$\mathrm{h}_{\gamma,\beta}(\mathrm{A})=\left\{\langle x,\left(\gamma\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}},\left(\mu_A(x)^{\delta}+\beta\pi_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}.$$

Since $\alpha\leq\gamma$ we have $\left(\alpha\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\leq\left(\gamma\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}$ and so $\mathrm{h}_{\alpha,\beta}(\mathrm{A})\subset \mathrm{h}_{\gamma,\beta}(\mathrm{A})$.

The proof of (iii) is similar to that (ii). The proofs of (iv), (v) and (vi) are obvious.

Definition 4.2. Letting $\alpha,\beta\in[0,1]$ and $\mathrm{A}\in\mathrm{GIFS_B}$, we define the some operators of as follows

i. $\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})=\left\{\langle x,\left(\mu_A(x)^{\delta}+\alpha(1-\mu_A(x)^{\delta}-\beta\nu_A(x)^{\delta})\right)^{\frac{1}{\delta}},\left(\beta\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}$,

ii. $\mathrm{j}^{*}_{\alpha,\beta}(\mathrm{A})=\left\{\langle x,\left(\nu_A(x)^{\delta}+\alpha(1-\beta\mu_A(x)^{\delta}-\nu_A(x)^{\delta})\right)^{\frac{1}{\delta}},\left(\beta\mu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}$,

iii. $\mathrm{H}^{*}_{\alpha,\beta}(\mathrm{A})=\left\{\langle x,\left(\alpha\mu_A(x)^{\delta}\right)^{\frac{1}{\delta}},\left(\nu_A(x)^{\delta}+\beta(1-\alpha\mu_A(x)^{\delta}-\nu_A(x)^{\delta})\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}$,

iv. $\mathrm{h}^{*}_{\alpha,\beta}(\mathrm{A})=\left\{\langle x,\left(\alpha\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}},\left(\mu_A(x)^{\delta}+\beta(1-\mu_A(x)^{\delta}-\alpha\nu_A(x)^{\delta})\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}$.

Theorem 4.8. For every $\mathrm{A}\in\mathrm{GIFS_B}$, and $\alpha,\beta,\gamma\in[0,1]$

i. $\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})\in\mathrm{GIFS_B}$,

ii. $\alpha\leq\gamma\Rightarrow\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})\subset\mathrm{J}_{\gamma,\beta}(\mathrm{A})$,

iii. $\beta\leq\gamma\Rightarrow\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})\supset\mathrm{J}^{*}_{\alpha,\gamma}(\mathrm{A})$,

iv. $\mathrm{J}^{*}_{1,1}(A)=\Diamond\ A$,

v. $\mathrm{J}^{*}_{0,1}(A)=A$.

Proof. (i) Since

$$\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})=\left\{\langle x,\left(\mu_A(x)^{\delta}+\alpha(1-\mu_A(x)^{\delta}-\beta\nu_A(x)^{\delta})\right)^{\frac{1}{\delta}},\left(\beta\nu_A(x)^{\delta}\right)^{\frac{1}{\delta}}\rangle : x\in X\right\}.$$

and

$$\mu_{\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})}(x)^{\delta}+\nu_{\mathrm{J}^{*}_{\alpha,\beta}(\mathrm{A})}(x)^{\delta}=((\mu_A(x)^{\delta}+\alpha(1-\mu_A(x)^{\delta}-\beta\nu_A(x)^{\delta}))^{\frac{1}{\delta}})^{\delta}+\left(\beta^{\frac{1}{\delta}}\nu_A(x)\right)^{\delta},$$

$$=\left(\mu_A(x)^{\delta}+\alpha(1-\mu_A(x)^{\delta}-\beta\nu_A(x)^{\delta}\right)+\beta\nu_A(x)^{\delta},$$

$$=\left((1-\alpha)\mu_A(x)^{\delta}+\beta(1-\alpha)(\nu_A(x)^{\delta}\right)+\alpha,$$

$$\leq \left((1-\alpha)\mu_A(x)^\delta + (1-\alpha)(\nu_A(x)^\delta\,)\right) + \alpha \leq 1,$$

we have $\mathrm{J}^*_{\alpha,\beta}\,(\mathrm{A}) \in \mathrm{GIFS_B}$.

(ii) Note that

$$\mathrm{J}^*_{\alpha,\beta}\,(\mathrm{A}) = \left\{\langle x, \left(\mu_A(x)^\delta + \alpha(1-\mu_A(x)^\delta - \beta\nu_A(x)^\delta)\,\right)^{\frac{1}{\delta}}, \left(\beta\nu_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

and

$$\mathrm{J}^*_{\gamma,\beta}\,(\mathrm{A}) = \left\{\langle x, \left(\mu_A(x)^\delta + \gamma(1-\mu_A(x)^\delta - \beta\nu_A(x)^\delta)\,\right)^{\frac{1}{\delta}}, \left(\beta\nu_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\}.$$

Since $\alpha \leq \gamma$ we have

$$\left(\mu_A(x)^\delta + \alpha(1-\mu_A(x)^\delta - \beta\nu_A(x)^\delta)\,\right)^{\frac{1}{\delta}} \leq \left(\mu_A(x)^\delta + \gamma\left(1-\mu_A(x)^\delta - \beta\nu_A(x)^\delta\right)\right)^{\frac{1}{\delta}},$$

and so $\mathrm{J}^*_{\alpha,\beta}(\mathrm{A}) \subset \mathrm{J}_{\gamma,\beta}(\mathrm{A})$.

