Dynamics of a Stochastic Predator-Prey Coupled System with Modified Leslie-Gower and Holling Type II Schemes

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Article Info

Received: 31/07/2017
Accepted: 23/12/2017

Keywords
Stochastic permanence
Stationary distribution
Lyapunov functional

Abstract

We study a stochastic predator-prey system with modified Leslie- Gower and Holling type II functional response among n patches. The existence and uniqueness as well as boundedness of solution are obtained. Furthermore, we obtain sufficient conditions for stochastic permanence, and by the Lyapunov functional technique we obtain sufficient conditions for the existence of the stationary distribution. Finally, we illustrate our conclusions through numerical simulations.

1. INTRODUCTION

We consider an autonomous stochastic predator-prey model with modified Leslie-Gower and Holling type II functional response in which preys disperse among n patches. As, the effect of dispersion on the species survival plays an important role in the study of dynamic biology and ecology, the analysis of mathematical models of populations dispersing among patches in a heterogeneous environment has been the subject of several recent papers [10,11].

In this paper, we are interested in a predator-prey model with modified Leslie-Gower and Holling type II functional response studied in the deterministic case by M.A. Aziz-Alaoui and M. Daher-Okiye [1] and with stochastic perturbation by [3,8]. In our case, we introduce the coupling as well as the stochastic perturbation to take into account the effect of randomly fluctuating and stochastically perturbed intrinsic growth rate. The key method used in this paper is the analysis of Lyapunov functions [9]. The system that we consider is

\[
\begin{align*}
    dx_i &= (x_i (1 - x_i - \frac{a_i y_i}{k_{1i} + x_i}) + \sum_{j=1}^{n} c_{ij} (x_j - x_i))dt + \sigma_{1i} x_i dW_{1i}(t) \\
    dy_i &= b_i y_i (1 - \frac{y_i}{k_{2i} + x_i}) + \sigma_{2i} y_i dW_{2i}(t), \quad i = 1,2,\ldots,n.
\end{align*}
\]

(1)

Here, \(x_i, y_i\) denote respectively the densities of preys and predators on the patch \(i\), and the parameters \(a_i, k_{1i}, k_{2i}\) and \(b_i\) are positive constants as in [1]. The constants \(c_{ij}\) correspond to the dispersal rate from patch \(j\) to \(i\), the processes \(W_i = (W_{1i}, W_{2i}), 1 \leq i \leq n\), are independent standard Brownian motions defined in the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), and \(\sigma_{1i}\) and \(\sigma_{2i}\) are positive constants. For this system, we begin to show the existence and uniqueness of the global positive solution with any initial
positive value in section 2. In section 3, we prove that when the noise is small enough the population system is stochastically permanent. In section 4, by the Lyapunov functional technique we obtain sufficient conditions for the existence of the stationary distribution of system (1). Finally, in section 5, we make numerical simulations to confirm the effect of white noise and the diffusion coefficient on the species.

2. EXISTENCE AND UNIQUENESS OF THE BOUNDED SOLUTION

Throughout this paper, we denote by $\mathbb{R}_{+}^{2n}$ the positive cone in $\mathbb{R}^{2n}$, that is

$$\mathbb{R}_{+}^{2n} = \{(x,y) \in \mathbb{R}^{2n}: (x_i, y_i) > (0,0) \text{ for all } 1 \leq i \leq n\},$$

and for $X = (x,y) \in \mathbb{R}^{2n}$, its norm is denoted by $|X| = \sum_{i=1}^{n}(x_i^2, y_i^2)^{\frac{1}{2}}$.

For simplicity, we define $\sigma = \max_{1 \leq i,j \leq n} \sigma_{ij}$, and for any constant sequence $(c_{ij})_{1 \leq i,j \leq n}$, we get $\tilde{c} = \max_{1 \leq i,j \leq n} c_{ij}$.

**Theorem 2.1** For any initial condition $(x_0, y_0) = ((x_{01}, y_{01}), ..., (x_{0n}, y_{0n})) \in \mathbb{R}_{+}^{2n}$, the system (1) admits a unique solution $(x_i(t), y_i(t)) = ((x_1(t), y_1(t)), ..., (x_n(t), y_n(t)))$, defined for all $t \geq 0$ a.s., and this solution remains in $\mathbb{R}_{+}^{2n}$.

