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Dynamics of a Stochastic Predator-Prey Coupled System with Modified Leslie-Gower and Holling Type II Schemes

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Abstract

We study a stochastic predator-prey system with modified Leslie- Gower and Holling type II functional response among n patches. The existence and uniqueness as well as boundedness of solution are obtained. Furthermore, we obtain sufficient conditions for stochastic permanence, and by the Lyapunov functional technique we obtain sufficient conditions for the existence of the stationary distribution. Finally, we illustrate our conclusions through numerical simulations.

1. INTRODUCTION

We consider an autonomous stochastic predator-prey model with modified Leslie-Gower and Holling type II functional response in which preys disperse among n patches. As, the effect of dispersion on the species survival plays an important role in the study of dynamic biology and ecology, the analysis of mathematical models of populations dispersing among patches in a heterogeneous environment has been the subject of several recent papers [10,11].

In this paper, we are interested in a predator-prey model with modified Leslie-Gower and Holling type II functional response studied in the deterministic case by M.A. Aziz-Alaoui and M. Daher-Okiye [1] and with stochastic perturbation by [3,8]. In our case, we introduce the coupling as well as the stochastic perturbation to take into account the effect of randomly fluctuating and stochastically perturbed intrinsic growth rate. The key method used in this paper is the analysis of Lyapunov functions [9]. The system that we consider is

$$\begin{cases} dx_i = (x_i\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i}\right) + \sum_j^n c_{ij}(x_j - x_i))dt + \sigma_{1i} x_i dW_{1i}(t) \\ dy_i = b_i y_i \left(1 - \frac{y_i}{k_{2i}+x_i}\right) + \sigma_{2i} y_i dW_{2i}(t), \quad \forall i = 1,2,\dots,n. \end{cases} \tag{1}$$

Here, x_i, y_i denote respectively the densities of preys and predators on the patch i, and the parameters a_i, k_{1i}, k_{2i} and b_i are positive constants as in [1]. The constants c_{ij} correspond to the dispersal rate from patch j to i, the processes $W_i = (W_{1i}, W_{2i})$, $1 \le i \le n$, are independent standard Brownian motions defined in the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathrm{P})$, and σ_{1i} and σ_{2i} are positive constants. For this system, we begin to show the existence and uniqueness of the global positive solution with any initial

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positive value in section 2. In section 3, we prove that when the noise is small enough the population system is stochastically permanent. In section 4, by the Lyapunov functional technique we obtain sufficient conditions for the existence of the stationary distribution of system (1). Finally, in section 5, we make numerical simulations to confirm the effect of white noise and the diffusion coefficient on the species.

2. EXISTENCE AND UNIQUENESS OF THE BOUNDED SOLUTION

Throughout this paper, we denote by $\mathbb{R}_+^{2n}$ the positive cone in $\mathbb{R}^{2n}$, that is

$\mathbb{R}_+^{2n} = \{(x,y) \in \mathbb{R}_+^{2n}: (x_i,y_i) > (0,0)$ for all $1 \leq i \leq n\}$, and for $X = (x,y) \in \mathbb{R}^{2n}$, its norm is denoted by $|X| = \sum_{i=1}^{n}(x_i^2, y_i^2)^{\frac{1}{2}}$.

For simplicity, we define $\check{\sigma} = \max_{1\leq i,j\leq n} \sigma_{ij}$, and for any constant sequence $(c_{ij})_{1\leq i,j\leq n}$, we get $\check{c} = \max_{1\leq i,j\leq n} c_{ij}$.

Theorem 2.1 *For any initial condition* $(x_0,y_0) = \big((x_{01},y_{01}),\dots,(x_{0n},y_{0n})\big) \in \mathbb{R}_+^{2n}$, *the system (1) admits a unique solution* $\big(x_i(t),y_i(t)\big) = \Big(\big(x_1(t),y_1(t)\big),\dots,\big(x_n(t),y_n(t)\big)\Big)$, *defined for all* $t \geq 0$ *a.s, and this solution remains in* $\mathbb{R}_+^{2n}$.

Proof. Let τ_e be the explosion time of the solution of (1). We have to prove that $\big(x_i(t),y_i(t)\big) \in \mathbb{R}_+^{2n}$ for every $t \in [0,\tau_e[$ and that $\tau_e = \infty$ a.s. The proof of the first assertion is adapted from [7]. Indeed, the coefficients in system (1) are locally Lipschitz, so there exists a unique local solution for all $t \in [0,\tau_e[$ and for all $\big(x_i(0),y_i(0)\big) \in \mathbb{R}_+^{2n}$. To show that this solution is global, it suffices to show that $\tau_e = \infty$. For that, let $k_0 > 0$ be large enough, such that $(x_{0i},y_{0i}) \in \left[\frac{1}{k_0},k_0\right] \times \left[\frac{1}{k_0},k_0\right]$. For each integer $k \geq k_0$ we define th stopping time

$\tau_k = \inf\left\{t \in [0,\tau_\varepsilon): x \notin \left(\frac{1}{k},k\right)\ ou\ yx \notin \left(\frac{1}{k},k\right) \text{ for some } i = 1,\dots,n\right\}$.

The sequence (τ_k) is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty}\tau_k$, whence $\tau_\infty \leq \tau_e$, (in fact, as $\big(x(t),y(t)\big) > 0$ a.s., we have $\tau_\infty = \tau_e$). It suffices to prove that $\tau_\infty = \infty$ a.s.. Assume that this statement is false, then there exist $T > 0$ and $\varepsilon \in]0,1[$ such that $P(\{\tau_\infty \leq T\}) > \varepsilon$. Since (τ_k) is increasing we have

$P(\{\tau_\infty \leq T\}) > \varepsilon$.

Now, consider the positive definite function $V\colon \mathbb{R}_+^{2n} \to \mathbb{R}_+^{2n}$ given by

$V(x,y) = \sum_{i=1}^{n}(x_i + 1 - log x_i + y_i + 1 - log y_i)$.