The proof of (iii) is similar to that (ii). The proofs of (iv) and (v) are obvious.

Theorem 4.9. For every $\mathrm{A} \in \mathrm{GIFS_B}$, and $\alpha, \beta, \gamma \in [0,1]$

i. $\mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A}) \in \mathrm{GIFS_B}$,
ii. $\alpha \leq \gamma \Rightarrow \mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A}) \subset \mathrm{j}^*_{\gamma,\beta}(\mathrm{A})$,
iii. $\beta \leq \gamma \Rightarrow \mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A}) \supset \mathrm{j}^*_{\alpha,\gamma}(\mathrm{A})$,
iv. $\mathrm{j}^*_{\alpha,\beta}\,(\overline{\mathrm{A}}) = \mathrm{J}^*_{\alpha,\beta}\,(\mathrm{A})$,
v. $\overline{\Box A} = \mathrm{j}^*_{1,1}\,(A)$,
vi. $\bar{A} = \mathrm{j}^*_{0,1}\,(A)$.

Proof. (i) Since

$$\mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A}) = \left\{\langle x, \left(\nu_A(x)^\delta + \alpha(1-\beta\mu_A(x)^\delta - \nu_A(x)^\delta)\,\right)^{\frac{1}{\delta}}, \left(\beta\mu_A(x)^\delta\right)^{\frac{1}{\delta}}\rangle : x \in X\right\},$$

and

$$\mu_{\mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A})}(x)^\delta + \nu_{\mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A})}(x)^\delta = ((\nu_A(x)^\delta + \alpha(1-\beta\mu_A(x)^\delta - \nu_A(x)^\delta)\,)^{\frac{1}{\delta}})^\delta + \left(\beta^{\frac{1}{\delta}}\,\mu_A(x)\right)^\delta,$$

$$= \left((1-\alpha)\nu_A(x)^\delta + (1-\alpha)\beta\mu_A(x)^\delta\right) + \alpha,$$

$$\leq (1-\alpha)\nu_A(x)^\delta + (1-\alpha)\mu_A(x)^\delta + \alpha \leq 1,$$

we have $\mathrm{j}^*_{\alpha,\beta}\,(\mathrm{A}) \in \mathrm{GIFS_B}$. The rest of the proof is similar to that of Theorem4.8.

Theorem 4.10. For every $\mathrm{A} \in \mathrm{GIFS_B}$, and $\alpha, \beta, \gamma \in [0,1]$

i. $\mathrm{H}^*_{\alpha,\beta}\,(\mathrm{A}) \in \mathrm{GIFS_B}$,
ii. $\alpha \leq \gamma \Rightarrow \mathrm{H}^*_{\alpha,\beta}\,(\mathrm{A}) \subset \mathrm{H}^*_{\gamma,\beta}\,(\mathrm{A})$,
iii. $\beta \leq \gamma \Rightarrow \mathrm{H}^*_{\alpha,\beta}\,(\mathrm{A}) \supset \mathrm{H}^*_{\alpha,\gamma}\,(\mathrm{A})$,
iv. $\mathrm{H}^*_{1,0}\,(\mathrm{A}) = \mathrm{A}$,

v. $H^{*}_{1,1}$ (A) $= \Box A$.

The proof of Theorem 4.10 is similar to that of Theorem 4.8.

Theorem 4.11. For every $A \in GIFS_B$, and for every three real numbers $\alpha, \beta, \gamma \in [0,1]$

i. $h^{*}_{\alpha,\beta}$ (A) $\in GIFS_B$,

ii. $\alpha \le \gamma \Rightarrow h^{*}_{\alpha,\beta}$ (A) $\subset h^{*}_{\gamma,\beta}$ (A),

iii. $\beta \le \gamma \Rightarrow h^{*}_{\alpha,\beta}$ (A) $\supset h^{*}_{\alpha,\gamma}$ (A),

iv. $h^{*}_{\alpha,\beta}$ $(\bar{A}) = H^{*}_{\alpha,\beta}$ (A),

v. $h^{*}_{1,0}$ (A) $= \bar{A}$,

vi. $h^{*}_{1,1}$ (A) $= \overline{\lozenge A}$.

The proof of Theorem 4.11 is similar to that of Theorem 4.8.

5. CONCLUTIONS

We have introduced modal types of operators over $GIFS_B$ and proved their relationships. An open problem is: definition of level operator, negation operator and other operators over $GIFS_B$ and the study of their properties.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

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