



A New Distribution Family Constructed By Polynomial Rank Transmutation

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Abstract

In this study, a new polynomial rank transmutation is proposed with the help of the bivariate Farlie-Gumbel Morgenstern distribution family. The distribution family obtained by this transmutation is considered to be an alternative to the distribution families obtained by quadratic rank transmutation (QRT). Various properties of the introduced family are studied. Two real data sets are taken into account to show that this family is an alternative to the QRT distribution family.

1. INTRODUCTION

Numerous studies have been conducted by many authors using the quadratic rank transformation proposed by [14]. However, many other distributions have been derived with the help of families such as Marshall-Olkin generated family (MO-G) by [10], beta generated family by [7], transformed-transformer family by [2] and Weibull-G by [5]. The overall aim is to show the modelling performance of the generated distributions on real-data.

We present a transformation, similar to the polynomial rank transformation proposed by [14], using both the convex combinations of the distributions of order statistics and the conditional bivariate Farlie-Gumbel-Morgenstern distribution. We discuss details in the motivation section.

2. MOTIVATION

Quadratic rank transmutation introduced by [14] in subsection 4.2, leads to the occurrence of numerous studies. We can express the idea behind this definition as follows: Let us consider two-component systems (series and parallel) where component lifetimes are identically distributed as F . Then failure distributions of the lifetimes of series and parallel systems are $2F - F^2$ and F^2 , respectively. Note that failure distribution of component lifetime, F lies between these two distribution functions. Namely, the ordering amongst three distribution functions is $F^2 \leq F \leq 2F - F^2$. Accordingly, F can be represented by a convex combination of $2F - F^2$ and F^2 as $\delta(2F - F^2) + (1 - \delta)F^2$, where δ is a convex combination

parameter belonging to interval $[0,1]$. In particular, F is obtained if the delta is taken as $\frac{1}{2}$. In addition, it

is possible to obtain numerous distribution functions. Let us call the distribution function H obtained by this convex transformation. Then H can be represented by the following form:

$$H = 2\delta F - (2\delta - 1)F^2.$$

By re-parameterizing $2\delta-1=\lambda$ where $\lambda \in [-1,1]$, H becomes $(1+\lambda)F-\lambda F^2$. Thus, we achieve the quadratic rank transmutation given in equation (48) by [14]. Now, let us focus on the polynomial structure given in equation (66) by [14]. This definition may overlap with the method of [9]. The idea of our study is based on the method which was suggested by [9] for obtaining positively quadrant dependent bivariate distribution family. Accordingly, let $F(x)$ be a distribution function, and $\psi(x)$ be a continuous function defined on the same support. Then we define a function $H(x)$ as

$$H(x) = F(x) + \psi(x) \quad (1)$$

The function $\psi(x)$ must satisfy the following conditions in order to say that $H(x)$ is a distribution function:

- (i) $\lim_{x \rightarrow -\infty} \psi(x) = 0$ and $\lim_{x \rightarrow \infty} \psi(x) = 0$
- (ii) $\frac{d}{dx} F(x) + \frac{d}{dx} \psi(x) \geq 0$

For example, $\psi(x)$ is $\theta F(x)(1-F(x))$ where θ takes the values in the interval $[-1,1]$. In addition, [16] have proposed two new distribution families by considering specific terms for $\psi(x)$. In this case, $H(x)$ indicates the transmuted distribution proposed by [14]. In this study, we are transformed the bivariate Farlie-Gumbel Morgenstern distribution into a univariate distribution by using conditional distribution. In the light of the information from [14] and [9], we call this new transmuted distribution family as Polynomial Transmuted Distribution family (PT-D).

3. CONSTRUCTION OF THE NEW FAMILY OF DISTRIBUTION

Let us consider bivariate Farlie Gumbel Morgenstern Family of distribution

$$H(x, y) = F(x)G(y) \left[1 + \lambda \bar{F}(x)\bar{G}(y) \right]$$

where, F and G are the marginal distribution functions and $\bar{F} = 1-F$ and $\bar{G} = 1-G$ are the marginal survival functions of X and Y , respectively and λ is an association parameter lies interval $[-1,1]$, [8]. Suppose that second component is still working at time t , *i.e.* $Y > t$ is given, hence conditional probability of X that failures before the time t can be given by

$$\Pr(X \leq t | Y > t) = \frac{F(t) - F(t)G(t) \left[1 + \lambda \bar{F}(t)\bar{G}(t) \right]}{\bar{G}(t)} = F(t) \left[1 + \lambda \bar{F}(t)G(t) \right]. \quad (2)$$