Applying Itô's formula, we get

$$dV(x,y) = \sum_{i=1}^{n}\left((x_i-1)\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i}\right) + \frac{\sigma_{1i}^2}{2} + \sum_{j=1}^{n} c_{ij}\left(x_j - x_i + 1 - \frac{x_j}{x_i}\right)\right.$$
$$\left. + b_i(y_i-1)\left(1 - \frac{y_i}{k_{2i}+x_i}\right) + \frac{\sigma_{2i}^2}{2}\right)dt + \sum_{i=1}^{n}\sigma_{1i}(x_i-1)dW_{1i} + \sum_{i=1}^{n}\sigma_{2i}(y_i-1)dW_{2i}.$$

The positivity of $x_i(t)$ and $y_i(t)$ implies

$$dV(x,y) \le \sum_{i=1}^{n}\left(2x_i + \left(a_i + b_i + \frac{1}{k_{2i}}\right)y_i + \frac{\sigma_{1i}^2+\sigma_{2i}^2}{2} + \sum_{j=1}^{n} c_{ij}x_j\right)dt + \sum_{i=1}^{n}\sigma_{1i}(x_i-1)dW_{1i} + \sum_{i=1}^{n}\sigma_{2i}(y_i-1)dW_{2i}.$$

Let $C_{1i} = a_i + b_i + \frac{1}{k_{2i}}$, $C_{2i} = \frac{\sigma_{1i}^2+\sigma_{2i}^2}{2}$, $i = 1 \dots, n$. Using [2, lemma 4,1], we can write

$$2x_i + C_{1i}y_i \le 4(x_i + 1 - \log x_i) + 2C_{1i}\,(y_i + 1 - \log y_i) \le C \le C_{3i}V_i(x_i,y_i),$$

where $C_{3i} = \max(4, 2C_{1i})$, hence

$$dV(x,y) \le C_4\big(1 + V(x,y)\big)dt + \sum_{i=1}^{n}\sigma_{1i}(x_i-1)dW_{1i} + \sum_{i=1}^{n}\sigma_{2i}(y_i-1)dW_{2i}.$$

Where $C_4 = \max(C_{2i}, C_{3i}), i = 1, \dots, n$.

Integrating both sides from 0 to $\tau_k \wedge T$, and taking expectations, we get

$$E\,V\big(x(\tau_k\wedge T), y(\tau_k\wedge T)\big) \le V(x_0,y_0) + C_4 T + C_4\int_0^T E\,V\big(x(\tau_k\wedge T), y(\tau_k\wedge T)\big)\,dt.$$

By Gronwall's inequality, this yield

$$E\,V\big(x(\tau_k\wedge T), y(\tau_k\wedge T)\big) \le C_5, \tag{2}$$

where C_5 is the finite constant given by

$$C_5 = (V(x_0,y_0) + C_4 T)e^{C_4 T}. \tag{3}$$

Let $\Omega_k = \{\tau_k \le T\}$. We have $P(\Omega_k) \ge \varepsilon$, thus for all $\omega \in \Omega_k$, there exists at least one element of $x(\tau_k,\omega), y(\tau_k,\omega)$ which is equal either to k or to $\frac{1}{k}$, hence

$$V(x(\tau_k), y(\tau_k)) \ge (k + 1 - \log k) \wedge (\tfrac{1}{k} + 1 + \log k).$$

Therefore, by (2),

$$C_5 \ge E\big[1_{\Omega_k}V\big(x(\tau_k,\omega), y(\tau_k,\omega)\big)\big] \ge \varepsilon\left[(k + 1 - \log k) \wedge \left(\frac{1}{k} + 1 + \log k\right)\right],$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$, we get $C_5 = \infty$, which contradicts (3). So, we must have $\tau_\infty = \infty$ a.s. □

3. STOCHASTIC PERMANENCE

Theorem 2.1 shows that the solution of System (1) will remain in the positive cone $\mathbb{R}_+^{2n}$ with probability 1. To discuss how the solution varies in $\mathbb{R}_+^{2n}$, first we recall the definition of stochastically ultimate boundedness and stochastic permanence.

Definition 3.1 The solutions of system (1) are said to be stochastically ultimately bounded, if for any $\epsilon \in]0,1[$, there is a positive constant $\gamma(=\gamma(\varepsilon))$, such that for any initial value $X(0) \in \mathbb{R}_+^{2n}$, the solution of system (1) has the property that.

$$\limsup_{t\to\infty} \mathrm{P}\{X(t) > \gamma\} < \epsilon.$$

Definition 3.2 The system (1) is said to be stochastically permanent, if for any $\epsilon \in]0,1[$, there exists a pair of positive constants $\delta(\epsilon)$ and $\gamma(\epsilon)$ such that the solution of (1) with any initial value $\mathrm{X}(0) \in \mathbb{R}_+^{2\mathrm{n}}$, has the property that

$\liminf_{t\to\infty} \mathrm{P}\{X(t) \geq \delta\} \geq 1-\epsilon, \quad \liminf_{t\to\infty} \mathrm{P}\{X(t) \leq \gamma\} \geq 1-\epsilon.$

We start with a technical lemma

Lemma 3.3 *For any initial value* $(x(0), y(0)) \in \mathbb{R}_+^{2\mathrm{n}}$, *the solution of the system (1) satisfies* $\mathrm{E}[\sum_{i=1}^n (x_i(t)^p + y_i(t)^p)] \leq 2n\left(\frac{\beta}{\alpha}\right)^p$, $p > 1$.

Proof. Applying Itô's formula, we get

$$d\left(x_i^p\right) = p\left(x_i^p\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i} + \frac{p-1}{2}\sigma_{1i}^2\right) + \sum_{j=1}^n c_{ij}\left(x_j - x_i\right)\right)dt + p\sigma_{1i}x_i^p dW_{1i}(t)$$

$$= p\left(x_i^p\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i} + \frac{p-1}{2}\sigma_{1i}^2 - \sum_{j=1}^n c_{ij}\right) + \sum_{j=1}^n c_{ij}x_j\, x_i^{p-1}\right)dt$$

$$+ p\sigma_{1i}x_i^p dW_{1i}(t)$$

$$\leq p\left(x_i^p\left(1 - x_i + \frac{p-1}{2}\sigma_{1i}^2 - \sum_{j=1}^n c_{ij}\right) + \sum_{j=1}^n c_{ij}(\frac{(p-1)x_i^p}{p} + \frac{x_j^p}{p})\right)dt$$

$$+ p\sigma_{1i}x_i^p dW_{1i}(t)$$

$$\leq p\left(x_i^p\left(1 - x_i + \frac{p-1}{2}\sigma_{1i}^2 - \frac{1}{p}\sum_{j=1}^n c_{ij}\right) + \frac{1}{p}\sum_{j=1}^n c_{ij}x_j^p\right)dt + p\sigma_{1i}x_i^p dW_{1i}(t)$$

$$d\left(y_i^p\right) = p y_i^p\left(b_i(1 - \frac{y_i}{k_{2i}+x_i}) + \frac{p-1}{2}\sigma_{2i}^2\right)dt + p\sigma_{2i}y_i^p dW_{2i}(t)$$