**Proof.** Let $\tau_{e}$ be the explosion time of the solution of (1). We have to prove that $(x_i(t), y_i(t)) \in \mathbb{R}_{+}^{2n}$ for every $t \in [0,\tau_{e}]$ and that $\tau_{e} = \infty$ a.s. The proof of the first assertion is adapted from [7]. Indeed, the coefficients in system (1) are locally Lipschitz, so there exists a unique solution for all $t \in [0,\tau_{e}]$ and for all $(x_i(0), y_i(0)) \in \mathbb{R}_{+}^{2n}$. To show that this solution is global, it suffices to show that $\tau_{e} = \infty$. For that, let $k_0 > 0$ be large enough, such that $(x_{0i}, y_{0i}) \in \left[\frac{1}{k_0}, k_0\right] \times \left[\frac{1}{k_0}, k_0\right]$. For each integer $k \geq k_0$ we define the stopping time

$$\tau_k = \inf\left\{t \in [0,\tau_{e}]: x \notin \left(\frac{1}{k}, k\right) \text{ ou } y \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1, ..., n\right\}.$$

The sequence $(\tau_k)$ is increasing as $k \to \infty$. Set $\tau_{\infty} = \lim_{k \to \infty} \tau_k$, whence $\tau_{\infty} \leq \tau_{e}$, (in fact, as $(x(t), y(t)) > 0$ a.s., we have $\tau_{\infty} = \tau_{e}$). It suffices to prove that $\tau_{\infty} = \infty$ a.s.. Assume that this statement is false, then there exist $T > 0$ and $\varepsilon \in ]0,1[$ such that $P(\{\tau_{\infty} \leq T\}) > \varepsilon$. Since $(\tau_k)$ is increasing we have $P(\{\tau_{\infty} \leq T\}) > \varepsilon$.

Now, consider the positive definite function $V: \mathbb{R}_{+}^{2n} \to \mathbb{R}_{+}^{2n}$ given by

$$V(x, y) = \sum_{i=1}^{n}(x_i + 1 - \log x_i + y_i + 1 - \log y_i).$$

Applying Itô’s formula, we get

$$dV(x, y) = \sum_{i=1}^{n}\left(x_i - 1\right)\left(1 - x_i - \frac{a_i y_i}{k_{1i} + x_i}\right) + \frac{\sigma_{1i}^2}{2} + \sum_{j=1}^{n}c_{ij}\left(x_j - x_i + 1 - \frac{x_j}{x_i}\right)$$

$$+ b_i\left(y_j - 1\right)\left(1 - \frac{y_j}{k_{2i} + x_i}\right) + \frac{\sigma_{2i}^2}{2}dt + \sum_{i=1}^{n}\sigma_{1i}(x_i - 1)dW_{1i} + \sum_{i=1}^{n}\sigma_{2i}(y_i - 1)dW_{2i}.$$

The positivity of $x_i(t)$ and $y_i(t)$ implies...
\[
\begin{align*}
dV(x,y) & \leq \sum_{i=1}^{n} \left( 2x_i + \left( a_i + b_i + \frac{1}{k_{zi}} \right)y_i + \frac{\sigma_i^2 + \sigma_{zi}^2}{2} + \sum_{j=1}^{n} c_{ij}x_j \right) dt + \sum_{i=1}^{n} \sigma_i(x_i - 1)dW_{1i} \\
& \quad + \sum_{i=1}^{n} \sigma_{zi}(y_i - 1)dW_{2i}.
\end{align*}
\]

Let \( C_{1i} = a_i + b_i + \frac{1}{k_{zi}}, \ C_{2i} = \frac{\sigma_i^2 + \sigma_{zi}^2}{2}, \ i = 1, \ldots, n. \) Using [2, lemma 4.1], we can write

\[
2x_i + C_{1i}y_i \leq 4(x_i + 1 - \log x_i) + 2C_{2i} (y_i + 1 - \log y_i) \leq C
\]

where \( C_{3i} = \max(4,2C_{1i}), \) hence

\[
dV(x,y) \leq C_4(1 + V(x,y))dt + \sum_{i=1}^{n} \sigma_i(x_i - 1)dW_{1i} + \sum_{i=1}^{n} \sigma_{zi}(y_i - 1)dW_{2i}.
\]

Where \( C_4 = \max(C_{2i},C_{3i}), \ i = 1, \ldots, n. \)

Integrating both sides from 0 to \( \tau_k \wedge T, \) and taking expectations, we get

\[
E[V(x(\tau_k \wedge T),y(\tau_k \wedge T))] \leq V(x_0,y_0) + C_4 T + C_4 \int_0^T E[V(x(\tau_k \wedge T),y(\tau_k \wedge T))] dt.
\]

By Gronwall’s inequality, this yield

\[
E[V(x(\tau_k \wedge T),y(\tau_k \wedge T))] \leq C_5,
\]

where \( C_5 \) is the finite constant given by

\[
C_5 = (V(x_0,y_0) + C_4 T)e^{C_4 T}.
\]

Let \( \Omega_k = \{ \tau_k \leq T \}. \) We have \( P(\Omega_k) \geq \varepsilon, \) thus for all \( \omega \in \Omega_k, \) there exists at least one element of \( x(\tau_k,\omega), y(\tau_k,\omega) \) which is equal either to \( k \) or to \( \frac{k}{1+k}, \) hence

\[
V(x(\tau_k),y(\tau_k)) \geq (k + 1 - \log k) \wedge \left( \frac{1}{k} + 1 + \log k \right).
\]

Therefore, by (2),

\[
C_5 \geq E\left[1_{\Omega_k}V(x(\tau_k,\omega),y(\tau_k,\omega))\right] \geq \varepsilon \left[(k + 1 - \log k) \wedge \left( \frac{1}{k} + 1 + \log k \right)\right],
\]

where \( 1_{\Omega_k} \) is the indicator function of \( \Omega_k. \) Letting \( k \to \infty, \) we get \( C_5 = \infty, \) which contradicts (3). So, we must have \( \tau_\infty = \infty \) a.s.

