

On The Robust $\mathcal{D}$ -Stability of The Polytope of Matrices

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Matrisler Politopunun Gürbüz $\mathcal{D}$ -Kararlılığı Üzerine

Şerife YILMAZ* 

Department of Mathematics and Science Education, Faculty of Education, Burdur Mehmet Akif Ersoy University, Türkiye



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Abstract

In this study, the robust $\mathcal{D}$ -stability of a matrix polytope, defined as the convex hull of a finite set of $\mathcal{D}$ -stable matrices, is considered. A sufficient condition for the robust $\mathcal{D}$ -stability of the matrix polytope is derived for the region $\mathcal{D} \subset \mathbb{C}$ defined as quadratic matrix inequalities (QMI). This condition is given by linear matrix inequalities (LMIs). The solution of the LMIs is addressed by optimizing the maximal eigenvalue function of symmetric matrices using convex optimization methods. Illustrative examples are provided to demonstrate the practical applications of the proposed stability condition.

Keywords: Matrix Polytope; Robust $\mathcal{D}$ -Stability; QMI Region; Linear Matrix Inequality; Convex Programming.

Öz

Bu çalışmada, sonlu sayıda $\mathcal{D}$ -kararlı matrislerin konveks zarfı şeklinde verilen bir matris politopunun gürbüz $\mathcal{D}$ -kararlılığı ele alınmıştır. $\mathcal{D} \subset \mathbb{C}$ bölgesi, kuadratik matris eşitsizlikleri (KME) ile tanımlanan bir bölge olmak üzere, matrisler politopunun gürbüz $\mathcal{D}$ -kararlılığı için bir yeter koşul elde edilmiştir. Bu koşul doğrusal matris eşitsizlikleri (DME) ile verilmiştir. DME'lerin çözümü konveks optimizasyon yöntemleri kullanılarak simetrik matrislerin maksimal özdeğer fonksiyonunun optimize edilmesiyle ele alınmıştır. Kararlılık koşulunun uygulamaları açıklayıcı örneklerle verilmiştir.

Anahtar Kelimeler: Matrisler Politopu; Gürbüz $\mathcal{D}$ -Kararlılık; KME Bölgesi; Lineer Matris Eşitsizliği; Konveks Programlama.

1. Introduction

The asymptotic stability of linear systems described by the equation $\dot{x} = Ax$ is an important concept in dynamical systems and control theory. The system's stability is determined by the location of the eigenvalues, which are the roots of the characteristic polynomial of the matrix A in the complex plane.

Let $\mathcal{D} \subset \mathbb{C}$ be a symmetric region of the complex plane to the real axis. A matrix $A \in \mathbb{R}^{n \times n}$ is called $\mathcal{D}$ -stable if all its eigenvalues lie within the set $\mathcal{D}$ (Barmish 1994). The Hurwitz and Schur stability regions are the most well-known of these regions. If the region $\mathcal{D}$ is open left half-plane, then the matrix A is called Hurwitz stable. If the region $\mathcal{D}$ is an open unit disk, then the matrix A is called Schur stable. $\mathcal{D}$ -stability problem of matrices and polynomials have been investigated in many works (see (Bhattacharyya et al. 1995, Fam and Meditch 1978, Hinrichsen and Pritchard 2005, Yedavalli 2014, Yılmaz 2022, Zaitsev 2024)).

For a matrix A to be Hurwitz stable, a necessary and sufficient condition is the existence of a positive definite matrix $P > 0$ such that the Lyapunov inequality must be satisfied:

$$A^T P + P A < 0.$$

Similarly, a matrix A is Schur stable if and only if there exists a positive definite matrix $P > 0$ such that the Stein inequality must be hold (see Khalil 2002, Yılmaz and Aksoy 2023):

$$A^T P A - P < 0.$$

The location of the eigenvalues in the complex plane of a matrix has been investigated for more specific regions $\mathcal{D}$. In Gutman (1981), a polynomial-type function has been adopted to describe the region of the complex plane in which roots are located. Necessary and sufficient conditions have been derived from the Kronecker and bialternate matrix products and positive definite matrices in Gutman (1981), Zrida and Bouazizi (2022), Chilali and Gahinet (1996).