Now, suppose also that lifetimes of two components are identical, namely $F = G$. In this case, the distribution given by eq. (2) can be transformed to the univariate case. Let this conditional distribution is represented by $H(t)$. Then $H(t)$ can be rewritten in following form

$$\begin{aligned} H(t) &= F(t) \left[1 + \lambda \bar{F}(t)F(t) \right] = (1+\lambda)F(t) - \lambda F(t) \left[1 - F(t)\bar{F}(t) \right] = (1+\lambda)F(t) - \lambda F(t) \left[1 - F(t) + F^2(t) \right] \\ &= (1+\lambda)F(t) - \lambda (F(t) - F^2(t) + F^3(t)) \end{aligned} \quad (3)$$

where $F(t)$ is the cdf of the base random variable. Observe that for $\lambda = 0$, we have baseline distribution. For $\lambda = -1$ we have the failure distribution of the lifetime of the system which defined by $T = \min\{X_1, \max\{X_2, X_3\}\}$. Here, component lifetimes namely X_1, X_2 and X_3 are independent and distributed as F . For $\lambda = 1$ the cdf indicates that the failure probability of the system switching from three-component series system to three-component parallel system with respective switching probabilities $\frac{1}{3}$ and $\frac{2}{3}$. According to definition of the other rank transmutations described by [14] in Subsection 4.6,

the transmutation given by eq. (3) can only be viewed as the one of the other rank transmuted distributions. Based on this definition, $H(t)$ is called as polynomial transmuted distribution (PT-D). Recalling the thought given in the motivation section, we try to explain $H(t)$ in a different way. Then we have the following ordering such as

$$\left(\frac{1}{3} (3F(t) - 3F^2(t) + F^3(t)) + \frac{2}{3} F^3(t) \right) \leq F(t) \leq F(t) + F^2(t) - F^3(t).$$

Hence, a convex combination representation of cdf $H(t)$ is given by

$$\begin{aligned} H(t) &= \delta (F(t) + F^2(t) - F^3(t)) + (1 - \delta) (F(t) - F^2(t) + F^3(t)) = 2\delta F(t) - (2\delta - 1) (F(t) - F^2(t) + F^3(t)) \\ &= (1 + \lambda) F(t) - \lambda (F(t) - F^2(t) + F^3(t)) = F(t) + \lambda (F^2(t) - F^3(t)) \end{aligned} \quad (4)$$

where $\delta \in [0, 1]$ and $2\delta - 1 = \lambda \in [-1, 1]$. Note that, the latter representation in eq. (4) coincides with eq. (1) by considering $\psi(t) = \lambda (F^2(t) - F^3(t))$.

In this way, a number of more polynomial transmuted distributions can be suggested which may be useful in practice.

4. REPRESENTATIONS OF CDF, PDF AND SURVIVAL FUNCTION OF PT-D

Let X_1, \dots, X_n be a sequence of independent and identically distributed random variables with distribution F . Let $X_{j:n}$ denotes the j th order statistics from a sample of size n and $F_{j:n}$ denotes the distribution of $X_{j:n}$. Note that $\sum_{j=1}^n F_{j:n} = nF$. Then $H(t)$ can be represented by

$$\begin{aligned} H(t) &= (1 + \lambda) F(t) - \lambda (F(t) - F^2(t) + F^3(t)) = (1 + \lambda) F(t) - \frac{\lambda}{3} (F_{1:3}(t) + 2F_{3:3}(t)) \\ &= F(t) + \frac{\lambda}{3} (F_{2:3}(t) - F_{3:3}(t)). \end{aligned} \quad (5)$$

Hence, the corresponding pdf is given by

$$\begin{aligned} h(t) &= f(t) [1 + \lambda F(t) (2 - 3F(t))] = (1 + \lambda) f(t) - \frac{\lambda}{3} (f_{1:3}(t) + 2f_{3:3}(t)) \\ &= f(t) + \frac{\lambda}{3} (f_{2:3}(t) - f_{3:3}(t)). \end{aligned} \quad (6)$$

According to eq. (4), survival function of PT-random variable T is given by

$$\begin{aligned} \bar{H}(t) &= \bar{F}(t) [1 - \lambda F^2(t)] = (1 + \lambda) \bar{F}(t) - \frac{\lambda}{3} (\bar{F}_{1:3}(t) + 2\bar{F}_{3:3}(t)) \\ &= \bar{F}(t) + \frac{\lambda}{3} (\bar{F}_{2:3}(t) - \bar{F}_{3:3}(t)). \end{aligned} \quad (7)$$

5. MOMENT GENERATING FUNCTION AND RAW MOMENTS OF A PT RANDOM VARIABLE

In this section, the moment generating function and k th raw moments of PT-random variable are presented. The moment generating function of PT-random variable is given as

$$M_T(v) = (1 + \lambda)M_X(v) - \frac{\lambda}{3} [M_{X_{1:3}}(v) + 2M_{X_{3:3}}(v)] = M_X(v) + \frac{\lambda}{3} [M_{X_{2:3}}(v) - M_{X_{3:3}}(v)], \quad (8)$$

and k th raw moments of PT-random variable is given as

$$E[T^k] = (1 + \lambda)E[X^k] - \frac{\lambda}{3} [E[X_{1:3}^k] + 2E[X_{3:3}^k]] = E[X^k] + \frac{\lambda}{3} [E[X_{2:3}^k] - E[X_{3:3}^k]]. \quad (9)$$

6. HAZARD RATE FUNCTION OF PT RANDOM VARIABLE

Besides the monotonically increasing or decreasing hazard rates, the bathtub, the reverse-bathtub, W-shaped, N-shaped, etc. hazard rates are also important for modeling lifetime data sets.

Additional parameters of the newly defined family of distributions can give different shape to their hazard rate functions. In this section, the hazard rate function for the PT-D family is defined and then compared with the hazard rate function of the transmuted distribution family. The monotonicity property of the hazard rate function is then examined according to the parameter λ .