3. STOCHASTIC PERMANENCE

Theorem 2.1 shows that the solution of System (1) will remain in the positive cone \( \mathbb{R}_{+}^{2n} \) with probability 1. To discuss how the solution varies in \( \mathbb{R}_{+}^{2n}, \) first we recall the definition of stochastically ultimate boundedness and stochastic permanence.

**Definition 3.1** The solutions of system (1) are said to be stochastically ultimately bounded, if for any \( \varepsilon \in [0,1], \) there is a positive constant \( \gamma(=\gamma(\varepsilon)), \) such that for any initial value \( X(0) \in \mathbb{R}_{+}^{2n}, \) the solution of system (1) has the property that.
\limsup_{t \to \infty} P\{X(t) > \gamma\} < \varepsilon.

**Definition 3.2** The system (1) is said to be stochastically permanent, if for any \( \varepsilon \in [0, 1] \), there exists a pair of positive constants \( \delta(\varepsilon) \) and \( \gamma(\varepsilon) \) such that the solution of (1) with any initial value \( X(0) \in \mathbb{R}_+^n \), has the property that

\[
\liminf_{t \to \infty} P\{X(t) \geq \delta\} \geq 1 - \varepsilon, \quad \liminf_{t \to \infty} P\{X(t) \leq \gamma\} \geq 1 - \varepsilon.
\]

We start with a technical lemma

**Lemma 3.3** For any initial value \((x(0), y(0)) \in \mathbb{R}_+^n\), the solution of the system (1) satisfies

\[
E[\sum_{i=1}^n (x_i(t))^p + y_i(t)^p] \leq 2n \left(\frac{\beta}{p}\right)^p, \quad p > 1.
\]

**Proof.** Applying Itô’s formula, we get

\[
d(x_i^p) = p \left( x_i^p \left( 1 - x_i - \frac{a_iy_i}{k_{i1}^*x_i} + \sum_{j=1}^n c_{ij}(x_j - x_i) \right) + \sum_{j=1}^n c_{ij}(x_j - x_i) \right) dt + p \sigma_{1i} x_i^p dW_{i1}(t)
\]

\[
= p \left( x_i^p \left( 1 - x_i - \frac{a_iy_i}{k_{i1}^*x_i} + \sum_{j=1}^n c_{ij}(x_j - x_i) \right) + \sum_{j=1}^n c_{ij}(x_j - x_i) \right) dt + p \sigma_{1i} x_i^p dW_{i1}(t)
\]

\[
\leq p \left( x_i^p \left( 1 - x_i + \frac{p-1}{2} \sigma_{i1}^2 - \sum_{j=1}^n c_{ij} \right) + \sum_{j=1}^n c_{ij}(x_j - x_i) \right) dt + p \sigma_{1i} x_i^p dW_{i1}(t)
\]

\[
d(y_i^p) = p y_i^p \left( b_1 \left( 1 - y_i \right) + \frac{p-1}{2} \sigma_{2i}^2 \right) dt + p \sigma_{2i} y_i^p dW_{2i}(t)
\]

then

\[
d(\sum_{i=1}^n (x_i^p + y_i^p)) \leq p \sum_{i=1}^n \left[ x_i^p \left( 1 - x_i + \frac{p-1}{2} \sigma_{i1}^2 + \sum_{j=1}^n c_{ij} \right) + y_i^p \left( b_1 \left( 1 - y_i \right) + \frac{p-1}{2} \sigma_{2i}^2 \right) \right] dt + p \sum_{i=1}^n \sigma_{1i} x_i^p dW_{i1}(t) + p \sum_{i=1}^n \sigma_{2i} y_i^p dW_{2i}(t)
\]