In Zrida and Bouazizi (2022), the concept of quadratic matrix inequalities (QMI) is introduced to characterize stability regions defined by

$$\mathcal{D} = \{z \in \mathbb{C}: Q + Sz + S^T \bar{z} + Rz\bar{z} < 0\} \quad (1)$$

where Q and R are symmetric matrices in $\mathbb{R}^{m \times m}$, with $R \geq 0$ and $S \in \mathbb{R}^{m \times m}$ (we take $R > 0$ in this paper to be

able to implement the Schur complement lemma). Various convex regions in the complex plane can be represented using QMI. Important properties of QMI regions include being open, convex, and symmetrical to the real axis. Furthermore, any intersection of QMI regions is also a QMI region. The robust $\mathcal{D}$ -stability of a parameter-dependent matrix family is determined using test points to assess whether LMIs are satisfied within specific intervals. This way, the matrix family's $\mathcal{D}$ -stability bound can be determined reliably.

In this paper, we build on existing results to provide sufficient conditions for the robust $\mathcal{D}$ -stability of matrix polytopes. Using the Schur complement lemma, we derive conditions that simplify the associated matrix inequality problem. To solve the convex optimization problem arising from these inequalities, we introduce an algorithm based on Kelley's method. The proposed conditions and algorithm are validated through illustrative examples, demonstrating their effectiveness in addressing robust $\mathcal{D}$ -stability.

The paper is structured as follows. Section 2 introduces the concept of robust $\mathcal{D}$ -stability for a polytope and provides the necessary background. Section 3 presents sufficient conditions for robust $\mathcal{D}$ -stability of matrix polytopes using the Schur complement lemma. Section 4 describes a solution algorithm for LMIs based on convex optimization and includes numerical examples to demonstrate its application.

2. Robust $\mathcal{D}$ -stability of a Polytope

A useful theorem for determining the eigenvalues of a matrix A within the set $\mathcal{D}$ (1) is given in Chilali and Gahinet (1996). This theorem provides the necessary and sufficient conditions in the form of Linear Matrix Inequalities (LMIs) for $\mathcal{D}$ -stability.

In the following theorem, the Kronecker product ($\otimes$) represents an operation that combines two matrices into a structured block matrix, preserving their algebraic structure.

Theorem 1 (Chilali and Gahinet (1996)): $A \in \mathbb{R}^{n \times n}$ is $\mathcal{D}$ -stable if and only if there exists a symmetric positive definite matrix $P > 0$ such that

$$Q \otimes P + S \otimes (AP) + S^T \otimes (PA^T) + R \otimes (APA^T) < 0. \quad (2)$$

We have provided the definition and some results concerning the $\mathcal{D}$ -stability of matrices. Consider a matrix family of n -dimensional square matrices denoted by $\mathcal{A}$. If every matrix within the family $\mathcal{A}$ is $\mathcal{D}$ -stable, then the family $\mathcal{A}$ is referred to as robustly $\mathcal{D}$ -stable. The well-

known matrix family is the polytope of matrices, and it is defined as the convex hull of two or more finite Hurwitz (Schur) matrices

$$\begin{aligned} \mathcal{A} &= \text{conv}\{A_1, A_2, \dots, A_N\} \\ \mathcal{A} &= \{A(\alpha) = \sum_{i=1}^N \alpha_i A_i : \alpha \in \Lambda\} \end{aligned} \quad (3)$$

where

$$\Lambda = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0 \right\}.$$

Edge Theorem is a recognized theorem in control theory that gives robust $\mathcal{D}$ -stability of a polytope of polynomials (Bartlett et al. 1988). However, this theorem is not valid for a matrix polytopes (Barmish et al. 1988).

The robust stability of the polytope of matrices has been studied using linear matrix inequalities (LMIs) techniques (Boyd et al. 1994). A necessary condition for $\mathcal{A}$ to be robustly Hurwitz (Schur) stable is the existence of a common solution $P > 0$ that satisfies the Lyapunov (Stein) inequalities:

$$A_i^T P + P A_i < 0 \quad (i = 1, 2, \dots, N)$$

$$(A_i^T P A_i - P < 0)$$

(see Shorten and Narendra 2002, Yilmaz et al. 2014, Yilmaz et al. 2016).