$$\leq p y_i^p\left(b_i\left(1 - \frac{y_i}{k_{2i}}\right) + \frac{p-1}{2}\sigma_{2i}^2\right)dt + p\sigma_{2i}y_i^p dW_{2i}(t),$$

then

$$d\left(\sum_{i=1}^n (x_i^p + y_i^p)\right) \leq p\sum_{i=1}^n\left[x_i^p\left(1 - x_i + \frac{p-1}{2}\sigma_{1i}^2 + \frac{2}{p}\sum_{j=1}^n c_{ij}\right) + y_i^p\left(b_i\left(1 - \frac{y_i}{k_{2i}}\right)\right.\right.$$

$$\left.\left. + \frac{p-1}{2}\sigma_{2i}^2\right)\right]dt + \sum_i^n \sigma_{1i}x_i^p dW_{1i}(t) + p\sum_i^n \sigma_{2i}y_i^p dW_{2i}(t)$$

$$\leq p\sum_{i=1}^n\left[-x_i^{p+1} - \frac{b_i}{k_{2i}}y_i^{p+1} + x_i^p\left(1 - \frac{p-1}{2}\sigma_{1i}^2 + \frac{2}{p}\sum_{j=1}^n c_{ij}\right) + y_i^p\left(b_i + \frac{p-1}{2}\sigma_{2i}^2\right)\right]dt$$

$$+ \sum_i^n \sigma_{1i}x_i^p dW_{1i}(t) + p\sum_i^n \sigma_{2i}y_i^p dW_{2i}(t).$$

Let $\beta_1 = p\ (1 + \frac{p-1}{2}\sigma_{1i}^2 + \frac{2}{p}\sum_{j=1}^n c_{ij})$, $\beta_2 = p\ (b_i + \frac{p-1}{2}\sigma_{1i}^2)$, $\alpha = \min(p, \frac{b_i}{k_{2i}})$, and $\beta = \max(\beta_1, \beta_2)$, then,

$$d\left(\sum_{i=1}^n (x_i^p + y_i^p)\right) \leq \left[-\alpha\sum_{i=1}^n (x_i^{p+1} + y_i^{p+1}) + \beta\sum_{i=1}^n (x_i^p + y_i^p)\right]dt$$

$$+ \sum_i^n \sigma_{1i}x_i^p dW_{1i}(t) + p\sum_i^n \sigma_{2i}y_i^p dW_{2i}(t).$$

Integrating both sides from 0 to t, and taking expectations, we get

$$\frac{d\mathrm{E}[\sum_{i=1}^{n}(x_i^p+y_i^p)]}{dt} \leq -\alpha \mathrm{E}\left[\sum_{i=1}^{n}(x_i^{p+1}+y_i^{p+1})\right] + \beta \mathrm{E}\left[\sum_{i=1}^{n}(x_i^p+y_i^p)\right]$$

$$\leq -(2n)^{\frac{-1}{p}}\alpha \mathrm{E}\left[\sum_{i=1}^{n}(x_i^p+y_i^p)\right]^{\frac{1+p}{p}} + \beta \mathrm{E}\left[\sum_{i=1}^{n}(x_i^p+y_i^p)\right]$$

$$\leq \mathrm{E}\left[\sum_{i=1}^{n}(x_i^p+y_i^p)\right]\left(\beta - \frac{\alpha}{(2n)^{\frac{1}{p}}}\mathrm{E}\left[\sum_{i=1}^{n}(x_i^p+y_i^p)\right]^{\frac{1}{p}}\right).$$

Let $Z(t) = \mathrm{E}\left[\sum_{i=1}^{n}(x_i^p+y_i^p)\right]$, so we have

$$\frac{dZ}{dt} \leq Z\left(\beta - \frac{\alpha}{(2n)^{\frac{1}{p}}}Z^{\frac{1}{p}}\right).$$

Since the solution of equation $\frac{d\bar{Z}}{dt} = \bar{Z}\left(\beta - \frac{\alpha}{(2n)^{\frac{1}{p}}}\bar{Z}^{\frac{1}{p}}\right)$ tends to $2n\left(\frac{\beta}{\alpha}\right)^p$, as $t \to \infty$, the Comparison Theorem, we get

$$\limsup_{t\to\infty} \mathrm{E}\left[\sum_{i=1}^{n}(x_i(t)^p+y_i(t)^p)\right] \leq 2n\left(\frac{\beta}{\alpha}\right)^p.$$

□

By Chebyshev's inequality and Lemma 3.3, the following result is straightforward.

Theorem 3.4 *The solutions of System (1) are stochastically ultimately bounded.*

Now, we impose a hypothesis,

Assumption 3.1 $\alpha_1 = \max_{1\leq i\leq n}\{1, b_i, \check{c}\} > \frac{\check{\sigma}}{2}$.

Lemma 3.5 *Under Assumption 3.1, for any initial value* $X_0 = (x_0, y_0) = ((x_{01}, y_{01}), \dots, (x_{0n}, y_{0n})) \in \mathbb{R}_+^{2n}$, *the solution* $X(t) = ((x_1(t), y_1(t)), \dots, (x_n(t), y_n(t)))$ *satisfies that*

$$\mathrm{Limsup}_{t\to\infty}\, \mathrm{E}\left[\frac{1}{|X(t)|^\theta}\right] \leq H,$$

where θ *is an arbitrary positive constant satisfying*

$$\alpha_1 > (\theta+1)\frac{\check{\sigma}^2}{2}, \tag{4}$$

and

$$H = \frac{n^\theta(a_2+4ka_1)}{4ka_1}\max\left\{1, \left(\frac{2a_1+a_2+\sqrt{a_2^2+4a_1a_2}}{2a_1}\right)^{\theta-2}\right\},$$

in which k is an arbitrary positive constant satisfying

$$0 < \frac{k}{\theta} < \alpha_1 - \check{\sigma}^2\frac{\theta+1}{2}, \tag{5}$$

while

$$a_1 = \alpha_1 - \breve{\sigma}^2 \frac{\theta+1}{2} - \frac{k}{\theta} > 0, \quad a_2 = \alpha_1 + \breve{\sigma}^2 + \frac{2k}{\theta} > 0.$$