Let \( \beta_1 = p \left( 1 + \frac{p-1}{2} \sigma_{i1}^2 + \frac{2}{p} \sum_{j=1}^n c_{ij} \right), \quad \beta_2 = p \left( b_1 + \frac{p-1}{2} \sigma_{2i}^2 \right), \quad \alpha = \min(p, \frac{b_1}{k_{2i}}), \quad \beta = \max(\beta_1, \beta_2), \) then,

\[
d(\sum_{i=1}^n (x_i^p + y_i^p)) \leq \left[ -\alpha \sum_{i=1}^n (x_i^{p+1} + y_i^{p+1}) + \beta \sum_{i=1}^n (x_i^p + y_i^p) \right] dt + p \sum_{i=1}^n \sigma_{1i} x_i^p dW_{i1}(t) + p \sum_{i=1}^n \sigma_{2i} y_i^p dW_{2i}(t).
\]
Integrating both sides from 0 to $t$, and taking expectations, we get

$$\frac{d}{dt} \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right] \leq -\alpha \mathbb{E}\left[\sum_{i=1}^{n}(x_i^{p+1} + y_i^{p+1})\right] + \beta \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right]$$

$$\leq -(2n) \frac{1}{\beta} \alpha \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right]^{1+p} + \beta \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right]$$

$$\leq \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right] \left( \beta - \frac{\alpha}{(2n)^p} \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right]^{1/p} \right),$$

Let $Z(t) = \mathbb{E}\left[\sum_{i=1}^{n}(x_i^p + y_i^p)\right]$, so we have

$$\frac{dZ}{dt} \leq Z \left( \beta - \frac{\alpha}{(2n)^p} Z^{1/p} \right).$$

Since the solution of equation $\frac{dZ}{dt} = Z \left( \beta - \frac{\alpha}{(2n)^p} Z^{1/p} \right)$ tends to $2n \left( \frac{\beta}{\alpha} \right)^p$, as $t \to \infty$, the Comparison Theorem, we get

$$\limsup_{t \to \infty} \mathbb{E}\left[\sum_{i=1}^{n}(x_i(t)^p + y_i(t)^p)\right] \leq 2n \left( \frac{\beta}{\alpha} \right)^p.$$ 

By Chebyshev’s inequality and Lemma 3.3, the following result is straightforward.

**Theorem 3.4** The solutions of System (1) are stochastically ultimately bounded.

Now, we impose a hypothesis,

**Assumption 3.1** $\alpha_1 = \max_{1 \leq i \leq n} \{a_i, b_i, c_i\} > \frac{\sigma}{2}$

**Lemma 3.5** Under Assumption 3.1, for any initial value $X_0 = (x_0, y_0) = ((x_{01}, y_{01}), ..., (x_{0n}, y_{0n})) \in \mathbb{R}^{2n}$, the solution $X(t) = ((x_1(t), y_1(t)), ..., (x_n(t), y_n(t)))$ satisfies that

$$\limsup_{t \to \infty} \mathbb{E}\left[\frac{1}{|X(t)|^p}\right] \leq H,$$

where $\theta$ is an arbitrary positive constant satisfying

$$\alpha_1 > (\theta + 1) \frac{\sigma^2}{2}, \quad (4)$$

and

$$H = n^\theta \left(a_2 + 4ka_1\right) \max \left\{1, \left(\frac{2a_1 + a_2 + \sqrt{a_2^2 + 4a_1a_2}}{2a_1}\right)^{\theta - 2}\right\},$$

in which $k$ is an arbitrary positive constant satisfying

$$0 < \frac{k}{\theta} < \alpha_1 - \frac{\sigma^2 \theta + 1}{2}. \quad (5)$$
where

\[ a_1 = a_1 - \sigma^2 \theta + \frac{1}{2} - \frac{k}{\theta} > 0, \quad a_2 = a_1 + \sigma^2 + \frac{2k}{\theta} > 0. \]

**Proof.** Let us define \( V(X) = \sum_{i=1}^{n}(x_i + y_i) \) for \( X \in \mathbb{R}^{2n}_+ \), and

\[
U(X) = \frac{1}{V(X(t))} \text{ on } t \geq 0.
\]

By the Itô formula, we get

\[
dU(t) = -U(t)^2 \sum_{i} \left( x_i \left( 1 - x_i - \frac{a_i y_i}{k_{x_i} + x_i} \right) + \sum_{j} c_{ij} (x_j - x_i) + b_i y_i \left( 1 - \frac{y_i}{k_{x_i} + x_i} \right) \right) dt
\]

\[
+ \sigma_1 x_i dW_1(t) + \sigma_2 y_1 dW_2(t) + U(t)^3 \sum_{i=1}^{n} \left[ (\sigma_1 x_i)^2 + (\sigma_2 y_i)^2 \right] dt
\]

\[
= LV(t) dt - U(t)^2 \sum_{i=1}^{n} \left[ \sigma_1 x_i dW_1(t) + \sigma_2 y_i dW_2(t) \right],
\]

where

\[
LU(t) = -U(t)^2 \sum_{i} \left( x_i \left( 1 - x_i - \frac{a_i y_i}{k_{x_i} + x_i} \right) + \sum_{j} c_{ij} (x_j - x_i) + b_i y_i \left( 1 - \frac{y_i}{k_{x_i} + x_i} \right) \right)
\]

\[
+ U(t)^3 \sum_{i=1}^{n} \left[ (\sigma_1 x_i)^2 + (\sigma_2 y_i)^2 \right].
\]