From the eq. (7) hazard rate function denoted by $r(t)$ can be defined by

$$r(t) = \frac{-d \log(\bar{H}(t))}{dt} = r_F(t) + \frac{2\lambda F(t)f(t)}{1 - \lambda F^2(t)} = r_F(t) \left[1 + 2\lambda \frac{F(t)\bar{F}(t)}{1 - \lambda F^2(t)} \right] \quad (10)$$

The subsequent theorems are given in order to be able to examine the monotonicity of the hazard rate function according to the hazard rate function of the base distribution:

Theorem1: Let $\varphi(u, \lambda) = \frac{-1 + \lambda u}{1 - \lambda u^2}$ be a continuous function defined on a set $\{(u, \lambda) : u \in [0, 1], \lambda \in [-1, 1]\}$.

Then it satisfies the following properties:

- i. $\varphi(u, \lambda)$ increases in λ ,
- ii. $\varphi(u, \lambda) = -1$ at $\lambda = 0$,
- iii. $\varphi(u, \lambda)$ increases in u at $\lambda = 1$,
- iv. $\frac{-1}{2}(1 + \sqrt{1 - \lambda}) \leq \varphi(u, \lambda) \leq -1$ if $\lambda \leq 0$, and $-1 \leq \varphi(u, \lambda) \leq \frac{-1}{2}(1 + \sqrt{1 - \lambda})$ if $\lambda \geq 0$.

Proof. (i) Since $\frac{\partial \varphi(u, \lambda)}{\partial \lambda} = \frac{u(1 - u)}{(1 - \lambda u^2)^2} \geq 0$ (i) is satisfied for all λ . (ii) and (iii) are obvious. (iv) Firstly the critical points of $\varphi(u, \lambda)$ are determined as below:

$$\frac{\partial \varphi(u, \lambda)}{\partial u} = \frac{\lambda^2 u^2 - 2\lambda u + \lambda}{(1 - \lambda u^2)^2} = \frac{(1 - \lambda u)^2 - (1 - \lambda)}{(1 - \lambda u^2)^2} = 0.$$

Hence, $u^* = \frac{1 - \sqrt{1 - \lambda}}{\lambda}$ is an appropriate point lies in an interval $[0, 1]$. Furthermore, $\varphi(0, \lambda) = \varphi(1, \lambda) = -1$.

Consider the inequality $1 - \lambda \geq 1$ for $\lambda \leq 0$. Hence, the inequality $\frac{-1}{2}(1 + \sqrt{1 - \lambda}) \leq -1$ holds for $\lambda \leq 0$. This latter inequality means that $\varphi(u^*, \lambda) \leq \varphi(0, \lambda)$. In this case, $\varphi(u, \lambda)$ is concave upward at the point u^* since the endpoints of the function $\varphi(u, \lambda)$ are the same. Therefore the first part is proved.

Now, let rewrite $\varphi(u, \lambda)$ in the following form

$$\varphi(u, \lambda) = \frac{-1}{u + \frac{1-u}{1-\lambda u}}.$$

The quantity $u + \frac{1-u}{1-\lambda u}$ is positive, and concave function of u on $(0,1)$ for $\lambda \geq 0$, so does $\varphi(u, \lambda)$. By noting that $\varphi(0, \lambda) = \varphi(1, \lambda) = -1$, u^* is a maximum point of $\varphi(u, \lambda)$ and maximum value at this point is $\varphi(u^*, \lambda) = \frac{-1}{2}(1 + \sqrt{1-\lambda})$.

Theorem2: Hazard rate function $r(t)$ has the following monotonicity properties.

- i. $r(t)$ increases in λ ,
- ii. If the base distribution F has an IHR (Increasing Hazard Rate) property or has a CFR (Constant Hazard Rate) property, PT-D has an IHR for $\lambda = 1$,
- iii. $(2 - \sqrt{1-\lambda})r_F(t) \leq r(t) \leq r_F(t)$ if $\lambda \leq 0$, and $r_F(t) \leq r(t) \leq (2 - \sqrt{1-\lambda})r_F(t)$ if $\lambda \geq 0$.

Proof. According to definition of $\varphi(u, \lambda)$, hazard rate function can be rewritten as $r(t) = r_F(t)[3 + 2\varphi(F(t), \lambda)]$. Hence proofs of (i)-(iii) are immediately followed from Theorem1.

If the results of [6] are noted, the shape of the hazard rate function of the transmuted random variable depends on the sign of λ . In other words, hazard rate function of transmuted random variable increases for $\lambda \leq 0$, and decreases for $\lambda \geq 0$. However, hazard rate function of PT random variable exhibits this situation only for $\lambda = 1$. Thus, it can be said that the hazard rate function is in somewhat more flexible structure. Because of this, PT-D can be used to model for non-monotonic lifetime data in terms of its hazard rate. As a consequence, PT-D can be useful for modeling life datasets with bathtub, and upside down bathtub shaped hazard rates.

7. GENERATING RANDOM NUMBER FROM PT-D

The number generation process is set-up with convex combination notation given in eq. (4). Hence, it is possible to describe the PT-D as a 3-component mixture distribution, so the number generation is easier.