The Theorem 1 regarding the robust $\mathcal{D}$ -stability of the matrix polytope defined in (3) can be stated as follows.

Theorem 2: $\mathcal{A}$ (3) is robustly $\mathcal{D}$ -stable if and only if there exists $P(\alpha) > 0$ such that

$$Q \otimes P(\alpha) + S \otimes (A(\alpha)P(\alpha)) + S^T \otimes (P(\alpha)A^T(\alpha)) + R \otimes (A(\alpha)P(\alpha)A^T(\alpha)) < 0 \quad (4)$$

for all $\alpha \in \Lambda$.

For the robust $\mathcal{D}$ -stability of $\mathcal{A}$, the existence of positive definite matrices $P(\alpha) > 0$ for each $\alpha \in \Lambda$ is required, thus resulting in problems of infinite dimension (Geromel et al. 1998, Oliveira and Peres 2005). In Theorem 2, it is stated that if such a $P(\alpha)$ satisfies inequality (4) for all α , the conclusion follows.

However, if we can establish the existence of a common positive definite matrix $P(\alpha) = P$ that satisfies the inequality uniformly for all α , we can avoid the complexity

associated with individually determining $P(\alpha)$ for each α . This reduction is applied to the case of matrix polytopes $\mathcal{A}$, where finding a common P that satisfies the condition for $A_1, A_2, \dots, A_N$.

In this paper, we provide sufficient conditions for the robust $\mathcal{D}$ -stability of the polytope of matrices $\mathcal{A}$. For this, if $P > 0$ exists for the matrix $A_1, A_2, \dots, A_N$ matrices, then this solution satisfies the matrix inequalities in Theorem 2 for all matrices in the polytope of matrices $\mathcal{A}$. We obtained this result using the Schur complement lemma. Many studies are in the literature on solving linear matrix inequalities (Polyak et al. 2021). Here, we use the Kelley method to calculate the minimum of the convex function $\phi: \mathbb{R}^r \rightarrow \mathbb{R}$, defined on the set of symmetric matrices and represented by r -dimensional vectors. When solving the minimum problem using the iterative process, the constructed algorithm effectively addresses the matrix inequalities if the ϕ function takes a negative value within the set of positive matrices.

3. Common Solution to Matrix Polytope

In this section, we give sufficient conditions for the robust $\mathcal{D}$ -stability of the matrix polytope $\mathcal{A}$. To achieve this, we will linearize the expression $A(\alpha)P(\alpha)A(\alpha)^T$ on the left side of inequality (4) using Schur's complementary lemma.

Lemma 1 (Geromel 1998): Consider the following matrix

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}$$

where $M_1 \in \mathbb{R}^{n \times n}$, $M_2 \in \mathbb{R}^{n \times m}$, and $M_3 \in \mathbb{R}^{m \times m}$ with M_3 invertible. Then $M > 0$ if and only if

$$M_3 > 0 \text{ and } M_1 - M_2 M_3^{-1} M_2^T > 0.$$

Lemma 2: There exists a $P > 0$ satisfying equation (2) if and only if there exists a $P > 0$ such that

$$\begin{bmatrix} -Q \otimes P - S \otimes (AP) - S^T \otimes (PA^T) & R \otimes (AP) \\ R \otimes (PA^T) & R \otimes P \end{bmatrix} > 0. \quad (5)$$

Proof: Let $M_1 = -(Q \otimes P + S \otimes (AP) + S^T \otimes (PA^T))$, $M_2 = R \otimes (AP)$ and $M_3 = R \otimes P$. Since $P > 0$ and $R > 0$, all eigenvalues of P and R are positive (Horn and Johnson 2013, p. 438). On the other hand, the matrix $R \otimes P$ is symmetric, and its eigenvalues are the product of the eigenvalues of P and R (Bernstein 2009, p. 401). Hence,