Under Assumption 1.3, we can choose a positive constant \( \theta \) such that it obeys (4). By the Itô formula again, we have

\[
d[(1 + U(t)^\theta)] = (-\theta (1 + U(t))^{\theta-1} U(t)^2 \sum_{i=1}^{n} \left( x_i \left( 1 - x_i - \frac{a_i y_i}{k_{x_i} + x_i} \right) 
\right.
\]

\[
+ \sum_{j} c_{ij} (x_j - x_i) + b_i y_i \left( 1 - \frac{y_i}{k_{x_i} + x_i} \right) + (\theta (1 + U(t))^{\theta-1} U(t)^3
\]

\[
+ \frac{\theta(\theta-1)}{2} (1 + U(t))^{\theta-2} U(t)^4 \sum_{i=1}^{n} \left[ (\sigma_1 x_i)^2 + (\sigma_2 y_i)^2 \right] \right) dt
\]

\[
- \theta (1 + U(t))^{\theta-1} U(t)^2 \sum_{i=1}^{n} \left[ \sigma_1 x_i dW_1(t) + \sigma_2 y_i dW_2(t) \right].
\]

Now, choose \( k > 0 \) sufficiently small such that it satisfies (5). Thus, by the Itô formula, we get

\[
d[e^{kt}(1 + U(t)^\theta)] = ke^{kt}(1 + U(t))^\theta + e^{kt} d(1 + U(t))^\theta
\]

\[
= e^{kt} d(1 + U(t))^{\theta-2} \left[ (k(1 + U(t))^2 + J(t)) \right] dt
\]

\[
- \theta (1 + U(t))^{\theta-1} U(t)^2 \sum_{i=1}^{n} \left[ \sigma_1 x_i dW_1(t) + \sigma_2 y_i dW_2(t) \right],
\]

where
\[ J(t) = -\theta (1 + U(t))^{\theta - 1} U(t)^2 \sum_{i=1}^{n} \left( x_i \left( 1 - x_i - \frac{a_i y_i}{k_{i1} + x_i} \right) + \sum_j c_{ij} (x_j - x_i) + b_i y_i \left( 1 - \frac{y_i}{k_{2i} + x_i} \right) \right) + \left( \theta (1 + U(t))^{\theta - 1} U(t)^3 \right) + \begin{array}{c} \theta (\theta - 1) \\ 2 \end{array} \left( 1 + U(t) \right)^{\theta - 2} U(t)^4 \right) \sum_{i=1}^{n} \left[ (\sigma_{1i} x_i)^2 + (\sigma_{2i} y_i)^2 \right]. \]

We thus obtain
\[ J(t) \leq -\theta \left( \alpha_1 - \tilde{\sigma}^2 \frac{\theta + 1}{2} \right) U(t)^2 + \theta \left( \alpha_1 + \tilde{\sigma}^2 \right) U(t), \]
where \( \alpha_1 \) has been defined in the statement of the lemma. Substituting this into (6) yields
\[ d \left[ e^{\kappa t} (1 + U(t))^\theta \right] \leq e^{\kappa t} (1 + U(t))^{\theta - 2} \left( k (1 + U(t))^2 - \theta \left( \alpha_1 - \tilde{\sigma}^2 \frac{\theta + 1}{2} \right) U(t)^2 \right) + \theta \left( \alpha_1 + \tilde{\sigma}^2 - \frac{2k}{\theta} \right) U(t) + k \right) dt \]
\[ \leq e^{\kappa t} (1 + U(t))^{\theta - 2} \left( -\theta \left( \alpha_1 - \tilde{\sigma}^2 \frac{\theta + 1}{2} - \frac{k}{\theta} \right) U(t)^2 \right) + \theta \left( \alpha_1 + \tilde{\sigma}^2 - \frac{2k}{\theta} \right) \left( U(t) + k \right) \leq H_1, \]
on \( U(t) > 0 \), where
\[ H_1 = \frac{a_2 + 4k a_1}{4 k a_1} \max \left\{ 1, \left( \frac{2a_1 + a_2 + \sqrt{a_2^2 + 4a_1 a_2}}{2a_1} \right)^{\theta - 2} \right\}, \]
and \( \alpha_1, a_2 \) have been defined in the lemma. Thus
\[ d \left[ e^{\kappa t} (1 + U(t))^\theta \right] \leq H_1 e^{\kappa t} dt - \theta e^{\kappa t} (1 + U(t))^{\theta - 1} U(t)^2 \sum_{i=1}^{n} \sigma_{1i} x_i dW_{1i}(t) + \sigma_{2i} y_i dW_{2i}(t). \]
This implies
\[ E \left[ e^{\kappa t} (1 + U(t))^\theta \right] \leq (1 + U(t))^\theta + \frac{H_1}{k} e^{\kappa t}. \]
Then
\[ \limsup_{t \to \infty} E \left[ U(t)^\theta \right] \leq \limsup_{t \to \infty} \left[ (1 + \theta(t))^\theta \right] \leq \frac{H_1}{k}. \]
For $X \in \mathbb{R}^{2n}_+$, note that
\[
(\sum_{i=1}^{2n}(x_i + y_i))^\theta \leq (\max_{1 \leq i \leq 2n}(x_i + y_i))^\theta \leq n^\theta (\max_{1 \leq i \leq 2n}(x_i + y_i))^\theta \leq n^\theta |X|^\theta.
\]
Consequently,
\[
\text{Limsup}_{t \to \infty} E \left[ \frac{1}{|X(t)|^\theta} \right] \leq n^\theta \frac{H}{k} \leq H.
\]