Step 1: Generate three random numbers from the baseline distribution, namely $X_1, X_2, X_3 \sim F$,

Step 2: Generate a random number U from uniform distribution on $(0,1)$,

Step 3: If $\lambda = 0$, a random number T from F is $F^{-1}(U)$,

Step 4: If $U \leq \left(\frac{1+\lambda}{2}\right)$, a random number T from PT-D is $\min\{X_1, \max\{X_2, X_3\}\}$,

Step 5: If $U \leq \left(\frac{1+\lambda}{2}\right) + \left(\frac{1-\lambda}{2}\right)\left(\frac{1}{3}\right)$, $T = \min\{X_1, X_2, X_3\}$ otherwise, $T = \max\{X_1, X_2, X_3\}$.

In the next section, the specific distributions chosen for this family are considered.

8. SPECIAL CASES: PT-WEIBULL and PT-EXPONENTIAL DISTRIBUTIONS

The Weibull distribution is widely used in reliability and lifetime data analysis due to its flexibility. The values of the shape and the scale parameters effect on distributional characteristics such as the shape of

the pdf curve, the reliability and the hazard rate functions. For this reason, the first specific base distribution is taken as Weibull. The second specific base distribution is considered as exponential distribution since exponential distribution makes the hazard rate function of the PT-D as bathtub or reversed bathtub shaped.

8.1 The Cdf's and Pdf's of PT-Weibull and PT-Exponential Distributions

According to eq. (5), the cumulative distribution function of PT-Weibull (PT-W) can be respectively given by

$$H_{PT-W}(t) = 1 - e^{-\left(\frac{t}{\sigma}\right)^\eta} + \lambda \left(1 - e^{-\left(\frac{t}{\sigma}\right)^\eta} \right)^2 e^{-\left(\frac{t}{\sigma}\right)^\eta} = (1-\lambda)F_{Weib(\sigma,\eta)}(t) + 2\lambda F_{Weib(\sigma/2^\eta,\eta)}(t) - \lambda F_{Weib(\sigma/3^\eta,\eta)}(t). \quad (11)$$

Note that H_{PT-W} can be represented by the weighted sum of Weibull distributions with common shape parameter η and respective scale parameters σ , $\frac{\sigma}{2^\eta}$ and $\frac{\sigma}{3^\eta}$. Similarly, the cdf of PT-E can be represented by the weighted sum of exponential distribution with scale parameters σ , $\frac{\sigma}{2}$ and $\frac{\sigma}{3}$ as follows:

$$H_{PT-E}(t) = 1 - e^{-\frac{t}{\sigma}} + \lambda \left(1 - e^{-\frac{t}{\sigma}} \right)^2 e^{-\frac{t}{\sigma}} = (1-\lambda)F_{Exp(\sigma)}(t) + 2\lambda F_{Exp(\sigma/2)}(t) - \lambda F_{Exp(\sigma/3)}(t). \quad (12)$$

The associated probability density functions of PT-W and PT-E are given by

$$\begin{aligned} h_{PT-W}(t) &= \eta \frac{t^{\eta-1}}{\sigma^\eta} e^{-\left(\frac{t}{\sigma}\right)^\eta} \left[1 - \lambda \left(1 - 4e^{-\left(\frac{t}{\sigma}\right)^\eta} + 3e^{-2\left(\frac{t}{\sigma}\right)^\eta} \right) \right] \\ &= (1-\lambda)\eta \frac{t^{\eta-1}}{\sigma^\eta} e^{-\left(\frac{t}{\sigma}\right)^\eta} + \lambda \left(4\eta \frac{t^{\eta-1}}{\sigma^\eta} e^{-2\left(\frac{t}{\sigma}\right)^\eta} - 3\eta \frac{t^{\eta-1}}{\sigma^\eta} e^{-3\left(\frac{t}{\sigma}\right)^\eta} \right) \\ &= (1-\lambda)f_{Weib(\sigma,\eta)}(t) + 2\lambda f_{Weib(\sigma/2^\eta,\eta)}(t) - \lambda f_{Weib(\sigma/3^\eta,\eta)}(t), \quad \sigma > 0, \eta > 0, \lambda \in [-1,1], \end{aligned} \quad (13)$$

and

$$\begin{aligned} h_{PT-E}(t) &= \frac{1}{\sigma} e^{-\frac{t}{\sigma}} \left[1 - \lambda \left(1 - 4e^{-\frac{t}{\sigma}} + 3e^{-\frac{2t}{\sigma}} \right) \right] \\ &= (1-\lambda)\frac{1}{\sigma} e^{-\frac{t}{\sigma}} + \lambda \left(4\frac{1}{\sigma} e^{-\frac{2t}{\sigma}} - 3\frac{1}{\sigma} e^{-\frac{3t}{\sigma}} \right) \\ &= (1-\lambda)f_{Exp(\sigma)}(t) + 2\lambda f_{Exp(\sigma/2)}(t) - \lambda f_{Exp(\sigma/3)}(t), \quad \sigma > 0, \lambda \in [-1,1]. \end{aligned} \quad (14)$$