$M_3 > 0$. By utilizing the properties of the Kronecker product (see Bernstein 2009, Chapter 7), we can derive the following equations:

$$\begin{aligned} M_2 M_3^{-1} M_2^T &= [R \otimes (AP)][R \otimes P]^{-1}[R \otimes (AP)]^T \\ &= [R \otimes (AP)][R^{-1} \otimes P^{-1}][R^T \otimes (PA^T)] \\ &= [R \otimes (AP)][(R^{-1}R) \otimes (P^{-1}PA^T)] \\ &= [R \otimes (AP)][I_n \otimes A^T] \\ &= R \otimes (APA^T). \end{aligned}$$

Here, we have used the Kronecker product property

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

to simplify the expressions in the above derivation. This property allows the product of Kronecker terms to be distributed as the product of their respective matrices.

Therefore,

$$\begin{aligned} M_1 - M_2 M_3^{-1} M_2^T &= -Q \otimes P - S \otimes (AP) - \\ &\quad S^T \otimes (PA^T) - R \otimes (APA^T) > 0. \end{aligned}$$

□

Theorem 3: Let the matrix polytope $\mathcal{A}$ (3) be given and $P > 0$ satisfies

$$\begin{aligned} Q \otimes P + S \otimes (A_i P) + S^T \otimes (PA_i^T) \\ + R \otimes (A_i P A_i^T) < 0 \end{aligned} \quad (6)$$

for $i = 1, 2, \dots, N$. In that case, the inequality (6) is satisfied for all $A \in \mathcal{A}$. Consequently, the matrix polytope $\mathcal{A}$ is robustly $\mathcal{D}$ -stable.

Proof: For each $i = 1, 2, \dots, N$, a $P > 0$ exists such as equation (6) holds. Let $M(A) = [M_{ij}(A)]$ be a block matrix with

$$\begin{aligned} M_{11}(A) &= -[Q \otimes P + S \otimes (AP) + S^T \otimes (PA^T)], \\ M_{12}(A) &= R \otimes (AP), \\ M_{21}(A) &= R^T \otimes (PA^T), \\ M_{22}(A) &= R \otimes P. \end{aligned}$$

By Lemma 2,

$$M(A_i) = \begin{bmatrix} M_{11}(A_i) & M_{12}(A_i) \\ M_{21}(A_i) & M_{22}(A_i) \end{bmatrix} > 0, \quad (7)$$

for $i = 1, 2, \dots, N$. If both sides of the inequality (7) are multiplied by α_i and summed from $i = 1$ to N

$$M(A(\alpha)) = \begin{bmatrix} M_{11}(A(\alpha)) & M_{12}(A(\alpha)) \\ M_{21}(A(\alpha)) & M_{22}(A(\alpha)) \end{bmatrix} > 0$$

where $\alpha \in \Lambda$. According to Lemma 2,

$$Q \otimes P + S \otimes (A(\alpha)) + S^T \otimes (PA^T(\alpha)) + R \otimes (A(\alpha)PA^T(\alpha)) < 0$$

is satisfied for every $\alpha \in \Lambda$, which proves that $\mathcal{A}$ is robustly $\mathcal{D}$ -stable. $\square$

We will convert the given condition into a convex minimization problem to assess a matrix polytope's robust $\mathcal{D}$ -stability, utilizing the sufficient condition provided in Theorem 3.

4. A Solution Algorithm for LMIs Using a Convex Optimization Method

In this section, we present a convex optimization method for finding a positive definite matrix $P > 0$ that satisfies the inequality (6) for $\mathcal{D}$ -stable matrices $\{A_1, A_2, \dots, A_N\}$.

The maximal and minimal eigenvalues of a symmetric matrix $S \in \mathbb{R}^{n \times n}$ are defined as follows:

$\lambda_{\max}(S) = \max_{\|v\|=1} v^T S v$, $\lambda_{\min}(S) = \min_{\|v\|=1} v^T S v$
(see Horn and Johnson 2013, p. 235). If the unit eigenvector corresponding to the largest eigenvalue is $\tilde{v}$ then

$$\lambda_{\max}(S) = \tilde{v}^T S \tilde{v}$$

A similar equality holds for the minimum eigenvalue.