**Theorem 3.6** Under condition (4), the System (1) is stochastically permanent.

**Proof.** Let $X(t)$ be the solution of System (1) with any given positive initial value $X(0) \in \mathbb{R}^{2n}_+$. By Lemma 3.5, we have
\[
\text{Limsup}_{t \to \infty} E \left[ \frac{1}{|X(t)|^\theta} \right] \leq H.
\]
For $X(t) \in \mathbb{R}^{2n}_+$ and for any $\epsilon > 0$, let $\delta = \left(\frac{\epsilon}{H}\right)^{\frac{1}{\theta}}$, we get the following
\[
P\{X(t) < \delta\} = P\left\{ \frac{1}{(X(t))^\theta} < \frac{1}{\delta^\theta} \right\}
\leq \frac{E \left[ \frac{1}{|X(t)|^\theta} \right]}{\delta^\theta}
\leq \delta^\theta H = \epsilon.
\]
Hence
\[
\text{Limsup}_{t \to \infty} P\{X(t) < \delta\} \leq \epsilon,
\]
and this implies
\[
\text{Limsup}_{t \to \infty} P\{X(t) \geq \delta\} \geq 1 - \epsilon.
\]
The other property of Definition 3.2 follows from Theorem 3.4. 

4. **STATIONARY DISTRIBUTION**

In this section, we investigate that there is a stationary distribution for System (1) instead of asymptotically stable equilibria and list some results on the stationary distribution (see Khasminskii [[4], pp.106-125]). Let $X(t)$ be a homogeneous Markov process in $E_i$ ($E_i$ denotes $l$-space) described by the stochastic equation
\[
dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t)
\]
The diffusion matrix is
\[
A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).
\]

**Assumption 4.1** There exists a bounded domain $U \subset E_i$ with regular boundary $\Gamma$, having the following
properties.

(i) In the domain \( U \) and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix \( A(x) \) is bounded away from zero.

(ii) If \( x \in E_1 \setminus U \), the mean time \( \tau \) for which a path starting at \( x \) and reaching the set \( U \) is finite, and \( \sup_{x \in K} E_x \tau < \infty \) for every compact subset \( K \subset E_1 \).

**Lemma 4.1** (see [4]). If Assumption 4.1 holds, then the Markov process \( X(t) \) has a stationary distribution \( \mu(A) \). Let \( f(\cdot) \) be a function integrable with respect to the measure \( \mu \). Then

\[
P_x \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(X(t)) dt = \int_{E_1} f(x) \mu(dx) \right\} = 1
\]

for all \( x \in E_1 \).

**Theorem 4.2** Let \( M_1 = -k_{1i} + \frac{\alpha y_i}{k_{1i}} - \frac{\alpha_i}{z} + \frac{\gamma_i}{2k_{2i}} \), \( M_2 = -1 - \frac{\alpha_i}{z} + \frac{\gamma_i}{2k_{2i}} \). Assume that \( M_1 < 0 \) and \( M_2 < 0 \). Then there is a stationary distribution \( \mu(A) \) for System (1).

**Proof.** Let us set

\[
V_i(x_i, y_i) = (k_{1i} + x_i) \left( x_i - x_i^* - \log \frac{x_i}{x_i^*} \right) + \frac{k_{2i} + x_i^*}{b_i} \left( y_i - y_i^* - \log \frac{y_i}{y_i^*} \right).
\]

By Itô’s formula, we have

\[
dV_i = \left( (k_{1i} + x_i) \left( x_i - x_i^* \right) \left( 1 - x_i - \frac{\alpha y_i}{k_{1i} + x_i} + \sum_j^m c_{ij} \left( \frac{x_j}{x_i} - 1 \right) \right) + \frac{x_i^* \sigma_i^2}{z} \right)
\]

\[
+ \left( (k_{2i} + x_i^*) \left( y_i - y_i^* \right) \left( 1 - \frac{\gamma_i}{k_{2i} + x_i^*} + \frac{\gamma_i \sigma_i^2}{z} \right) \right) dt
\]

\[
+ (k_{1i} + x_i^*) (x_i - x_i^*) \sigma_{2i} dW_{1i}(t) + (k_{2i} + x_i^*) (y_i - y_i^*) \sigma_{2i} dW_{2i}(t)
\]