Let us now discuss the possible shapes of h_{PT-W} and h_{PT-E} as follows:

$$\lim_{t \rightarrow 0} h_{PT-W}(t) = \lim_{t \rightarrow 0} f_W(t) = \begin{cases} 0, & \eta > 1 \\ \frac{1}{\sigma}, & \eta = 1 \\ \infty, & \eta < 1 \end{cases}, \quad \lim_{t \rightarrow +\infty} h_{PT-W}(t) = (1-\lambda) \lim_{t \rightarrow +\infty} f_W(t) = 0,$$

and

$$\lim_{t \rightarrow 0} h_{PT-E}(t) = \lim_{t \rightarrow 0} f_E(t) = \frac{1}{\sigma}, \quad \lim_{t \rightarrow +\infty} h_{PT-E}(t) = (1-\lambda) \lim_{t \rightarrow +\infty} f_E(t) = 0.$$

In order to visualize the variety of shapes, some plots of the h_{PT-W} and h_{PT-E} are given by Figure 1.

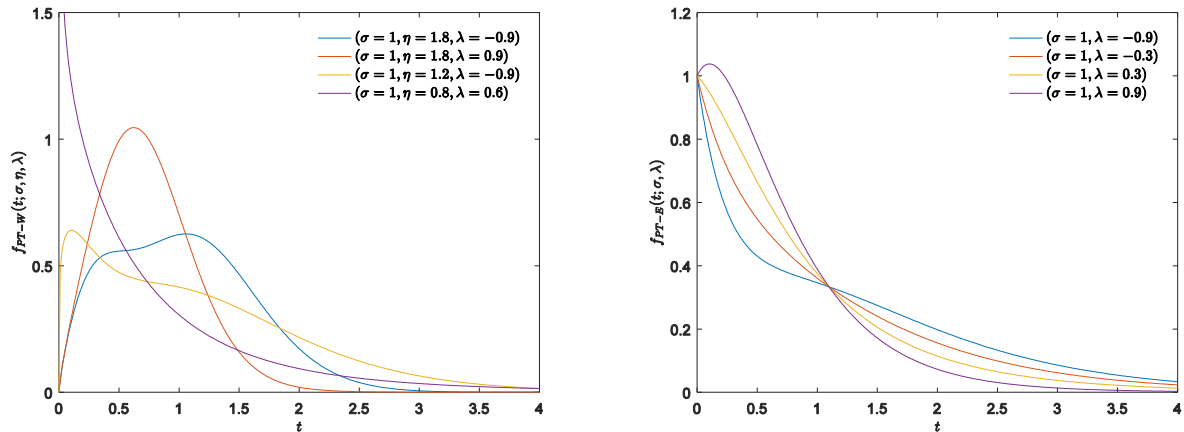


Figure 1. The pdfs' of various PT-W and PT-E distributions.

As can be seen, the PT-W distribution is successful in modeling bimodal data sets. We support this on the illustrative sample dataset in the application section.

8.2 The Survival and Hazard Rate Functions of PT-W and PT-E Distributions

From the eq. (7) and eq. (11)- (12) the survival functions of PT-W and PT-E distributions are respectively given by

$$\bar{H}_{PT-W}(t) = e^{-\left(\frac{t}{\sigma}\right)^\eta} \left[1 - \lambda \left(1 - e^{-\left(\frac{t}{\sigma}\right)^\eta} \right)^2 \right] = (1-\lambda) \bar{F}_{Weib(\sigma, \eta)}(t) + 2\lambda \bar{F}_{Weib(\sigma/2^\eta, \eta)}(t) - \lambda \bar{F}_{Weib(\sigma/3^\eta, \eta)}(t) \tag{15}$$

and

$$\bar{H}_{PT-E}(t) = e^{-\frac{t}{\sigma}} \left[1 - \lambda \left(1 - e^{-\frac{t}{\sigma}} \right)^2 \right] = (1-\lambda) \bar{F}_{Exp(\sigma)}(t) + 2\lambda \bar{F}_{Exp(\sigma/2)}(t) - \lambda \bar{F}_{Exp(\sigma/3)}(t) \tag{16}$$

Hence, hazard rate functions of PT-W and PT-E distributions are obtained from the eq. (10) and the eq. (15)-(16) as follows:

$$r_{PT-W}(t) = \eta \frac{t^{\eta-1}}{\sigma^\eta} \left[1 + 2\lambda \frac{e^{-\left(\frac{t}{\sigma}\right)^\eta} - e^{-2\left(\frac{t}{\sigma}\right)^\eta}}{1 - \lambda \left(1 - e^{-\left(\frac{t}{\sigma}\right)^\eta} \right)^2} \right], \tag{17}$$

and

$$r_{PT-E}(t) = \frac{1}{\sigma} \left[1 + 2\lambda \frac{e^{-\frac{t}{\sigma}} - e^{-2\frac{t}{\sigma}}}{1 - \lambda \left(1 - e^{-\frac{t}{\sigma}} \right)^2} \right]. \tag{18}$$