Proof. Let us define $V(X) = \sum_{i=1}^{n}(x_i + y_i)$ for $X \in \mathbb{R}_+^{2n}$, and

$$U(X) = \frac{1}{V(X(t))} \; on \; t \geq 0.$$

By the Itô formula, we get

$$dU(t) = -U(t)^2 \sum_i^n \left(\left(x_i\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i}\right) + \sum_j^n c_{ij}(x_j - x_i) + b_i y_i\left(1 - \frac{y_i}{k_{2i}+x_i}\right)\right) dt\right.$$

$$\left. + \sigma_{1i} x_i dW_{1i}(t) + \sigma_{2i} y_i dW_{2i}(t)\right) + U(t)^3 \sum_{i=1}^{n}[(\sigma_{1i}x_i)^2 + (\sigma_{2i}y_i)^2]\, dt$$

$$= LV(t)dt - U(t)^2 \sum_{i=1}^{n}[\sigma_{1i}x_i dW_{1i}(t) + \sigma_{2i}y_i dW_{2i}(t)],$$

where

$$LU(t) = -U(t)^2 \sum_i^n \left(\left(x_i\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i}\right) + \sum_j^n c_{ij}(x_j - x_i) + b_i y_i\left(1 - \frac{y_i}{k_{2i}+x_i}\right)\right)\right.$$

$$+U(t)^3 \sum_{i=1}^{n}[(\sigma_{1i}x_i)^2 + (\sigma_{2i}y_i)^2].$$

Under Assumption 1.3, we can choose a positive constant θ such that it obeys (4). By the Itô formula again, we have

$$d\left[(1+U(t))^\theta\right] = (-\theta(1+U(t))^{\theta-1}U(t)^2 \sum_{i=1}^{n}\left(x_i\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i}\right)\right.$$

$$\left.+ \sum_j^n c_{ij}(x_j - x_i) + b_i y_i\left(1 - \frac{y_i}{k_{2i}+x_i}\right)\right) + \left(\theta(1+U(t))^{\theta-1}U(t)^3\right.$$

$$\left.+\frac{\theta(\theta-1)}{2}(1+U(t))^{\theta-2}U(t)^4\right)\sum_{i=1}^{n}[(\sigma_{1i}x_i)^2 + (\sigma_{2i}y_i)^2]\Big)\, dt$$

$$-\theta\big(1+U(t)\big)^{\theta-1}U(t)^2 \sum_{i=i}^{n}[\sigma_{1i}x_i dW_{1i}(t) + \sigma_{2i}y_i dW_{2i}(t)].$$

Now, choose $k > 0$ sufficiently small such that it satisfies (5). Thus, by the Itô formula, we get

$$d\left[e^{kt}(1+U(t))^\theta\right] = ke^{kt}\big(1+U(t)\big)^\theta + e^{kt}d\big(1+U(t)\big)^\theta$$

$$= e^{kt}d\big(1+U(t)\big)^{\theta-2}\left(\left(k\big(1+U(t)\big)^2 + J(t)\right)dt\right. \tag{6}$$

$$\left.-\theta\big(1+U(t)\big)^{\theta-1}U(t)^2 \sum_{i=1}^{n}[\sigma_{1i}x_i dW_{1i}(t) + \sigma_{2i}y_i dW_{2i}(t)]\right),$$

where

$$J(t) = -\theta(1+U(t))^{\theta-1}U(t)^2 \sum_{i=1}^{n} \left(x_i \left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i} \right) \right.$$

$$\left. + \sum_j^n c_{ij}(x_j - x_i) + b_i y_i \left(1 - \frac{y_i}{k_{2i}+x_i} \right) \right) + \left(\theta(1+U(t))^{\theta-1}U(t)^3 \right.$$

$$\left. + \frac{\theta(\theta-1)}{2}(1+U(t))^{\theta-2}U(t)^4 \right) \sum_{i=1}^{n} [(\sigma_{1i}x_i)^2 + (\sigma_{2i}y_i)^2].$$

We thus obtain

$$J(t) \le -\theta\left(\alpha_1 - \breve{\sigma}^2 \frac{\theta+1}{2}\right) U(t)^2 + \theta(\alpha_1 + \breve{\sigma}^2)U(t),$$

where α_1 has been defined in the statement of the lemma. Substituting this into in (6) yields

$$d[e^{kt}(1+U(t))^{\theta}] \le e^{kt}\big(1+U(t)\big)^{\theta-2}\left(k\big(1+U(t)\big)^2 - \theta\left(\alpha_1 - \breve{\sigma}^2 \frac{\theta+1}{2}\right)U(t)^2\right.$$

$$\left. +\theta(\alpha_1 + \breve{\sigma}^2)U(t)\right)dt - \theta e^{kt}\big(1+U(t)\big)^{\theta-1}U(t)^2 \sum_{i=1}^{n} \sigma_{1i}x_i dW_{1i}(t)$$

$$-\theta e^{kt}\big(1+U(t)\big)^{\theta-1}U(t)^2 \sum_{i=1}^{n} \sigma_{2i}y_i dW_{2i}(t)$$

$$= e^{kt}\big(1+U(t)\big)^{\theta-2}\left(-\theta\left(\alpha_1 - \breve{\sigma}^2 \frac{\theta+1}{2} - \frac{k}{\theta}\right)U(t)^2\right.$$

$$\left. +\theta\left(\alpha_1 + \breve{\sigma}^2 - \frac{2k}{\theta}\right)U(t) + k\right)dt$$

$$-\theta e^{kt}\big(1+U(t)\big)^{\theta-1}U(t)^2 \sum_{i=1}^{n} [\sigma_{1i}x_i dW_{1i}(t) + \sigma_{2i}y_i dW_{2i}(t)].$$

It is not difficult to see that

$$\big(1+U(t)\big)^{\theta-2}\left(-\theta\left(\alpha_1 - \breve{\sigma}^2 \frac{\theta+1}{2} - \frac{k}{\theta}\right)U(t)^2 + \theta\left(\alpha_1 + \breve{\sigma}^2 - \frac{2k}{\theta}\right)U(t) + k\right) \le H_1,$$

on $U(t) > 0$, where

$$H_1 = \frac{a_2 + 4ka_1}{4ka_1} \max\left\{1, \left(\frac{2a_1 + a_2 + \sqrt{a_2^2 + 4a_1a_2}}{2a_1}\right)^{\theta-2}\right\},$$

and a_1, a_2 have been defined in the lemma. Thus

$$d[e^{kt}(1+U(t))^{\theta}] \le H_1 e^{kt}dt - \theta e^{kt}\big(1+U(t)\big)^{\theta-1}U(\square)^2 \sum_{i=1}^{n} [\sigma_{1i}x_i dW_{1i}(t)$$

$$+\sigma_{2i}y_i dW_{2i}(t)].$$

This implies

$$\mathrm{E}\left[e^{kt}\big(1+U(t)\big)^{\theta}\right] \le \big(1+U(t)\big)^{\theta} + \frac{H_1}{k}e^{kt}.$$

Then

$$\limsup_{t\to\infty} \mathrm{E}[U(t)^{\theta}] \le \limsup_{t\to\infty} \mathrm{E}\left[\big(1+\square(t)\big)^{\theta}\right] \le \frac{H_1}{k}.$$

For $X \in \mathbb{R}_+^{2n}$, note that

$(\sum_{i=1}^{n}(x_i + y_i))^\theta \leq (n \ \max_{1\leq i\leq n}(x_i + y_i))^\theta \leq n^\theta(\max_{1\leq i\leq n}(x_i + y_i)^2)^{\frac{\theta}{2}} \leq n^\theta|X|^\theta.$

Consequently,

$\text{Limsup}_{t\to\infty} \text{E}\left[\frac{1}{|X(t)|^\theta}\right] \leq n^\theta \frac{H_1}{k} \leq H,$ □

Theorem 3.6 *Under condition (4), the System (1) is stochastically permanent.*

Proof. Let $X(t)$ be the solution of System (1) with any given positive initial value $X(0) \in \mathbb{R}_+^{2n}$. By Lemma 3.5, we have

$\text{Limsup}_{t\to\infty} \text{E}\left[\frac{1}{|X(t)|^\theta}\right] \leq H.$

For $X(t) \in \mathbb{R}_+^{2n}$ and for any $\epsilon > 0$, let $\delta = \left(\frac{\epsilon}{H}\right)^{\frac{1}{\theta}}$, we get the following

$$\begin{aligned}
\text{P}\{X(t) < \delta\} &= \text{P}\left\{\frac{1}{(X(t))^\theta} < \frac{1}{\delta^\theta}\right\} \\
&\leq \frac{\text{E}\left[\frac{1}{|X(t)|^\theta}\right]}{\frac{1}{\delta^\theta}} \\
&\leq \delta^\theta H = \epsilon.
\end{aligned}$$

Hence

$\text{Limsup}_{t\to\infty} \text{P}\{X(t) < \delta\} \leq \epsilon,$

and this implies

$\text{Limsup}_{t\to\infty} \text{P}\{X(t) \geq \delta\} \geq 1 - \epsilon.$

The other property of Definition 3.2 follows from Theorem 3.4. □

4. STATIONARY DISTRIBUTION

In this section, we investigate that there is a stationary distribution for System (1) instead of asymptotically stable equilibria and list some results on the stationary distribution (see Khasminskii [[4], pp.106-125]). Let $X(t)$ be a homogeneous Markov process in E_l (E_l denotes l-space) described by the stochastic equation

$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t)$

The diffusion matrix is

$$A(x) = \left(a_{ij}(x)\right), \quad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Assumption 4.1 There exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties.

- (i) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

- (ii) If $x \in E_l \setminus U$, the mean time τ for which a path starting at x and reaching the set U is finite, and $\sup_{x\in K} E_x \tau < \infty$ for every compact subset $K \subset E_l$.

Lemma 4.1 *(see [4]). If Assumption 4.1 holds, then the Markov process $X(t)$ has a stationary distribution $\mu(A)$. Let $f(.)$ be a function integrable with respect to the measure μ. Then*

$$\mathrm{P}_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f\big(X(t)\big)dt = \int_{E_l} f(x)\mu(dx)\right\} = 1$$

for all $x \in E_l$.

Theorem 4.2 *Let $M_1 = -k_{1i} + \frac{a_i y_i^*}{k_{1i}} - \frac{a_i}{2} + \frac{y_i^*}{2k_{2i}}$, $M_2 = -1 - \frac{a_i}{2} + \frac{y_i^*}{2k_{2i}}$. Assume that $M_1 < 0$ and $M_2 < 0$. Then there is a stationary distribution $\mu(.)$ for System (1).*

Proof. Let us set

$$V_i(x_i, y_i) = (k_{1i} + x_i^*)\left(x_i - x_i^* - \log\frac{x_i}{x_i^*}\right) + \frac{k_{2i}+x_i^*}{b_i}\left(y_i - y_i^* - \log\frac{y_i}{y_i^*}\right).$$

By Itô's formula, we have

$$dV_i = \left((k_{1i} + x_i^*)\big((x_i - x_i^*)\left(1 - x_i - \frac{a_i y_i}{k_{1i}+x_i} + \sum_j^n c_{ij}\left(\frac{x_j}{x_i} - 1\right)\right) + \frac{x_i^*\sigma_{1i}^2}{2}\right)$$

$$+(k_{2i} + x_i^*)\left((y_i - y_i^*)\left(1 - \frac{y_i}{k_{2i}+x_i}\right) + \frac{y_i^*\sigma_{1i}^2}{2b_i}\right)\Bigg)dt$$

$$+(k_{1i} + x_i^*)(x_i - x_i^*)\sigma_{1i}dW_{1i}(t) + (k_{2i} + x_i^*)(y_i - y_i^*)\sigma_{2i}dW_{2i}(t)$$

$$= LV_i dt + (k_{1i} + x_i^*)(x_i - x_i^*)\sigma_{1i}dW_{1i}(t) + (k_{2i} + x_i^*)(y_i - y_i^*)\sigma_{2i}dW_{2i}(t),$$

where

$$LV_i = (k_{1i} + x_i^*)(x_i - x_i^*)\left(-x_i - \frac{a_i y_i}{k_{1i}+x_i} + \sum_j^n c_{ij}\left(\frac{x_j}{x_i} - \frac{x_j^*}{x_i^*}\right) + x_i^* + \frac{a_i y_i^*}{k_{1i}+x_i^*}\right)$$

$$+(k_{1i} + x_i^*)\frac{x_i^*\sigma_{1i}^2}{2} + (k_{2i} + x_i^*)\left[(y_i - y_i^*)\left(\frac{y_i^*}{k_{2i}+x_i^*} - \frac{y_i}{k_{2i}+x_i}\right) + \frac{y_i^*\sigma_{2i}^2}{2}\right]$$

$$= \left(-k_{1i} - x_i^* + \frac{a_i y_i}{k_{1i}+x_i}\right)(x_i - x_i^*)^2 - (y_i - y_i^*)^2$$

$$+\left(-a_i + \frac{y_i}{k_{2i}+x_i}\right)(x_i - x_i^*)(y_i - y_i^*) + (k_{1i} + x_i^*)\sum_j^n c_{ij}\left(\frac{x_j}{x_i} - \frac{x_j^*}{x_i^*}\right)$$

$$+(k_{1i} + x_i^*)\frac{x_i^*\sigma_{1i}^2}{2} + (k_{2i} + x_i^*)\frac{y_i^*\sigma_{2i}^2}{2}.$$

The positivity of $x_i(t)$ and Cauchy-Schwarz's inequality imply

$$LV_i \leq \left(-k_{1i} + \frac{a_i y_i^*}{k_{1i}} - \frac{a_i}{2} + \frac{y_i}{2k_{2i}}\right)(x_i - x_i^*)^2 + \left(-1 - \frac{a_i}{2} + \frac{y_i}{2k_{2i}}\right)(y_i - y_i^*)^2$$

$$+(k_{1i} + x_i^*) \sum_j^n c_{ij} (x_i - x_i^*) \left(\frac{x_j}{x_i} - \frac{x_j^*}{x_i^*}\right) + (k_{1i} + x_i^*) \frac{x_i^* \sigma_{1i}^2}{2} + (k_{2i} + x_i^*) \frac{y_i^* \sigma_{2i}}{2}$$

$$\leq -\left(k_{1i} - \frac{a_i y_i^*}{k_{1i}} + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(x_i - x_i^*)^2 - \left(1 + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(y_i - y_i^*)^2$$

$$+(k_{1i} + x_i^*) \sum_j^n c_{ij} x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} - \frac{x_j x_i^*}{x_j^* x_i} + 1\right) + (k_{1i} + x_i^*) \frac{x_i^* \sigma_{1i}^2}{2} + (k_{2i} + x_i^*) \frac{y_i^* \sigma_{2i}^2}{2}$$

$$\leq -\left(k_{1i} - \frac{a_i y_i^*}{k_{1i}} + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(x_i - x_i^*)^2 - \left(1 + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(y_i - y_i^*)^2$$

$$+(k_{1i} + x_i^*) \sum_j^n c_{ij} x_j^* \left(-\left(-\frac{x_j}{x_j^*} + \log\frac{x_j}{x_j^*}\right) + \left(-\frac{x_i}{x_i^*} + \log\frac{x_i}{x_i^*}\right) + \log\frac{x_j x_i^*}{x_j^* x_i} - \frac{x_j x_i^*}{x_j^* x_i} + 1\right)$$

$$+(k_{1i} + x_i^*) \frac{x_i^* \sigma_{1i}^2}{2} + (k_{2i} + x_i^*) \frac{y_i^* \sigma_{2i}^2}{2}$$

$$\leq -\left(k_{1i} - \frac{a_i y_i^*}{k_{1i}} + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(x_i - x_i^*)^2 - \left(1 + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(y_i - y_i^*)^2$$

$$+(k_{1i} + x_i^*) \sum_j^n c_{ij} x_j^* \left(-\left(-\frac{x_j}{x_j^*} + \log\frac{x_j}{x_j^*}\right) + \left(-\frac{x_i}{x_i^*} + \log\frac{x_i}{x_i^*}\right)\right)$$

$$+(k_{1i} + x_i^*) \frac{x_i^* \sigma_{1i}^2}{2} + (k_{2i} + x_i^*) \frac{y_i^* \sigma_{2i}^2}{2}.$$

From [6, Theorem 2.3], we have

$$\sum_j^n c_{ij} x_j^* \left(-\frac{x_j}{x_j^*} + \log\frac{x_j}{x_j^*}\right) = \sum_j^n c_{ij} x_j^* \left(-\frac{x_i}{x_i^*} + \log\frac{x_i}{x_i^*}\right).$$

So

$$LV_i \leq -\left(k_{1i} - \frac{a_i y_i^*}{k_{1i}} + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(x_i - x_i^*)^2 - \left(1 + \frac{a_i}{2} - \frac{y_i}{2k_{2i}}\right)(y_i - y_i^*)^2$$

$$+(k_{1i} + x_i^*) \frac{x_i^* \sigma_{1i}^2}{2} + (k_{2i} + x_i^*) \frac{y_i^* \sigma_{2i}^2}{2}.$$

Let $\lambda = (k_{1i} + x_i^*) \frac{x_i^* \sigma_{1i}^2}{2} + (k_{2i} + x_i^*) \frac{y_i^* \sigma_{2i}^2}{2}$, $M_1 = -k_{1i} + \frac{a_i y_i^*}{k_{1i}} - \frac{a_i}{2} + \frac{y_i^*}{2k_{2i}}$ and $M_2 = -1 - \frac{a_i}{2} + \frac{y_i^*}{2k_{2i}}$. We have

$$LV_i \leq M_1 (x_i - x_i^*)^2 + M_2 (y_i - y_i^*)^2 + \lambda.$$

Now, if λ satisfies,

$$\lambda < \min\{M_1 (x_i^*)^2, M_2 (y_i^*)^2\},$$

then the ellipsoid

$$M_1 (x_i - x_i^*)^2 + M_2 (y_i - y_i^*)^2 + \lambda = 0,$$

lies entirely in $\mathbb{R}_+^{2n}$. We can take U to be any neighborhood of the ellipsoid with $\overline{U} \subseteq E_l = \mathbb{R}_+^{2n}$, so for

$x \in E_l \setminus U$, we have $LV \leq N$ (N is a nonpositive constant), which implies that Condition (ii) is satisfied. Besides, there is a $M > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = \sum_{i=1}^{n}\left(\sigma_{1i}^2 x_i^2 \xi_i^2 + \sigma_{2i}^2 y_i^2 \xi_i^2\right) \geq M|\xi|^2,$$

for all $(x_1, y_1, \dots, x_n, y_n) \in \bar{U}$, $\xi \in \mathbb{R}^n$, thus Condition (i) is satisfied. Therefore, by Lemma (4.1) the system (1.1) has a stable stationary distribution $\mu(.)$ and it is ergodic. □

5. NUMERICAL SIMULATIONS

To illustrate our results, we consider the two-species nonlinear dispersal system with n = 2.

$$\begin{aligned}
dx_1 &= \left[x_1\left(1 - x_1 - \frac{a_1 y_1}{k_{11} + x_1}\right) + c_{12}(x_2 - x_1)\right]dt + \sigma_{11}x_1 dW_{11}(t),\\
dy_1 &= b_1 y_1\left(1 - \frac{y_1}{k_{21}+x_1}\right) + \sigma_{21} y_1 dW_{21}(t),\\
dx_2 &= \left[x_2\left(1 - x_2 - \frac{a_2 y_2}{k_{12}+x_2}\right) + c_{21}(x_1 - x_2)\right]dt + \sigma_{12}x_2 dW_{12}(t),\\
dy_2 &= b_2 y_2\left(1 - \frac{y_2}{k_{21}+x_2}\right) + \sigma_{22} y_2 dW_{22}(t).
\end{aligned} \tag{7}$$

We numerically simulate the solution of System (7). By the Milstein scheme mentioned in [[5], p345], which turns out to be an order 1.0 strong Taylor scheme, we consider the following discretized system:

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} + \left[x_{1,k}\left(1 - x_{1,k} - \frac{a_1 y_{1+1}}{k_{11}+x_{1,k}}\right) + c_{12}\left(x_{2,k} - x_{1,k}\right)\right]h\\
&\quad + \sigma_{11}x_{1,k}\sqrt{h}\xi_{1,k}^2 + \frac{1}{2}\sigma_{11}^2 x_{1,k}\left(h\xi_{1,k}^2 - h\right),\\
y_{1,k+1} &= y_{1,k} + b_1 y_{1,k}\left(1 - \frac{y_{1,k}}{k_{21}+x_{1,k}}\right)h + \sigma_{21}y_{1,k}\sqrt{h}\xi_{2,k}^2 + \frac{1}{2}\sigma_{21}^2 y_{1,k}\left(h\xi_{1,k}^2 - h\right),\\
x_{2,k+1} &= x_{2,k} + \left[x_{2,k}\left(1 - x_{2,k} - \frac{a_2 y_{2+k}}{k_{12}+x_{2,k}}\right) + c_{21}\left(x_{1,k} - x_{2,k}\right)\right]h\\
&\quad + \sigma_{12}x_{2,k}\sqrt{h}\xi_{3,k}^2 + \frac{1}{2}\sigma_{12}^2 x_{2,k}\left(h\xi_{3,k}^2 - h\right),\\
y_{2,k+1} &= y_{2,k} + b_2 y_{2,k}\left(1 - \frac{y_{2,k}}{k_{22}+x_{2,k}}\right)h + \sigma_{22}y_{2,k}\sqrt{h}\xi_{4,k}^2 + \frac{1}{2}\sigma_{22}^2 y_{2,k}\left(h\xi_{4,k}^2 - h\right).
\end{aligned} \tag{8}$$

For the numerical simulations, we choose $a_1 = 0.4$, $k_{11} = 0.08$, $k_{21} = 0.2$, $b_1 = 0.1$, $a_2 = 0.5$, $k_{12} =$ 0.4, $k_{22} = 0.25$, $b_2 = 1$, and the time step $h = 0.01$. In Figure 1, we assume that $c_{12} = 1.5$, $c_{21} = 1.6$, with $\sigma_{11} = 0.3$, $\sigma_{21} = 0.01$, and $\sigma_{21} = 0.01$, $\sigma_{22} = 0.01$. The initial value $(x_1(0), y_1(0))$ = ((0.55, 0, 6), (0.5, 0, 61)). In this case, the deterministic model has a globally stable equilibrium point $(x^*, y^*) = \left((x_1^*, y_1^*), (x_2^*, y_2^*)\right)$ = ((0.55, 0.75), (0.576, 0.825)). Obviously, Assumption 3.1 holds the system (7) is stochastically permanent. Also, the conditions of Theorem (4.2) are satisfied, so there is a stationary distribution. In Figure 2, we choose $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, and $\sigma_{21} = 0.01$, $\sigma_{22} = 0.2$. The populations of (x_1, y_1) and (x_2, y_2) suffer relatively large white noise.

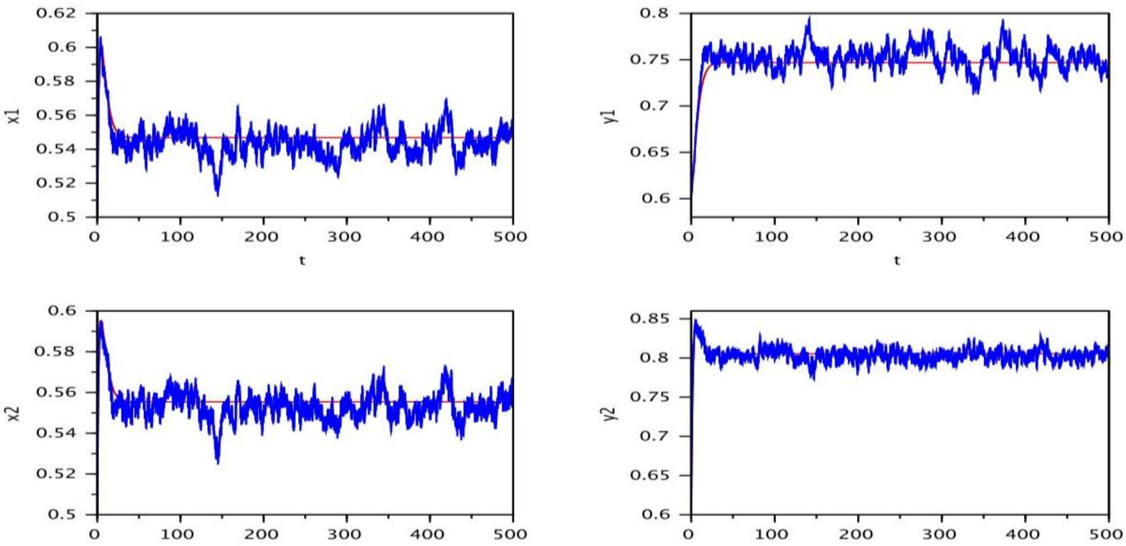

Figure 1. *Trajectories of the solutions of stochastic system (7) represented by the blue curves and the corresponding deterministic system represented by the red lines. The stochastic system is stochastically permanent.*