\[
= LV_i dt + (k_{1i} + x_i^*) (x_i - x_i^*) \sigma_{2i} dW_{1i}(t) + (k_{2i} + x_i^*) (y_i - y_i^*) \sigma_{2i} dW_{2i}(t),
\]

where

\[
LV_i = (k_{1i} + x_i^*) (x_i - x_i^*) \left( -x_i - \frac{\alpha y_i}{k_{1i} + x_i} + \sum_j^m c_{ij} \left( \frac{x_j}{x_i} - \frac{x_j^*}{x_i^*} \right) + \frac{\gamma_i \sigma_i^2}{z} \right)
\]

\[
+ (k_{1i} + x_i^*) \frac{x_i^* \sigma_i^2}{z} + (k_{2i} + x_i^*) \left[ (y_i - y_i^*) \left( \frac{y_i^*}{k_{2i} + x_i^*} - \frac{y_i}{k_{2i} + x_i} \right) + \frac{\gamma_i \sigma_i^2}{z} \right]
\]

\[
= \left( -k_{1i} - x_i^* + \frac{\alpha y_i}{k_{1i} + x_i} \right) (x_i - x_i^*)^2 - (y_i - y_i^*)^2
\]

\[
+ \left( \alpha_i + \frac{\gamma_i}{k_{2i} + x_i^*} \right) (x_i - x_i^*) (y_i - y_i^*) + (k_{1i} + x_i^*) \sum_j^m c_{ij} \left( \frac{x_j}{x_i} - \frac{x_j^*}{x_i^*} \right)
\]

\[
+ (k_{1i} + x_i^*) \frac{x_i^* \sigma_i^2}{z} + (k_{2i} + x_i^*) \frac{\gamma_i \sigma_i^2}{z}.
\]

The positivity of \( x_i(t) \) and Cauchy-Schwarz’s inequality imply
$LV_i \leq \left(-k_{1i} - \frac{a_i y_{1i}}{k_{2i}} + \frac{a_i}{2} + \frac{y_{1i}}{2k_{2i}}\right)(x_i - x_i^*)^2 + \left(-1 - \frac{a_i}{2} + \frac{y_{1i}}{2k_{2i}}\right)(y_i - y_i^*)^2$

$+ (k_{1i} + x_i^*) \sum_j c_{ij} (x_i - x_j^*) \left(\frac{x_j^i - x_j^i}{x_i^i} + (k_{1i} + x_i^*) \frac{x_j^i}{x_i^i} \right) + (k_{2i} + x_i^*) \frac{y_i^2}{y_i^2}$

$\leq -\left(k_{1i} - \frac{a_i y_{1i}}{k_{2i}} + \frac{a_i}{2} - \frac{y_{1i}}{2k_{2i}}\right)(x_i - x_i^*)^2 - \left(1 + \frac{a_i}{2} - \frac{y_{1i}}{2k_{2i}}\right)(y_i - y_i^*)^2$

$+ (k_{1i} + x_i^*) \sum_j c_{ij} x_j^* \left(-\left(-\frac{x_j}{x_j^*} + \log \frac{x_j}{x_j^*}\right) + \left(-\frac{x_j}{x_j^*} + \log \frac{x_j}{x_j^*}\right)\right)$

$+ (k_{1i} + x_i^*) \frac{x_i^2}{x_i^2} + (k_{2i} + x_i^*) \frac{y_i^2}{y_i^2}$

From [6, Theorem 2.3], we have

$\sum_j c_{ij} x_j^* \left(-\frac{x_j}{x_j^*} + \log \frac{x_j}{x_j^*}\right) = \sum_j c_{ij} x_j^* \left(-\frac{x_j}{x_j^*} + \log \frac{x_j}{x_j^*}\right)$

So

$LV_i \leq -\left(k_{1i} - \frac{a_i y_{1i}}{k_{2i}} + \frac{a_i}{2} - \frac{y_{1i}}{2k_{2i}}\right)(x_i - x_i^*)^2 - \left(1 + \frac{a_i}{2} - \frac{y_{1i}}{2k_{2i}}\right)(y_i - y_i^*)^2$

$+ (k_{1i} + x_i^*) \frac{x_i^2}{x_i^2} + (k_{2i} + x_i^*) \frac{y_i^2}{y_i^2}$

Let $\lambda = (k_{1i} + x_i^*) \frac{x_i^2}{x_i^2} + (k_{2i} + x_i^*) \frac{y_i^2}{y_i^2}$, $M_1 = -k_{1i} + \frac{a_i y_{1i}}{k_{2i}} - \frac{a_i}{2} + \frac{y_{1i}}{2k_{2i}}$, and $M_2 = -1 - \frac{a_i}{2} + \frac{y_{1i}}{2k_{2i}}$