Let us now discuss the possible shapes of r_{PT-W} and r_{PT-E} as follows:

$$\lim_{t \rightarrow 0} r_{PT-W}(t) = \lim_{t \rightarrow 0} r_W(t) = \begin{cases} 0, & \eta > 1 \\ \frac{1}{\sigma}, & \eta = 1 \\ \infty, & \eta < 1 \end{cases}, \quad \lim_{t \rightarrow +\infty} r_{PT-W}(t) = \lim_{t \rightarrow +\infty} r_W(t) = \begin{cases} \infty, & \eta > 1 \\ \frac{1}{\sigma}, & \eta = 1, \lambda \neq 1 \\ \frac{2}{\sigma}, & \eta = 1, \lambda = 1 \\ 0, & \eta < 1 \end{cases},$$

and asymptotic behaviour of r_{PT-E} is obvious for the case $\eta = 1$, that is

$$\lim_{t \rightarrow 0} r_{PT-E}(t) = \frac{1}{\sigma}, \quad \lim_{t \rightarrow +\infty} r_{PT-E}(t) = \begin{cases} \frac{1}{\sigma}, & \lambda \neq 1 \\ \frac{2}{\sigma}, & \lambda = 1 \end{cases}.$$

Some plots of the r_{PT-W} and r_{PT-E} are given by Figure 2.

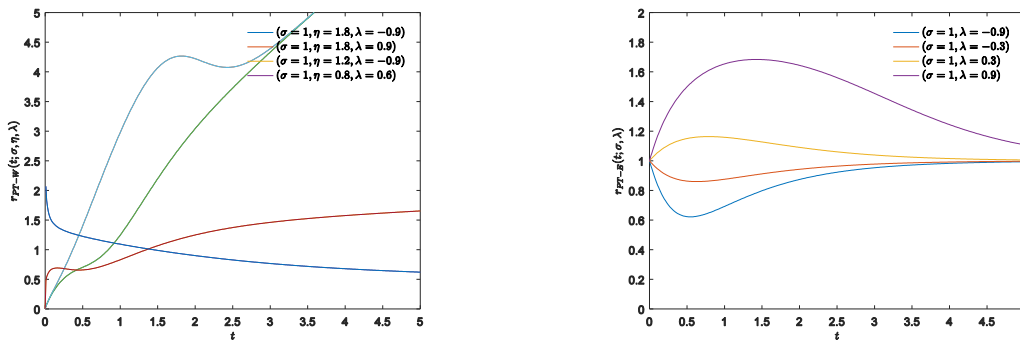


Figure 2. Plots of the r_{PT-W} and r_{PT-E} for some values of the parameters

8.3 Raw Moments of PT-W and PT-E Random Variables

By addressing to [11], the k th raw moments of the PT-W random variable can be derived from the eq. (13) as follows:

$$E_{PT-W} [T^k] = \sigma^k \Gamma \left(1 + \frac{k}{\eta} \right) \left[(1 - \lambda) + 2^{\frac{k}{\eta} + 1} \lambda - 3^{\frac{k}{\eta}} \lambda \right].$$

Hence k th raw moments of the PT-E random variable can be obtained by taking $\eta = 1$ in the latter equality as follows:

$$E_{PT-E} [T^k] = k! \sigma^k \left[(1 - \lambda) + \frac{\lambda}{2^{k-1}} - \frac{\lambda}{3^k} \right].$$

8.4 Maximum Likelihood Estimates of the Parameters of PT-W and PT-E Distributions

Let x_1, x_2, \dots, x_n be observations from the PT-W distribution with parameters σ , η and λ . Suppose that $\Theta = (\sigma, \eta, \lambda)'$ is the parameter vector. The log-likelihood function for Θ is given by

$$\ell(\Theta) = n \log \eta - n\eta \log \sigma + (\eta - 1) \sum_{j=1}^n \log t_j - \sum_{j=1}^n \left(\frac{t_j}{\sigma}\right)^\eta + \sum_{j=1}^n \log \left[1 - \lambda \left(1 - 4e^{-\left(\frac{t_j}{\sigma}\right)^\eta} + 3e^{-2\left(\frac{t_j}{\sigma}\right)^\eta} \right) \right].$$

To maximize $\ell(\Theta)$ with respect to σ , η and λ , we have the following system of non linear equations:

$$\frac{\partial \ell(\Theta)}{\partial \sigma} = \frac{\eta}{\sigma} \left\{ -n + \sigma^{-\eta} \left[\sum_{j=1}^n t_j^\eta + 2\lambda \sum_{j=1}^n \frac{t_j^\eta e^{-\left(\frac{t_j}{\sigma}\right)^\eta} \left(2 - 3e^{-\left(\frac{t_j}{\sigma}\right)^\eta} \right)}{1 - \lambda \left(1 - 4e^{-\left(\frac{t_j}{\sigma}\right)^\eta} + 3e^{-2\left(\frac{t_j}{\sigma}\right)^\eta} \right)} \right] \right\} = 0, \quad (19)$$

$$\frac{\partial \ell(\Theta)}{\partial \eta} = \frac{n}{\eta} - n \log \sigma + \sum_{j=1}^n \log t_j - \sum_{j=1}^n \left(\frac{t_j}{\sigma}\right)^\eta \log \left(\frac{t_j}{\sigma}\right) - 2\lambda \sum_{j=1}^n \frac{\left(\frac{t_j}{\sigma}\right)^\eta \log \left(\frac{t_j}{\sigma}\right) e^{-\left(\frac{t_j}{\sigma}\right)^\eta} \left(2 - 3e^{-\left(\frac{t_j}{\sigma}\right)^\eta} \right)}{1 - \lambda \left(1 - 4e^{-\left(\frac{t_j}{\sigma}\right)^\eta} + 3e^{-2\left(\frac{t_j}{\sigma}\right)^\eta} \right)} = 0, \quad (20)$$