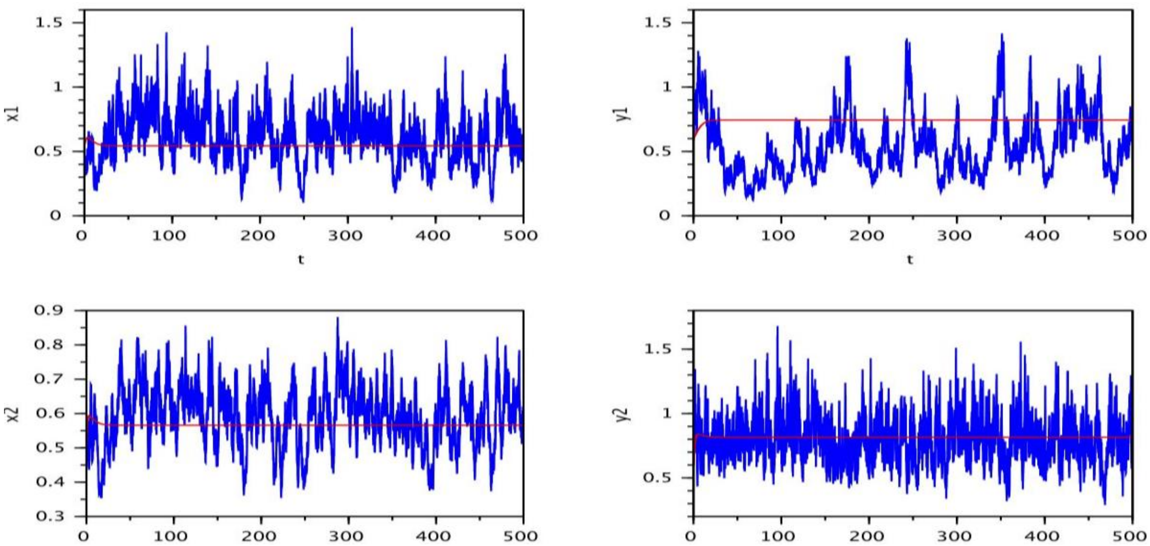

Figure 2. *Trajectories of the solutions of stochastic system (7) and the corresponding deterministic system, the blue lines and the red lines represent them, respectively.* $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, *and* $\sigma_{21} = 0.1$, $\sigma_{22} = 0.3$.

By comparing Figure 1, we can see that in Figure 2 the curves fluctuations are larger. In Figure 3, we choose $c_{12} = 0$, $c_{21} = 0.01$ and $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, and $\sigma_{21} = 0.1$, $\sigma_{22} = 0.3$. Figure 2 and 3 have the same white noise intensity but have different diffusion coefficients, because there is no diffusion effects, we can see that (x_1, y_1) will die.

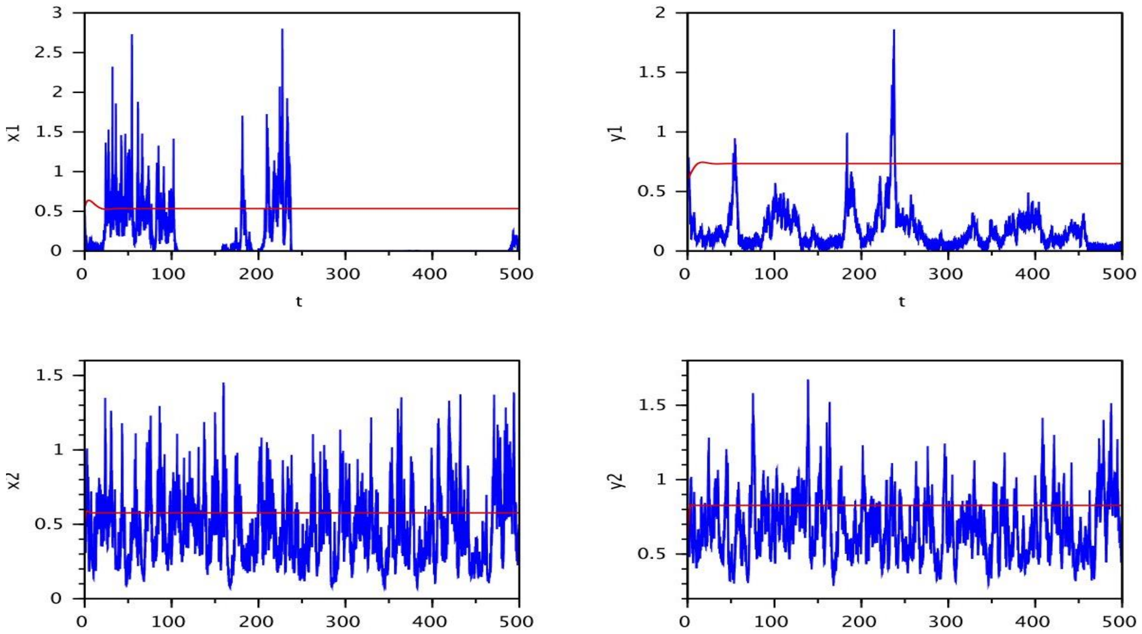

Figure 3. *Because there is no diffusion effects, we can see that* (x_1, y_1) *will die;* $c_{12} = 0$, $c_{21} = 0.01$ *and* $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, *and* $\sigma_{21} = 0.1$, $\sigma_{22} = 0.3$

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Aziz, M.A., Daher, O.M., “Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling- type II schemes”, Appl. Math. Lett., 16(7): 1069–1075, (2003).

[2] Dalal, N., Greenhalgh, D., Mao, X., “A stochastic model for internal HIV dynamics”, J. Math. Anal. Appl., 341(2): 1084–1101, (2008).

[3] Ji, C., Jiang, D., Shi, N., “Analysis of a predator- prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation”, J. Math. Anal. Appl., 359(2): 482–498, (2009).

[4] Khasminskii, R., “Stochastic stability of differential equations, volume 66 of Stochastic Modelling and Applied Probability”, Springer, Heidelberg, second edition, (2012).

[5] Kloeden, P.E., Platen, E., “Numerical solution of stochastic differential equations”, volume 23 of Applications of Mathematics (New York), Springer-Verlag, Berlin, (1992).

[6] Li, M.Y., Shuai, Z., “Global-stability problem for coupled systems of differential equations on networks”, J. Differential Equations, 248(1): 1–20, (2010).

[7] Li. X., Mao, X., “Population dynamical behavior of non- autonomous Lotka-Volterra competitive system with random perturbation”, Discrete Contin. Dyn. Syst., 24(2): 523–545, (2009).

[8] Lv, J., Wang, K., “Analysis on a stochastic predator-prey model with modified Leslie-Gower response”, Abstr. Appl. Anal., pages Art, ID 518719, 16, (2011).

[9] Mao, X., Yuan, C., Zou, J., “Stochastic differential delay equations of population dynamics”, J. Math. Anal. Appl., 304(1): 296–320, (2005).

[10] Zhang, L., Teng, Z., "The dynamical behavior of a predator- prey system with Gompertz growth function and impulsive dispersal of prey between two patches", Math. Methods Appl. Sci., 39(13): 3623–3639, (2016).

[11] Zu, L., Jiang, D., O'Regan, D., "Stochastic permanence, stationary distribution and extinction of a single-species nonlinear diffusion system with random perturbation", Abstr. Appl. Anal., pages Art, ID 320460, 14, (2014).