We have

$LV_i \leq M_1(x_i - x_i^*)^2 + M_2(y_i - y_i^*)^2 + \lambda$

Now, if $\lambda$ satisfies,

$\lambda \leq \min\{M_1(x_i^*)^2, M_2(y_i^*)^2\}$,

then the ellipsoid

$M_1(x_i - x_i^*)^2 + M_2(y_i - y_i^*)^2 + \lambda = 0$

lies entirely in $\mathbb{R}^n_+$. We can take $U$ to be any neighborhood of the ellipsoid with $\bar{U} \subseteq E_i = \mathbb{R}^n_+$, so for
We numerically simulate the solution of System (7). By the Milstein scheme mentioned in [5, p345], which turns out to be an order 1.0 strong Taylor scheme, we consider the following discretized system:

\[ x_{1,k+1} = x_{1,k} + \left[ x_{1,k} \left( 1 - x_{1,k} - \frac{a_1 y_{1,k}}{k_{11} + x_{1,k}} \right) + c_{12} (x_{2,k} - x_{1,k}) \right] h + \sigma_{11} x_{1,k} \sqrt{h} \xi_{1,k} + \frac{1}{2} \sigma_{11}^2 x_{1,k} \left( h \xi_{1,k}^2 - h \right), \]
\[ y_{1,k+1} = y_{1,k} + b_1 y_{1,k} \left( 1 - \frac{y_{1,k}}{k_{21} + x_{1,k}} \right) h + \sigma_{21} y_{1,k} \sqrt{h} \xi_{2,k} + \frac{1}{2} \sigma_{21}^2 y_{1,k} \left( h \xi_{2,k}^2 - h \right), \]
\[ x_{2,k+1} = x_{2,k} + \left[ x_{2,k} \left( 1 - x_{2,k} - \frac{a_2 y_{2,k}}{k_{12} + x_{2,k}} \right) + c_{21} (x_{1,k} - x_{2,k}) \right] h + \sigma_{12} x_{2,k} \sqrt{h} \xi_{3,k} + \frac{1}{2} \sigma_{12}^2 x_{2,k} \left( h \xi_{3,k}^2 - h \right), \]
\[ y_{2,k+1} = y_{2,k} + b_2 y_{2,k} \left( 1 - \frac{y_{2,k}}{k_{22} + x_{2,k}} \right) h + \sigma_{22} y_{2,k} \sqrt{h} \xi_{4,k} + \frac{1}{2} \sigma_{22}^2 y_{2,k} \left( h \xi_{4,k}^2 - h \right). \]

For the numerical simulations, we choose \( a_1 = 0.4, k_{11} = 0.08, k_{21} = 0.2, b_1 = 0.1, a_2 = 0.5, k_{12} = 0.4, k_{22} = 0.25, b_2 = 1, \) and the time step \( h = 0.01. \) In Figure 1, we assume that \( c_{12} = 1.5, c_{21} = 1.6, \) with \( \sigma_{11} = 0.3, \sigma_{21} = 0.01, \) and \( \sigma_{22} = 0.01. \) The initial value \( (x_1(0), y_1(0)) = ((0.55, 0, 6), (0.5, 0, 61)). \) In this case, the deterministic model has a globally stable equilibrium point \((x^*, y^*) = (x_1^*, y_1^*), (x_2^*, y_2^*)) = ((0.55, 0.75), (0.576, 0.825)). \) Obviously, Assumption 3.1 holds the system (7) is stochastically permanent. Also, the conditions of Theorem (4.2) are satisfied, so there is a stationary distribution. In Figure 2, we choose \( \sigma_{11} = 0.3, \sigma_{21} = 0.2, \) and \( \sigma_{21} = 0.01, \sigma_{22} = 0.2. \) The populations of \((x_1, y_1)\) and \((x_2, y_2)\) suffer relatively large white noise.
Figure 1. Trajectories of the solutions of stochastic system (7) represented by the blue curves and the corresponding deterministic system represented by the red lines. The stochastic system is stochastically permanent.

Figure 2. Trajectories of the solutions of stochastic system (7) and the corresponding deterministic system, the blue lines and the red lines represent them, respectively. $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, and $\sigma_{21} = 0.1$, $\sigma_{22} = 0.3$.

By comparing Figure 1, we can see that in Figure 2 the curves fluctuations are larger. In Figure 3, we choose $c_{12} = 0$, $c_{21} = 0.01$ and $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, and $\sigma_{21} = 0.1$, $\sigma_{22} = 0.3$. Figure 2 and 3 have the same white noise intensity but have different diffusion coefficients, because there is no diffusion effects, we can see that $(x_1, y_1)$ will die.
Because there is no diffusion effects, we can see that $(x_1, y_1)$ will die; $c_{12} = 0$, $c_{21} = 0.01$ and $\sigma_{11} = 0.3$, $\sigma_{21} = 0.2$, and $\sigma_{22} = 0.1$, $\sigma_{22} = 0.3$

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES