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = - \sum_{j=1}^n \frac{1 - 4e^{-\left(\frac{t_j}{\sigma}\right)^\eta} + 3e^{-2\left(\frac{t_j}{\sigma}\right)^\eta}}{1 - \lambda \left(1 - 4e^{-\left(\frac{t_j}{\sigma}\right)^\eta} + 3e^{-2\left(\frac{t_j}{\sigma}\right)^\eta} \right)} = 0. \quad (21)$$

This system of non-linear equations can be solved numerically. By using nonlinear optimization algorithms to numerically maximize the log likelihood function, ML estimates can be obtained. Similarly, to obtain ML estimates of the parameters of PT-E distribution, it is taken $\eta = 1$ in eq. (19) and (21).

9. APPLICATION

Throughout the application section, we use the Akaike Information Criterion(AIC) as a model selection criterion. Furthermore, Kolmogorov-Smirnov, Anderson-Darling and Cramér-von Mises statistics are taken into account as measures of Goodness-of-fit.

Akaike information criterion:

$$AIC = -2 \log L(\Theta; \underline{x}) + 2k,$$

where k is the size of the parameter vector Θ .

Goodness-of-fit statistics

Kolmogorov- Smirnov

$$K - S = \sup_x |H(x) - H_n(x)|,$$

where $H_n(x)$ is the empirical distribution function.

Anderson-Darling

$$\mathcal{A}^2 = - \left[n + \frac{1}{n} \sum_{j=1}^n \left[(2j-1) \log \left(H(t_{(j)}; \hat{\Theta}) \right) + (2(n-j)+1) \log \left(1 - H(t_{(j)}; \hat{\Theta}) \right) \right] \right]$$

(see, [15], [3] and [13]).

Cramér-von Mises

$$W^2 = \frac{1}{12n} + \sum_{j=1}^n \left(H(t_{(j)}; \hat{\theta}) - \frac{2j-1}{2n} \right)^2$$

where $t_{(j)}$ is the j th ordered sample, $t_1 \leq \dots \leq t_{(j-1)} \leq t_{(j)} \leq \dots \leq t_{(n)}$ (see, [13]).

The first data set illustrates the modelling performance of PT-W distribution, and the second data set is given to illustrate modelling success of PT-E distribution.

Data set 1: This data set represents the failure times of Kevlar 49/epoxy strands when the pressure is at 70% stress level. This data set is taken from [4] and given in Table 1.

The Weibull (W), transmuted Weibull (T-W) and polynomial transmuted Weibull (PT-W) distributions are fitted to the data and the MLEs of the parameters are computed. The values of Kolmogorov-Smirnov statistic (K-S), Akaike information criterion (AIC), Anderson-Darling statistic (A^2) and Cramér–von Mises statistic (W^2) are also given with these MLEs of the parameters in Table 2. A graphical comparison of the fitted models is displayed in Figure 3.

Table 1. Kevlar 49/epoxy strands failure times data (pressure at 70%).

1051	1337	1387	1921	1942	2322	3629	4006	4012	4063
4921	5445	5620	5817	5905	5956	6068	6121	6473	7501
7886	8108	8546	8666	8831	9106	9711	9806	10205	10396
10861	11026	11214	11362	11604	11608	11745	11762	11895	12044
13520	13670	14110	14496	15395	16179	17092	17568	17568	

This data set was previously used by [1]. According to their analyzes, they reported that this data is multimodal, platykurtic, and approximately symmetric (skewness = 0.1, kurtosis = -0.79). They show that Gumbel-Weibull distribution (GWD) provides an adequate fit to bimodal data (K-S= 0.0749).

Table 2. MLEs of the model parameters and values of goodness of fit statistics for models (Kevlar 49/epoxy data).

Model	Parameter Estimates	K-S	AIC	A^2	W^2
W	$\hat{\sigma} = 9906 \quad \hat{\eta} = 2$	0.0877	965.6987	0.5503	0.0778
T- W	$\hat{\sigma} = 9219.4 \quad \hat{\eta} = 1.9 \quad \hat{\lambda} = -0.3$	0.0839	967.3870	0.5028	0.0696
PT-W	$\hat{\sigma} = 8540.9 \quad \hat{\eta} = 2 \quad \hat{\lambda} = -1$	0.0604	964.8448	0.2638	0.0338

According to AIC values and measures of Goodness-of-fit, PT-W model is the best among the suggested models for fitting Kevlar 49/epoxy data. According to reported result of [1], PT-W distribution is as capable as GWD distribution for modelling bimodal data.