Let $x \in \mathbb{R}^r$ be $x^T = (x_1, x_2, \dots, x_r)$ and P be an $n \times n$ symmetric matrix defined as

$$P = P(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_r \end{bmatrix} \quad (8)$$

with $r = n(n+1)/2$.

Denote the left-hand side of inequality (6) by

$$F(x, i) = Q \otimes P + S \otimes (A_i P) + S^T \otimes (P A_i^T) + R \otimes (A_i P A_i^T).$$

Define the functions $\phi(x)$ and $\eta(x)$ as follows:

$$\begin{aligned} \phi(x) &= \max_{1 \leq i \leq N} \lambda_{\max}(F(x, i)) \\ \eta(x) &= \max_{1 \leq i \leq N, \|v\|=1} v^T F(x, i) v, \end{aligned} \quad (9)$$

$$\eta(x) = \min_{\|v\|=1} v^T P(x) v. \quad (10)$$

The functions $\phi(x)$ and $\eta(x)$ are both convex functions.

If there exists a $\tilde{x} \in \mathbb{R}^r$ such that $\phi(\tilde{x}) < 0$ and $\eta(\tilde{x}) > 0$, then the matrix $P = P(\tilde{x}) > 0$ satisfies the matrix inequality (6) for each $i = 1, 2, \dots, N$. Therefore, the matrix polytope $\mathcal{A} = \text{conv}\{A_1, A_2, \dots, A_N\}$ is robustly $\mathcal{D}$ -stable.

For the positive definite matrix $P(x) > 0$, we take x from the following box:

$$\mathcal{X} = [-1, 1] \times \cdots \times [-1, 1] = [-1, 1]^r,$$

since $\alpha > 0$ and $\alpha P(x) > 0$.

Consider the following convex minimization problem:

$$\begin{aligned} &\text{minimize} && \phi(x) \\ &\text{subject to} && \eta(x) > 0, \\ &&& x \in \mathcal{X}. \end{aligned} \quad (11)$$

To address this convex minimization problem, Kelley's cutting plane method is utilized (see Yılmaz et al. 2014). This method reformulates problem (11) as follows:

$$\begin{aligned} &c^T z \rightarrow \min \\ &c_1(z) \geq 0, \\ &c_2(z) \geq 0, \\ &-1 \leq x_i \leq 1 \quad (i = 1, 2, \dots, r) \end{aligned} \quad (12)$$

where $z = (x_1, x_2, \dots, x_r, L)^T$, $c = (0, \dots, 0, 1)^T$, $c_1(z) = L - \phi(x)$, and $c_2(z) = \min_{\|v\|=1} v^T P(x) v$.

Let z^0 be an initial starting point, and let $z^1, z^2, \dots, z^k$ represent subsequent iterations. In the first iteration, the cutting-plane algorithm tackles the following linear programming (LP) problem:

$$\begin{aligned} &\text{minimize} && L \\ &\text{subject to} && -h_1^T(z^0)z \geq -h_1^T(z^0)z^0 - c_1(z^0) \\ &&& -h_2^T(z^0)z \geq -h_2^T(z^0)z^0 - c_2(z^0) \\ &&& -1 \leq x_i \leq 1 \quad (i = 1, 2, \dots, r) \end{aligned} \quad (13)$$

where $h_j(z^i)$ denotes a gradient of $-c_j(z)$ at z^i ($j = 1, 2$).

Here, we have

$$h_1(z^i) = -\frac{\partial}{\partial z} (z_{r+1} - v^T F(x, i) v) \Big|_{v=\hat{v}, i=i}$$

where $\hat{i}$ is the index that maximizes $\lambda_{\max}(F(x, i))$ and $\hat{v}$ is a corresponding unit eigenvector.

Additionally, we also have:

$$h_2(z^i) = -\frac{\partial}{\partial z} (v^T P(x)v)|_{v=\hat{v}}$$

where $\hat{v}$ is the unit vector corresponding to $\lambda_{\min}(P(x))$.