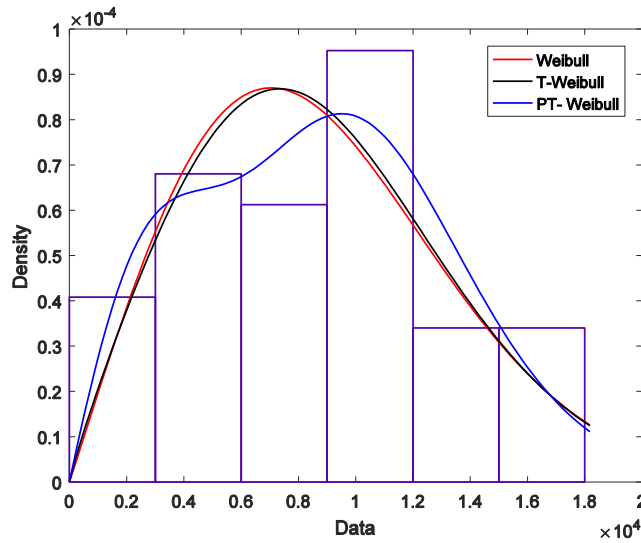


Figure 3. The histogram and the pdfs' of the fitted models for Kevlar 49/epoxy data.

Data set 2 The following data represent (naturally occurring) concentrations of uranium in ground water for a random sample of 100 Northwest Texas wells. This data set is originally reported by [12] and technical report is in Chapter 17 of [4]. Tabulated data set is given in Table 3 as below:

Table 3. Concentrations of uranium in ground water (sample size $n=100$).

8.0	13.7	4.9	3.1	78.0	9.7	6.9	21.7	26.8	56.2	25.3	4.4	29.8
22.3	9.5	13.5	47.8	29.8	13.4	21.0	26.7	52.5	6.5	15.8	21.2	13.2
12.3	5.7	11.1	16.1	11.4	18.0	15.5	35.3	9.5	2.1	10.4	5.3	11.2
0.9	7.8	6.7	21.9	20.3	16.7	2.9	124.2	58.3	83.4	8.9	18.1	11.9
6.7	9.8	15.1	70.4	21.3	58.2	25.0	5.5	14.0	6.0	11.9	15.3	7.0
13.6	16.4	35.9	19.4	19.8	6.3	2.3	1.9	6.0	1.5	4.1	34.0	17.6
18.6	8.0	7.9	56.9	53.7	8.3	33.5	38.2	2.8	4.2	18.7	12.7	3.8
8.8	2.3	7.2	9.8	7.7	27.4	7.9	11.1	24.7				

The subject data is fitted with both the exponential (E) and Transmuted-Exponential (T-E) distributions besides Polynomial Transmuted-Exponential (PT-E) distribution. The MLEs of the parameters, the values of Kolmogorov-Smirnov statistic (K-S), Akaike information criterion (AIC), Anderson-Darling statistic (A^2) and Cramér–von Mises statistic (W^2) are given in the Table 4. A graphical comparison of the fitted models is displayed in Figure 4.

Table 4. MLEs of the model parameters and values of goodness of fit statistics (uranium data).

Model	Parameter Estimates	K-S	AIC	A^2	W^2
E	$\hat{\sigma} = 19.4680$	0.0983	795.7544	1.6193	0.2313
T- E	$\hat{\sigma} = 14.4914 \quad \hat{\lambda} = -0.6526$	0.0846	795.6083	1.2651	0.2114
PT-E	$\hat{\sigma} = 24.1566 \quad \hat{\lambda} = 0.6448$	0.0771	794.0873	1.0416	0.1327

The values in Table 4 indicate that the PT-E distribution leads to a better fit than E and T-E distributions. The relative histogram and the fitted pdf of the models are plotted in Figure 4.

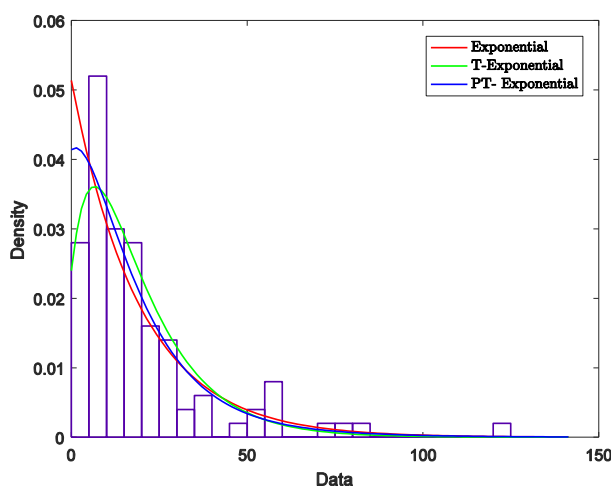


Figure 4. The histogram and the pdfs' of the fitted models for uranium data.

10. CONCLUSION

In this paper we propose and study a new class of distributions called the Polynomial Transmuted Family (PT-D). We investigate several structural properties such as the cumulative distribution function, the probability density function, the moment generating function, the raw moments, the survival and the hazard rate functions. We compare the hazard rate functions of PT and QRT distributions in the sense of monotonicity. For the special cases of PT-D family, Weibull and exponential distributions are considered as the base distribution. Some mathematical and statistical properties are given for PT-W and PT-E models. Two examples of real data sets prove empirically the importance and potentiality of the proposed family. In particular, it can be said that PT-W is successful in modeling the bimodal data sets.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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