Let the point z_*^k be the minimizer of the problem defined in (13). If z_*^k satisfies the inequality

$$\min\{c_1(z_*^k), c_2(z_*^k)\} \geq -\varepsilon,$$

where $\varepsilon > 0$ is a tolerance, then z_*^k is considered an approximate solution to the problem in (13).

If this condition is not satisfied, we define j^* as the index of the most negative $c_j(z_*^k)$. To update the constraints in (13) by adding the linear constraint:

$$c_{j^*}(z^{k+1}) - h_{j^*}^T(z^{k+1})(z - z^{k+1}) \geq 0, \quad (14)$$

and then repeat the process.

This algorithm can be adapted to our problem as follows:

Algorithm 1:

1. Select an initial point $z^0 = (x^0, L^0)^T$. Compute $\phi(x^0)$ and $c_2(z^0)$. If $\phi(x^0) < 0$ and $c_2(z^0) > 0$, complete. Otherwise, continue to the next step.
2. Find z_*^k by solving the LP problem in (13). If $c_1(z_*^k) > L^k$ and $c_2(z_*^k) > 0$, complete. Otherwise, proceed. Set $z^{k+1} = z_*^k$, update the constraints in (13) and repeat the procedure.

The stopping criteria for Algorithm 1 are implicitly defined through $c_1(z^k) > L^k$ and $c_2(z^k) > 0$. These conditions ensure that the solution satisfies the robust $\mathcal{D}$ -stability criteria. Specifically:

1. When $c_1(z^k) > L^k$, by the definition $c_1(z) = L - \phi(x)$, it follows that $\phi(x^k) < 0$. This implies that the left-hand side of the matrix inequality (6) is strictly negative, confirming the $\mathcal{D}$ -stability of the matrix family.
2. The condition $c_2(z^k) > 0$ ensures that the matrix $P(x^k)$ is positive definite, which is a necessary condition for stability.

Together, these criteria validate that the algorithm has reached a feasible and robustly $\mathcal{D}$ -stable solution.

Example 1: Consider the region $\mathcal{D}$ (1) defined by

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \\ 0 & -\cos \frac{\pi}{3} & \sin \frac{\pi}{3} \end{bmatrix}$$

(see Fig. 1). Let $\mathcal{A}$ be the convex hull of the $\mathcal{D}$ -stable matrices

$$A_1 = \begin{bmatrix} -0.35 & -0.05 & 0.15 \\ -0.15 & -0.6 & 0.05 \\ -0.05 & -0.1 & -0.65 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.45 & 0.67 & -0.45 \\ -0.33 & -2.13 & 1.05 \\ 0.34 & -2.8 & 1.5 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.5 & 0.02 & -0.05 \\ 0.07 & -0.52 & 0.05 \\ 0.02 & -0.02 & -0.47 \end{bmatrix}.$$

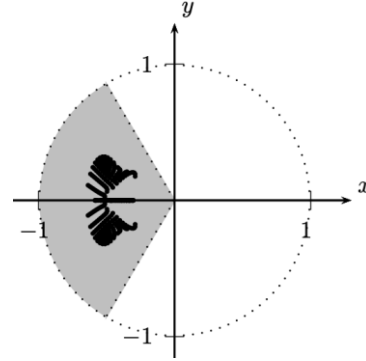


Figure 1. The region $\mathcal{D}$ and eigenvalues of the matrices within the polytope $\mathcal{A}$ in Example 1.

Let $z^0 = (x^0, L^0) = (1, 0, 0, 1, 0, 1, 1)^T$ be the starting point, and then apply the Algorithm 1. In this case, $c_2(z^0) = 1$, $\phi(x^0) = 9.465$ and the index $i^0 = 2$ yields this maximal eigenvalue.

Table 1. Iterative Results of Algorithm 1 for Example 1

k	L^k	$c_1(z^k)$	$c_2(z^k)$	i^k
0	1	-14.395	1	2
1	-31.731	-41.197	-2.561	2
2	-5.792	-9.450	-2.039	2
3	-1.499	-3.222	-1.834	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
26	-0.0026	-0.0043	0.0059	2
27	-0.0023	-0.0051	-0.0016	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
32	-0.0011	-0.0012	0.0028	1
33	-0.0010	-0.0009	0.0025	2

The solution of LP problem in step 33 is the point $z^{33} = (0.0673, -0.1402, -0.2445, 0.4850, 0.6453, 1, -0.0010)^T$ (see Table 1). Since $\phi(z^{33}) = -0.0001 < 0$ and $c_2(z^{33}) = 0.0025 > 0$, the matrix

$$P(z^{33}) = \begin{bmatrix} 0.0673 & -0.1402 & -0.2445 \\ -0.1402 & 0.4850 & 0.6453 \\ -0.2445 & 0.6453 & 1 \end{bmatrix}$$

satisfies the matrix inequalities (6) for $i = 1, 2, 3$. By Theorem 3, the family of matrices $\mathcal{A} = \text{conv}\{A_1, A_2, A_3\}$ is robustly $\mathcal{D}$ -stable.

Example 2 (Zrida and Bouazizi 2022): Consider the one-parameter matrix family $A(\delta) = A_0 + \delta A_g$ for $\delta \in \mathbb{R}$, where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, A_g = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

The region $\mathcal{D}(1)$ is defined by matrices

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is the intersection of a horizontal strip between $+j/2$ and $-j/2$ and the open left half-plane ($j = \sqrt{-1}$).

It has been established in Zrida and Bouazizi (2022) that the one-parameter matrix $A(\delta)$ is $\mathcal{D}$ -stable for $\delta \in (-1, -0.7321) \cup (2.7321, \infty)$ by analyzing the intervals in which the corresponding matrix function determines its eigenvalues.

In this example, we focus on the second stability interval $(2.7321, \infty)$ and demonstrate the robust $\mathcal{D}$ -stability of the matrix family within the subinterval $[2.74, 4]$ using our proposed method. We consider the $\mathcal{D}$ -stable matrices $A_1 = A(2.74)$ and $A_2 = A(4)$.

Starting with the initial point $z^0 = (1, 0, 1, 1)^T$, the Algorithm 1 gives the solution after 12 steps: $z^{12} = (0.26305, -0.49103, 1, -0.00191)^T$. Since $\phi(z^{12}) = -0.00017 < 0$ and $c_2(z^{12}) = 0.0176 > 0$, the matrix

$$P(z^{12}) = \begin{bmatrix} 0.26305 & -0.49103 \\ -0.49103 & 1 \end{bmatrix}$$

satisfies the inequalities (6) for $i = 1, 2$. By Theorem 3, the matrix $A(\alpha) = (1 - \alpha)A_1 + \alpha A_2$ is $\mathcal{D}$ -stable for all $\alpha \in [0, 1]$ (see Fig. 2).

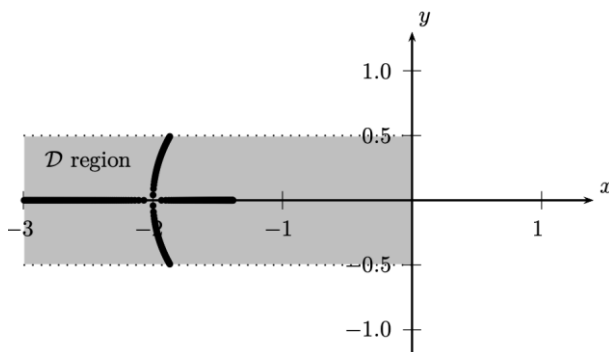


Figure 2. In Example 2, the eigenvalues of $A(\delta)$ for $\delta \in [2.74, 4]$.

4. Conclusions

This paper presents sufficient conditions for robust $\mathcal{D}$ -stability for matrix polytopes. We introduced a solution method utilizing the maximal eigenvalue function of symmetric matrices to solve LMIs. Through examples, the

application of these stability conditions is illustrated. Future work may explore extensions to broader classes of matrix families or additional structural constraints.

Declaration of Ethical Standards

The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement

The entire research, including the formulation of the problem, development of the theoretical framework, implementation of the algorithms, and writing of the manuscript, was conducted by Şerife Yılmaz.

Declaration of Competing Interest

The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability

All data generated or analyzed during this study are included in this published article.

5. References

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